4.5 : Extensions of the change of variable Technique ..

exp on Note.

Exercises :

Q49: let X_1, X_2, X_3 be i.i.d each with the distribution having p.d.f

$$f(x) = \begin{cases} e^{-x} & , \ 0 < x < \infty \\ 0 & , \ \text{elsewhere} \end{cases}$$

Show that $Y_1 = \frac{X_1}{X_1 + X_2}$, $Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}$, $Y_3 = X_1 + X_2 + X_3$.

are mutually indep.

$$X_1 = y_1 y_2 y_3$$

$$X_2 = y_2 y_3 - y_1 y_2 y_3 = y_2 y_3 (1 - y_1)$$

$$X_3 = y_3 - \cancel{y_1 y_2 y_3} - y_2 y_3 + \cancel{y_1 y_2 y_3} = (1 - y_2) y_3$$

$$J = \begin{vmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} & \frac{dx_3}{dy_3} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} & \frac{dx_2}{dy_3} \\ \frac{dx_3}{dy_1} & \frac{dx_3}{dy_2} & \frac{dx_3}{dy_3} \end{vmatrix} = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ -y_2 y_3 & y_3(1-y_1) & y_2(1-y_1) \\ 0 & -y_3 & 1-y_2 \end{vmatrix}$$

$$= y_2 y_3 \begin{vmatrix} y_3(1-y_1) & y_2(1-y_1) \\ -y_3 & 1-y_2 \end{vmatrix} + y_2 y_3 \begin{vmatrix} y_1 y_3 & y_1 y_2 \\ -y_3 & 1-y_2 \end{vmatrix} + 0$$

$$= y_2 y_3 \left(y_3 (1-y_1)(1-y_2) + y_2 y_3 (1-y_1) \right) + y_2 y_3 \left(y_1 y_3 (1-y_2) + y_1 y_2 y_3 \right)$$

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$$= y_2 y_3^2 (1-y_1)(1-y_2) + y_2^2 y_3^2 (1-y_1) + y_1 y_2 y_3^2 (1-y_2) + y_1 y_2^2 y_3^2$$

$$= y_2 y_3^2 .$$

Now $$f(x_1, x_2, x_3) = \begin{cases} e^{-x_1 - x_2 - x_3} & , \ (x_1, x_2, x_3) \in A \\ 0 & , \ \text{else} \end{cases}$$

$$A = \{ (x_1, x_2, x_3) : 0 < x_1 < \infty , \ 0 < x_2 < \infty , \ 0 < x_3 < \infty \}$$

$$B = \{ (y_1, y_2, y_3) : 0 < y_1 < 1 , \ 0 < y_2 < 1 , \ y_3 > 0 \}$$

$$g(y_1,y_2,y_3) = \begin{cases} f(y_1y_2y_3\ ,\ (1-y_1)y_2y_3\ ,\ (1-y_2)y_3)\ |J| & , (y_1,y_2,y_3)\in B \\ 0 & , \text{else} \end{cases}$$

$$= \begin{cases} e^{-y_1y_2y_3 - y_2y_3 + y_1y_2y_3 - y_3 + y_2y_3} \cdot y_2 y_3^2 & , (y_1,y_2,y_3)\in B \\ 0 & , \text{else} \end{cases}$$

$$= \begin{cases} e^{-y_3}\ y_2 y_3^2 & , (y_1,y_2,y_3)\in B \\ 0 & , \text{else} \end{cases}$$

to check the independent :

$\exists\ h_1(y_1) = 1 \quad , \quad 0 < y_1 < 1$

and $h_2(y_2) = y_2 \quad , \quad 0 < y_2 < 1$ s.t $g(y_1,y_2,y_3) = h_1(y_1)\,h_2(y_2)\,h_3(y_3)$.

and $h_3(y_3) = y_3^2 e^{-y_3} \quad , \quad y_3 > 0$

So Y_1, Y_2, Y_3 are mutually independent.

Q52: let X_1, X_2 and X_3 be three independent chi-square variables with r_1, r_2 and r_3 degrees of freedom, respectively

a. show that $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_1 + X_2$ are independent and that Y_2 is $\chi^2(r_1 + r_2)$.

$$X_1 \sim \chi^2(r_1) \longrightarrow f(x_1) = \frac{1}{\Gamma(\frac{r_1}{2})\, 2^{\frac{r_1}{2}}} x_1^{\frac{r_1}{2}-1} e^{-\frac{x_1}{2}}, \quad x_1 > 0$$

$$X_2 \sim \chi^2(r_2) \longrightarrow f(x_2) = \frac{1}{\Gamma(\frac{r_2}{2})\, 2^{\frac{r_2}{2}}} x_2^{\frac{r_2}{2}-1} e^{-\frac{x_2}{2}}, \quad x_2 > 0$$

$$\Rightarrow f(x_1, x_2) = f(x_1)f(x_2) = \begin{cases} \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})\, 2^{\frac{r_1+r_2}{2}}} x_1^{\frac{r_1}{2}-1} \cdot x_2^{\frac{r_2}{2}-1} e^{-\frac{x_1 + x_2}{2}}, & 0<x_1<\infty,\ 0<x_2<\infty \\ 0, & \text{else} \end{cases}$$

- $y_1 = \frac{x_1}{x_2} \longrightarrow x_1 = \frac{y_1 y_2}{1+y_1}$

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- $y_2 = x_1 + x_2 \longrightarrow x_2 = \frac{y_2}{1+y_1}$

- $$J = \begin{vmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} \end{vmatrix} = \begin{vmatrix} \frac{y_2}{(1+y_1)^2} & \frac{y_1}{1+y_1} \\ \frac{-y_2}{(1+y_1)^2} & \frac{1}{1+y_1} \end{vmatrix} = \frac{y_2}{(1+y_1)^3} + \frac{y_1 y_2}{(1+y_1)^3} = \frac{y_2(\cancel{1+y_1})}{(1+y_1)^{\cancel{3}\,2}} = \frac{y_2}{(1+y_1)^2}$$

- $A = \{x_1, x_2 : 0 < x_1 < \infty,\ 0 < x_2 < \infty\}$

  $B = \{y_1, y_2 : 0 < y_1 < \infty,\ 0 < y_2 < \infty\}$

- $$g(y_1, y_2) = \begin{cases} f\left(\frac{y_1 y_2}{1+y_1}, \frac{y_2}{1+y_1}\right) \left|\frac{y_2}{(1+y_1)^2}\right|, & y_1, y_2 \in B \\ 0, & \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})\, 2^{\frac{r_1+r_2}{2}}} \left(\frac{y_1 y_2}{1+y_1}\right)^{\frac{r_1}{2}-1} \left(\frac{y_2}{1+y_1}\right)^{\frac{r_2}{2}-1} e^{-\left(\frac{\frac{y_1 y_2}{1+y_1} + \frac{y_2}{1+y_1}}{2}\right)} \cdot \frac{y_2}{(1+y_1)^2}, & y_1, y_2 \in B \\ 0, & \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})\, 2^{\frac{r_1+r_2}{2}}} \left(\frac{y_1 y_2}{1+y_1}\right)^{\frac{r_1}{2}-1} \left(\frac{y_2}{1+y_1}\right)^{\frac{r_2}{2}} \frac{1}{1+y_1} e^{-\frac{y_2}{2}}, & y_1, y_2 \in B \\ 0, & \text{else} \end{cases}$$

to check its indep.

$$\exists \quad h_1(y_1) = \frac{1}{\Gamma(\frac{r_1}{2})\, 2^{\frac{r_1}{2}}} \left(\frac{y_1}{1+y_1}\right)^{\frac{r_1}{2}-1} \left(\frac{1}{1+y_1}\right)^{\frac{r_2}{2}-1}, \quad 0<y_1<\infty$$

and $$h_2(y_2) = \frac{1}{\Gamma(\frac{r_2}{2})\, 2^{\frac{r_2}{2}}}\, y_2^{\frac{r_1+r_2}{2}-2}\, e^{-y_2}, \quad 0<y_2<\infty$$

s.t. $h_1(y_1)\, h_2(y_2) = g(y_1, y_2)$

so Y_1 and Y_2 are independent.

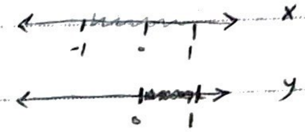
$\boxtimes$

Q53: if $f(x) = \begin{cases} \frac{1}{2} & , -1 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$ is the p.d.f of the random variable X, find the p.d.f of $Y = X^2$.

$A_1 = \{x : -1 < x < 0\}$

$A_2 = \{x : 0 < x < 1\}$

$B = \{y : 0 < y < 1\}$



$u_1(x) = x^2 \rightsquigarrow w_1(y) = \sqrt{y} \rightsquigarrow \bar{w}_1 = \frac{1}{2\sqrt{y}}$

$u_2(x) = x^2 \rightsquigarrow w_2(y) = -\sqrt{y} \rightsquigarrow \bar{w}_2 = \frac{-1}{2\sqrt{y}}$

Now,

$$g(y) = \begin{cases} f(w_1(y))\,|\bar{w}_1(y)| + f(w_2(y))\,|\bar{w}_2(y)| & , 0 < y < 1 \\ 0 & , \text{else} \end{cases}$$

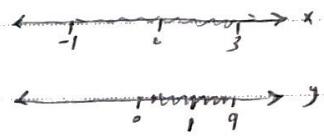
$$= \begin{cases} \frac{1}{2} \cdot \frac{1}{2\sqrt{y}} + \frac{1}{2} \cdot \frac{1}{2\sqrt{y}} & , 0 < y < 1 \\ 0 & , \text{else} \end{cases}$$

$$= \begin{cases} \frac{2}{4\sqrt{y}} & , 0 < y < 1 \\ 0 & , \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{2\sqrt{y}} & , 0 < y < 1 \\ 0 & , \text{else} \end{cases}$$

Q55: if X have p.d.f $f(x) = \begin{cases} \frac{1}{4} & , -1 < x < 3 \\ 0 & , \text{else} \end{cases}$ find the p.d.f of $y = x^2$.

$A_1 = \{x : -1 < x < 0\}$



$A_2 = \{x : 0 < x < 3\}$

$B_1 = \{y : 0 < y < 1\}$

$B_2 = \{y : 1 < y < 9\}$

$u_1(x) = x^2 \rightarrow w_1(y) = \sqrt{y} \quad , \quad 0 < x < 3$

$u_2(x) = x^2 \rightarrow w_2(y) = -\sqrt{y} \quad , \quad -1 < x < 0$

$$g(y) = \begin{cases} f(\sqrt{y}) \left| \frac{1}{2\sqrt{y}} \right| + f(-\sqrt{y}) \left| \frac{-1}{2\sqrt{y}} \right| & , y \in B \\ 0 & , \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{4} \cdot \frac{1}{2\sqrt{y}} + \frac{1}{4} \; \frac{1}{2\sqrt{y}} & , y \in B \\ 0 & , \text{else} \end{cases}$$

$\frac{2}{5\sqrt{y}}$

$$= \begin{cases} \frac{1}{4\sqrt{y}} & , y \in B \\ 0 & , \text{else} \end{cases}$$