

8.2 Trigonometric Integrals

(33)

$$\int \sin^m x \cos^n x dx$$

Case 1 m is odd $\Rightarrow m = 2k+1$ we use $\sin^2 x = 1 - \cos^2 x$

$$\sin^m x = \sin^{2k+1} x = [\sin^2 x]^k \sin x = [1 - \cos^2 x]^k \sin x$$

$$\text{Let } u = \cos x \Rightarrow du = -\sin x dx$$

Case 2 m is even and n is odd $\Rightarrow n = 2k+1$ we use $\cos^2 x = 1 - \sin^2 x$

$$\cos^n x = \cos^{2k+1} x = [\cos^2 x]^k \cos x = [1 - \sin^2 x]^k \cos x$$

$$\text{Let } u = \sin x \Rightarrow du = \cos x dx$$

Case 3 m and n are both even: we use

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Exp $\int \sin^3 x \cos^3 x dx = \int \sin^2 x \cos^3 x \sin x dx$ $u = \cos x$
 $du = -\sin x dx$

$$= \int (1 - \cos^2 x) \cos^3 x \sin x dx$$

$$= -\int (1 - u^2) u^3 du = -\int (u^3 - u^5) du = \frac{u^6}{6} - \frac{u^4}{4} + C$$

$$= \frac{1}{6} \cos^6 x - \frac{1}{4} \cos^4 x + C$$

Exp $\int \sin^2 x \cos^3 x dx = \int \sin^2 x \cos^2 x \cos x dx$ $u = \sin x$
 $du = \cos x dx$

$$= \int \sin^2 x (1 - \sin^2 x) \cos x dx = \int u^2 (1 - u^2) du$$

$$= \int (u^2 - u^4) du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$$

³
Exp $\int 16 \sin^2 x \cos^2 x \, dx = 16 \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right) dx$ (34)
 $= 4 \int (1 - \cos^2 2x) \, dx = 4x - 4 \int \cos^2 2x \, dx$
 $= 4x - 4 \int \left(\frac{1 + \cos 4x}{2} \right) dx = 4x - \frac{4}{2}x - 2 \int \cos 4x \, dx$
 $= 4x - 2x - \frac{1}{2} \sin 4x + C = 2x - \frac{\sin 4x}{2} + C$

Product of sin and cos

$$\int \sin mx \sin nx \, dx$$

$$\int \sin mx \cos nx \, dx$$

$$\int \cos mx \cos nx \, dx$$

$$\sin mx \sin nx = \frac{1}{2} [\cos(m-n)x - \cos(m+n)x]$$

$$\sin mx \cos nx = \frac{1}{2} [\sin(m-n)x + \sin(m+n)x]$$

$$\cos mx \cos nx = \frac{1}{2} [\cos(m-n)x + \cos(m+n)x]$$

Exp $\int \cos 3x \cos 4x \, dx = \int \frac{1}{2} [\cos(-x) + \cos 7x] \, dx$
 $= \frac{1}{2} \int (\cos x + \cos 7x) \, dx = \frac{1}{2} \sin x + \frac{1}{14} \sin 7x + C$

*Eliminating Square Roots:

Exp $\int_0^{\pi} \sqrt{1 - \sin^2 x} \, dx = \int_0^{\pi} \sqrt{\cos^2 x} \, dx = \int_0^{\pi} |\cos x| \, dx$
 $= \int_0^{\frac{\pi}{2}} \cos x \, dx - \int_{\frac{\pi}{2}}^{\pi} \cos x \, dx$
 $= \sin x \Big|_0^{\frac{\pi}{2}} - \sin x \Big|_{\frac{\pi}{2}}^{\pi}$
 $= \sin \frac{\pi}{2} + \sin \frac{\pi}{2} = 1 + 1 = 2$

Power of $\tan x$ and $\sec x$

(35)

$$\underline{\underline{\text{Exp}^*}} \int 4 \tan^3 x \, dx = 4 \int \tan^2 x \tan x \, dx$$

$$= 4 \int (\sec^2 x - 1) \tan x \, dx = 4 \int \sec^2 x \tan x \, dx - 4 \int \tan x \, dx$$

$$= 2 \tan^2 x - 4 \int \frac{\sin x}{\cos x} \, dx$$

$$= 2 \tan^2 x + 4 \ln |\cos x| + C$$

$$= 2 \tan^2 x - 2 \ln |\sec^2 x| + C$$

$$= 2 \tan^2 x - 2 \ln (1 + \tan^2 x) + C$$

$$\underline{\underline{\text{Exp}}} \int \sec^4 x \, dx = \int \sec^2 x \sec^2 x \, dx$$

$$= \int (1 + \tan^2 x) \sec^2 x \, dx$$

$$u = \tan x \\ du = \sec^2 x \, dx$$

$$= \int (1 + u^2) \, du$$

$$= u + \frac{u^3}{3} + C$$

$$= \tan x + \frac{1}{3} \tan^3 x + C$$

$$= \tan x + \frac{1}{3} [\sec^2 x - 1] \tan x + C$$

$$= \tan x - \frac{1}{3} \tan x + \frac{1}{3} \sec^2 x \tan x + C$$

$$= \frac{2}{3} \tan x + \frac{1}{3} \sec^2 x \tan x + C.$$
