Exp Assume \$20,000 in invested for 3 years at 10 % compounded annually

① How much interest is earned?

First Year $\Rightarrow P_1 = \$ 20{,}000 \Rightarrow I_1 = P_1 r t$

$= (20{,}000)\left(\frac{10}{100}\right)(1)$

$= \$ 2000$

Second Year $\Rightarrow P_2 = P_1 + I_1 = \$ 22{,}000 \Rightarrow I_2 = P_2 r t$

$= (22{,}000)\left(\frac{10}{100}\right)(1)$

$= \$ 2200$

Third Year $\Rightarrow P_3 = P_2 + I_2 = \$ 24{,}200 \Rightarrow I_3 = P_3 r t$

$= (24{,}200)\left(\frac{10}{100}\right)(1)$

$= \$ 2420$

Total Simple Interest $\Rightarrow I = I_1 + I_2 + I_3$

$= 2000 + 2200 + 2420$

$= 6620$ dollars

② Find the future value

or $S = P + I$ total

$$S = P_3 + I_3$$
$$= 24200 + 2420$$
$$= \$\ 26,620$$

or

$$S = P + I$$
$$= 20,000 + 6620$$
$$= \$\ 26,620$$

③ Is there any other method to calculate future value?

If principle P is invested at interest rate r per year compounded annually, then the future value S is given by

$$S = P(1+r)^{\overset{n \to \text{years}}{}}$$

$$= 20,000\left(1+\frac{10}{100}\right)^3$$
$$= 20,000(1+0.1)^3$$
$$= 20,000(1.1)^3$$
$$= \$\ 26,620$$

Exp If \$3000 is invested for 4 years

Exp If $3000 is invested ... at 9% compounded annually.

① How much the future value will be?

$$S = P(1+r)^n = 3000\left(1+\frac{9}{100}\right)^4$$

$$= 3000(1+0.09)^4 = 3000(1.09)^4$$

$$= \$4236$$

② How much interest is earned?

$$S = P + I$$

$$4236 = 3000 + I$$

$$I = 4236 - 3000$$

$$= \$1236$$

Remark t : time in years

we may divide the time t into m times per year per year

$m=1$

1 year

$m=1$ ⇒ the time t is compounded annually

$m=2$

1 year

$m=2$ ⇒ " " " " semiannually

$m=4$

1 year

$m=4$ ⇒ " " " " quarterly

$m=12$

1 year

$m=12$ ⇒ " " " " monthly

r : nominal annual rate (interest rate per year)

$\longleftarrow r \longrightarrow$

1 year

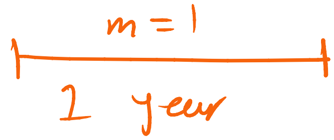
$i = \frac{r}{4}$

compounded quarterly

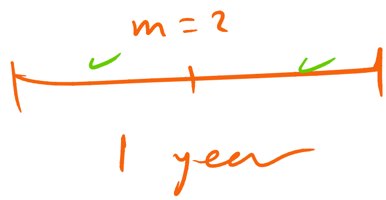
$\longleftarrow r \longrightarrow$

$i = \frac{r}{m}$ interest rate per period

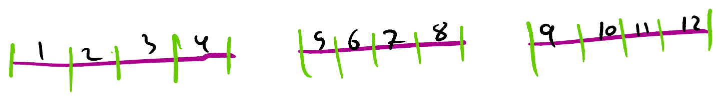
$n = 1$



$$n = m t$$
$$= (2)(1)$$
$$= 2$$



$t = 3$



3 years

$m = 4$

$$n = m t$$
$$= (4)(3) = 12$$

n → total number of compounding periods in t years

1 year

: 1 year

future Value $\qquad S = P(1+i)^{n}$

$$= P\left(1+\frac{r}{m}\right)^{mt}$$

Exp If \$ 5000 in invested at 6% nominal interest rate compounded quarterly for 5 years.

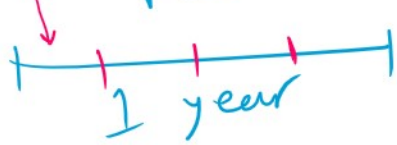
(1) Find the number of compounding periods per year

$m = 4 \qquad$ quarterly

(2) Find the number of componding periods for the investment

$n = mt = (4)(5) = \underline{\underline{20}}$ periods

(3) Find interest rate for each compounding period



m = 4

$r = \frac{6}{100} = 0.06$

$$100$$

$$i = \frac{r}{m} = \frac{0.06}{4} = 0.015$$

④ Find the future value

$$S = P(1+i)^{n} = P\left(1+\frac{r}{m}\right)^{mt}$$

$m \to 4$, $t \to 5$

$$= (5000)(1+0.015)^{20}$$

$$= (5000)(1.015)^{20}$$

$$= \$\,6734.28$$ to the nearest cent

⑤ Find interest that is totally earned

$$S = P + I$$

$$6734.28 = 5000 + I$$

$$I = 6734.28 - 5000$$

$$= \$\,1734.28$$

Exp Find the interest per period i and the number of compounding periods n

and the

and the number of compounding p
for the following investments

① 12 % $\underbrace{\text{compounded monthly}}_{m=12}$ $\underbrace{\text{for 7 years}}_{t=7}$

$$r = \frac{12}{100} = 0.12$$

$$i = \frac{r}{m} = \frac{0.12}{12} = 0.01$$

i=0.01

$$n = mt = 12(7) = 84 \text{ periods}$$

② 7.2 % $\underbrace{\text{compounded quarterly}}_{m=4}$ $\underbrace{\text{for 11 quarters}}_{t\,??}$

$$r = \frac{7.2}{100} = 0.072$$

1 year — 4 quarters

t = ? — 11 quarters

$$4t = (11)(1)$$

$$\boxed{t = \frac{11}{4} \text{ years}}$$

$$i = \frac{r}{m} = \frac{0.072}{4} = 0.018$$

$$n = mt = (\cancel{4})\left(\frac{11}{\cancel{4}}\right) = 11 \text{ periods}$$

Exp what amount must be invested now in order ... after 3 years if

Exp What amount must ... to have \$ 12,000 after 3 years if 6 % interest rate compounded semiannually

P ?? , $S = \$ 12,000$, $t = 3$, $r = \frac{6}{100}$ $m = 2$

0.06

$$S = P(1+i)^n$$

$$S = P\left(1+\frac{r}{m}\right)^{mt}$$

$$12,000 = P\left(1+\frac{0.06}{2}\right)^{(2)(3)}$$

$$= P(1+0.03)^6$$

$$12,000 = P(1.03)^6$$

$$\frac{12,000}{1.194052} = \frac{P\,\cancel{(1.194052)}}{\cancel{1.194052}}$$

$\Rightarrow P = \$ 10,049.81$ to the nearest cent

- How much interest is earned through this investment

$$S = P + I$$

$$S = P + I$$

$$12{,}000 = 10{,}049.81 + I$$

$$I = 12{,}000 - 10{,}049.81$$

$$= \$\,1950.19$$

1 year

$m = 2$ semiannually

$m = 4$ quarterly

$m = 12$ monthly

$m = 365$ daily

...

1 year

every instant

continuous compounding

Def (Future Value for continuous compounding)

If $\$P$ is invensted now for t year at nominal rate r compounded continuously then the future value is given by then the future value is given by

$$S = P e^{rt} , \quad e \approx 2.718$$

Exp Find the future value if \$ 1000 is invested for 20 years at 8 % compounded continuously

$t = 20$ $\quad r = \frac{8}{100} = 0.08$

$$S = P e^{rt}$$

$$= 1000 \, e^{(0.08)(20)}$$

$$= 1000 \, e^{1.6}$$

$$= 1000 \, (4.95303)$$

$$= \$ \, 4953.03$$

Exp What amount must be invested now at 6.5 % compounded continuously, so that it will be worth \$ 25,000 after 8 years

t → 8 years $\quad$ S $\quad$ P ??

6.5 ... .065

$\to 8$ years

$r = \frac{6.5}{100} = 0.065$

$S = P e^{rt}$ ✓

$25{,}000 = P e^{(0.065)(8)}$

$25{,}000 = P e^{0.52}$

$25{,}000 = P(1.68202765)$

$P = \frac{25\,000}{1.68202765} = \$14{,}863.01$

- How much interest the invester has earned?

$S = P + I$

$25{,}000 = 14{,}863.01 + I$

$I = 25{,}000 - 14863.01$

$= \$\,10{,}136.99$

Exp ① How much will you earn if you invest \$1000 for 5 years at 8% compounded continuously

$$S = Pe^{rt} = 1000\, e^{(0.08)(5)} = 1000\, e^{0.4}$$

$$= 1000\,(1.49182)$$

$$= \$\,1491.82$$

② How much will you earn if you invest \$1000 for 5 years at 8% compounded quarterly.

P → \$1000

years → $t = 5$

8% → $r = \frac{8}{100} = 0.08$

$m = 4$ → quarterly

$$S = P(1+i)^n = P\left(1+\frac{r}{m}\right)^{mt}$$

$$= 1000\left(1+\frac{0.08}{4}\right)^{(4)(5)}$$

$$= 1000\,(1+0.02)^{20}$$

$$= 1000\,(1.02)^{20}$$

$$= 1000\,(1.485947)$$

… 05.95

$= \$\, 1485.95$

③ Compare investments in ①, ②

Investment by compounding continuously has extra interest by

$$\underset{S_c}{1491.82} - \underset{S_q}{1485.95}$$

$$= \$\, 5.87$$

extra intrest by continuusly

④ How much interest is earned in ①, ②

① Compounded Continuusly

$$S = P + I_c$$

$$1491.82 = 1000 + I_c$$

$$I_c = \$\, 491.82$$

② Compounded Quarterly

$$S = P + I_q$$

$$1485.95 = 1000 + I_q$$

$$I_q = \$\, 485.95$$

$491.82 - 485.95$

Note that $I_c - I_q = 491.82 - 485.95$

$= \$\, 5.87$

extra interest

$$S = P(1+i)^n = P\left(1+\frac{r}{m}\right)^{mt}$$

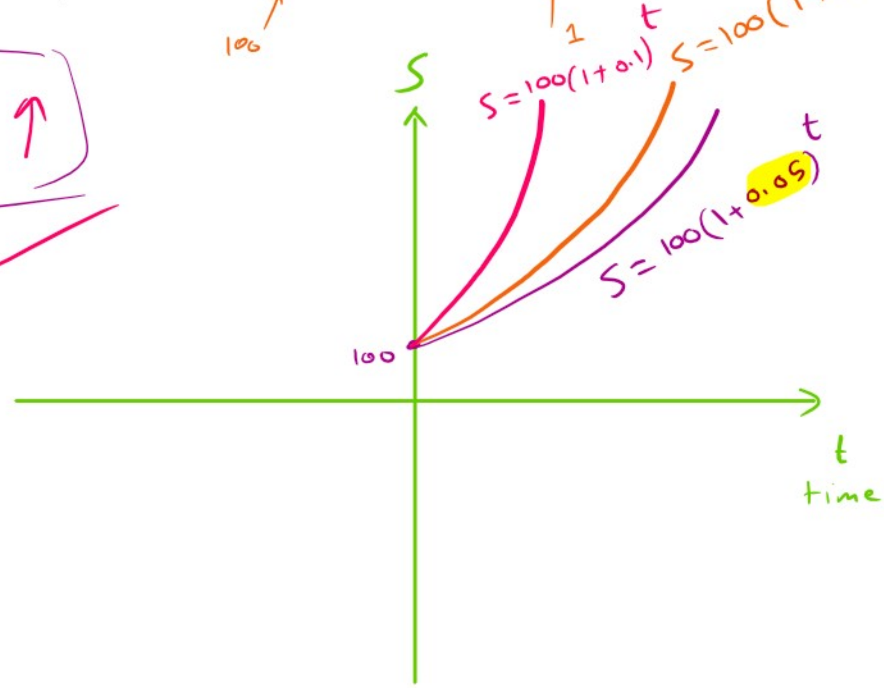
$P = 100$, $m = 1$

- As $\boxed{r\uparrow \Rightarrow S\uparrow}$

Assume • $P = \$\,100$

• $m = 1$

interest rate compounded annually



$$S = 100(1+r)^t$$

when $t = 0 \Rightarrow S = 100(1+r)^0 = 100(1) = 100$

Exp $\Longrightarrow$ Assume $\boxed{t=1}$, $\boxed{r=1}$, $\boxed{P=1}$

$$S = P\left(1+\frac{r}{m}\right)^{mt}$$

$$= 1\left(1+\frac{1}{m}\right)^{m}$$

$$S = \left(1+\frac{1}{m}\right)^{m}$$

- Annually $m=1 \Rightarrow S = \left(1+\frac{1}{1}\right)^{1} = 2$
- Monthly $m=12 \Rightarrow S = \left(1+\frac{1}{12}\right)^{12} = 2.6130\ldots$
- Daily $m = \underline{\underline{360}}$ Business Year $\Rightarrow S = \left(1+\frac{1}{360}\right)^{360} = 2.7145..$
  $(12)(30)$
- Hourly $\Rightarrow m = (360)(24) = 8640 \Rightarrow S = \left(1+\frac{1}{8640}\right)^{8640} = 2.71812..$
- Minute $\Rightarrow m = (8640)(60) = 518400 \Rightarrow S = \left(1+\frac{1}{518400}\right)^{518400} = 2.71827$

$$\downarrow$$

$$\boxed{e \cong 2.718}$$

Compounded continuously

$\searrow$

exponential function

mt

$$S = P\left(1+\frac{i}{m}\right)^{mt}$$

$$= P(1+r)^{n} \longrightarrow Pe^{rt}$$

