Exp Consider the function $f(x) = \begin{cases} x^2+k^2 , & x \geq \frac{1}{2} \\ k , & x < \frac{1}{2} \end{cases}$

Find the constant k if $\lim_{x \to \frac{1}{2}} f(x)$ exists

$$\lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^-} f(x)$$

$$\left(\frac{1}{2}\right)^2 + k^2 = k$$

$$k^2 - k + \frac{1}{4} = 0$$

$$\left(k - \frac{1}{2}\right)\left(k - \frac{1}{2}\right) = 0 \quad \Rightarrow k = \frac{1}{2}$$

Continuity

Def A function $f(x)$ is continuous at x_0 if

[1] $f(x_0)$ exists and [2] $\lim_{x \to x_0} f(x)$ exists and

[3] $\lim_{x \to x_0} f(x) = f(x_0)$

Def: f is cont. on $[a,b]$ if f is cont. on each poin in $[a,b]$

Exp
- All polynomials are cont. on $\mathbb{R} = (-\infty, \infty)$
- $\sin x$, $\cos x$, $|x|$, e^x are cont. on $\mathbb{R}$
- Rational functions $R(x) = \frac{f(x)}{g(x)}$ are cont. everywhere except where $g(x) = 0$

Exp $R(x) = \frac{x^2+x+1}{x^2-1}$ is cont. on $\mathbb{R} \setminus \{\pm 1\}$

Exp Suppose that $f(x) = \begin{cases} ax + b & , \ x \le 0 \\ x^2 + 3a - b & , \ 0 < x \le 2 \\ 3x - 5 & , \ 2 < x \end{cases}$

is cont. function. Find a, b and sketch f(x)

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$a(0) + b = (0)^2 + 3a - b$$

$$b = 3a - b$$

$$2b = 3a \quad (1)$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$(2)^2 + 3a - b = 3(2) - 5$$

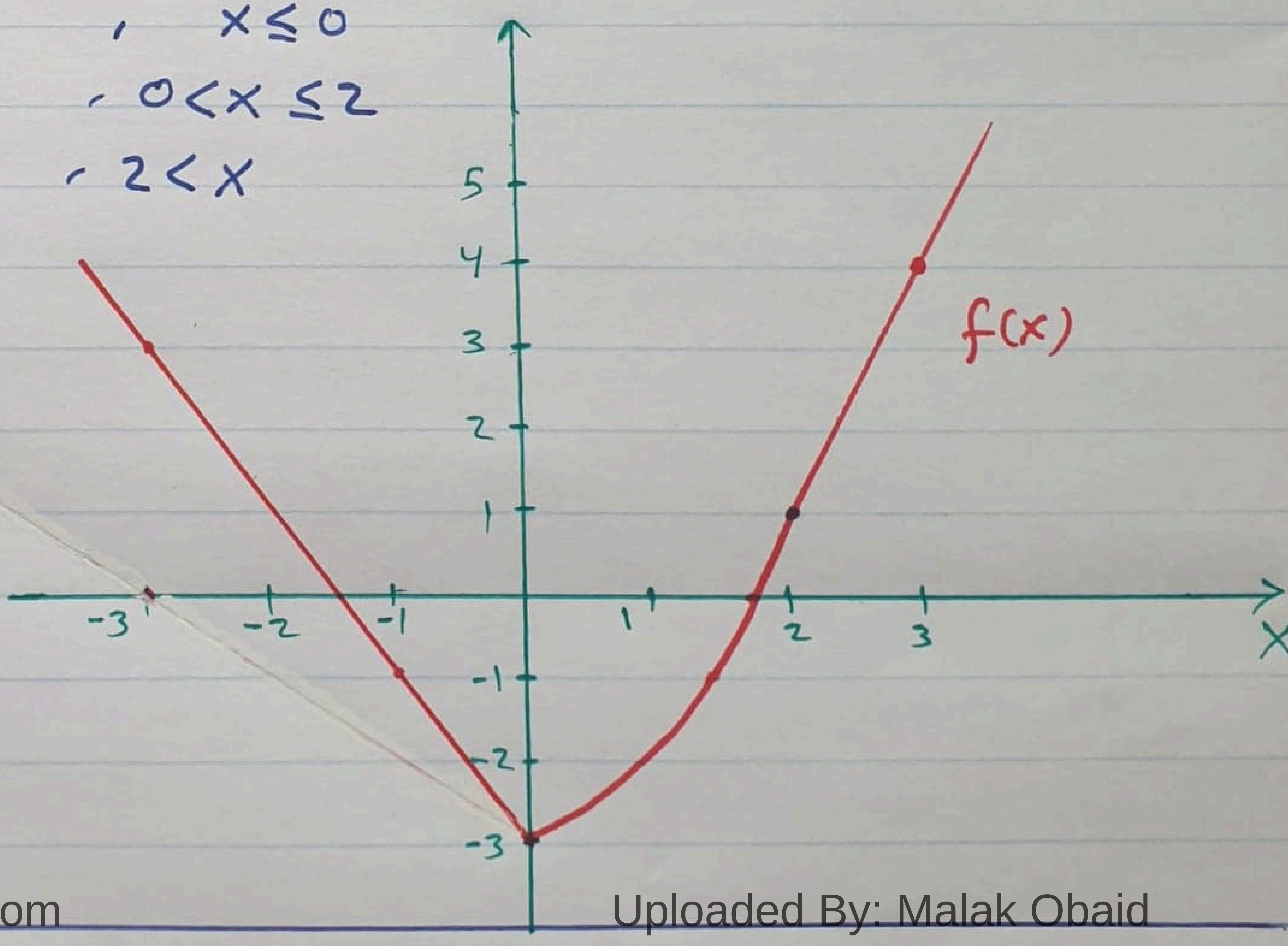
$$4 + 3a - b = 1$$

$$3a = b - 3 \quad (2)$$

$$2b = b - 3 \quad \Rightarrow b = -3$$

$$\Rightarrow a = -2$$

$$f(x) = \begin{cases} -2x - 3 & , \ x \le 0 \\ x^2 - 3 & , \ 0 < x \le 2 \\ 3x - 5 & , \ 2 < x \end{cases}$$



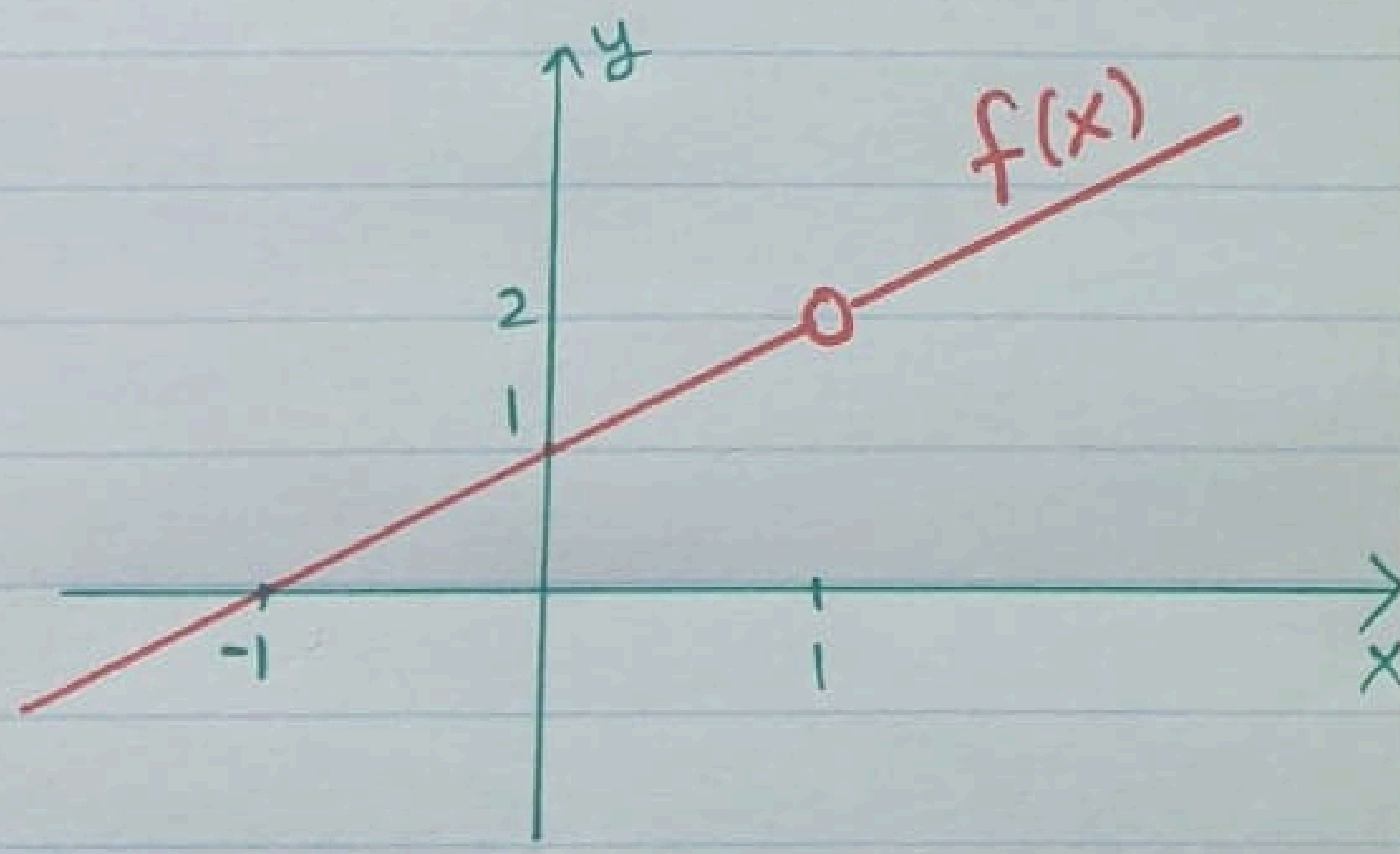
Removable Discontinuity

Exp $f(x) = \dfrac{x^2 - 1}{x - 1}$

$f(1) = \frac{0}{0}$

1. Find $\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{(x-1)(x+1)}{(x-1)} = \lim_{x \to 1} (x+1) = 2$

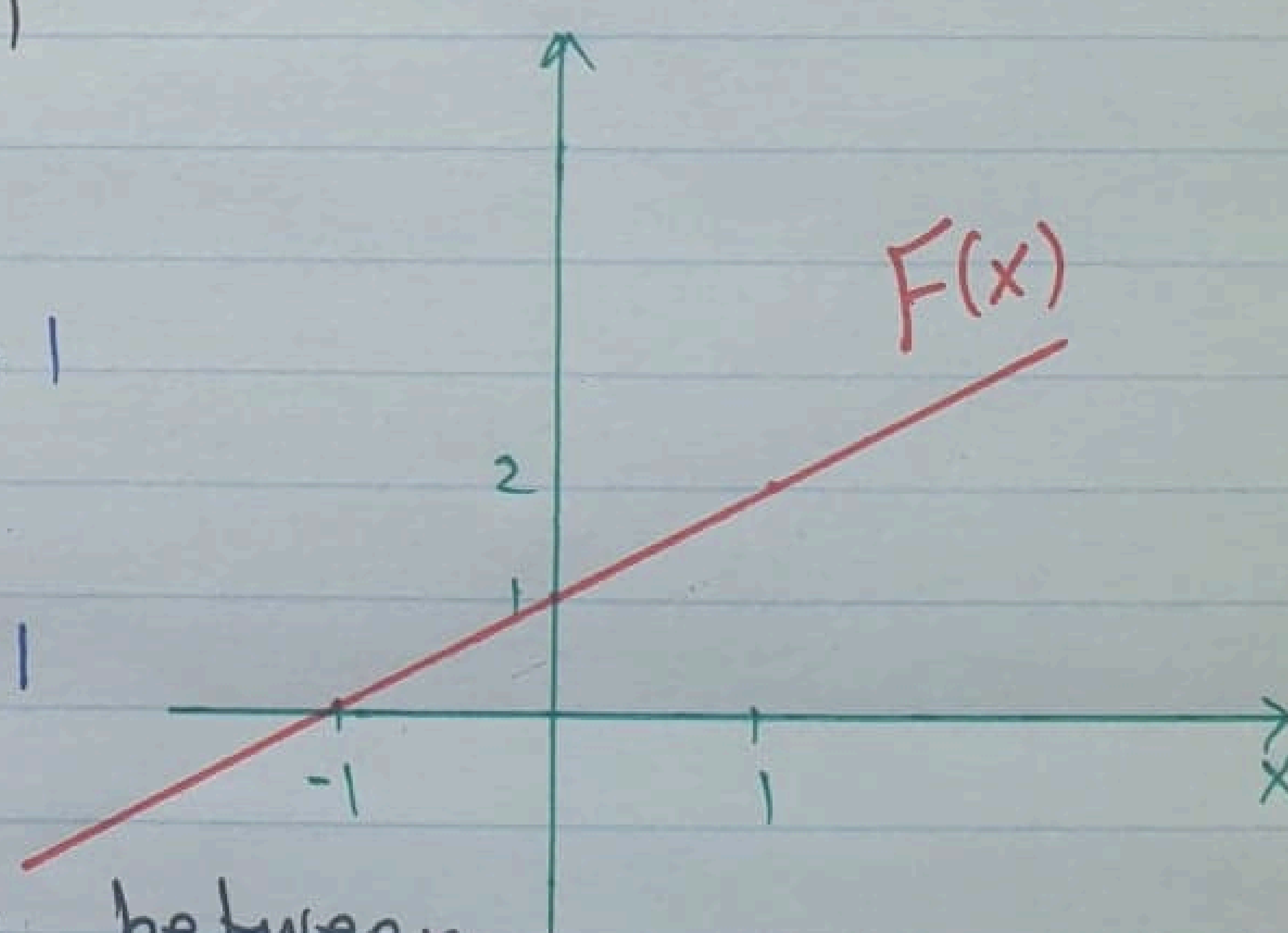
2. Sketch $f(x)$



4. sketch $F(x)$ where

$$F(x) = \begin{cases} f(x) & \text{if } x \neq 1 \\ \lim_{x \to 1} f(x) & \text{if } x = 1 \end{cases}$$

$$F(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$



5. What is the difference between $f(x)$ and $F(x)$

$f(x)$ is discont. at $x = 1$
$F(x)$ is cont. at $x = 1$

$x = 1$ is called removable discontinuity of $f(x)$
$F(x)$ is called continuous extension of $f(x)$ at $x = 1$

Exp Find cont. extension of $f(x) = \dfrac{x^2 - 5x + 6}{2x - 4}$ at $x = 2$

$$F(x) = \begin{cases} f(x) & \text{if } x \neq 2 \\ \lim\limits_{x \to 2} f(x) & \text{if } x = 2 \end{cases}$$

$$= \begin{cases} \dfrac{x^2 - 5x + 6}{2x - 4} & \text{if } x \neq 2 \\ \dfrac{-1}{2} & \text{if } x = 2 \end{cases}$$

$\lim\limits_{x \to 2} f(x) =$

$\lim\limits_{x \to 2} \dfrac{x^2 - 5x + 6}{2(x-2)}$

$\lim\limits_{x \to 2} \dfrac{(x-2)(x-3)}{2(x-2)}$

$= \lim\limits_{x \to 2} \frac{1}{2}(x-3) = \dfrac{-1}{2}$

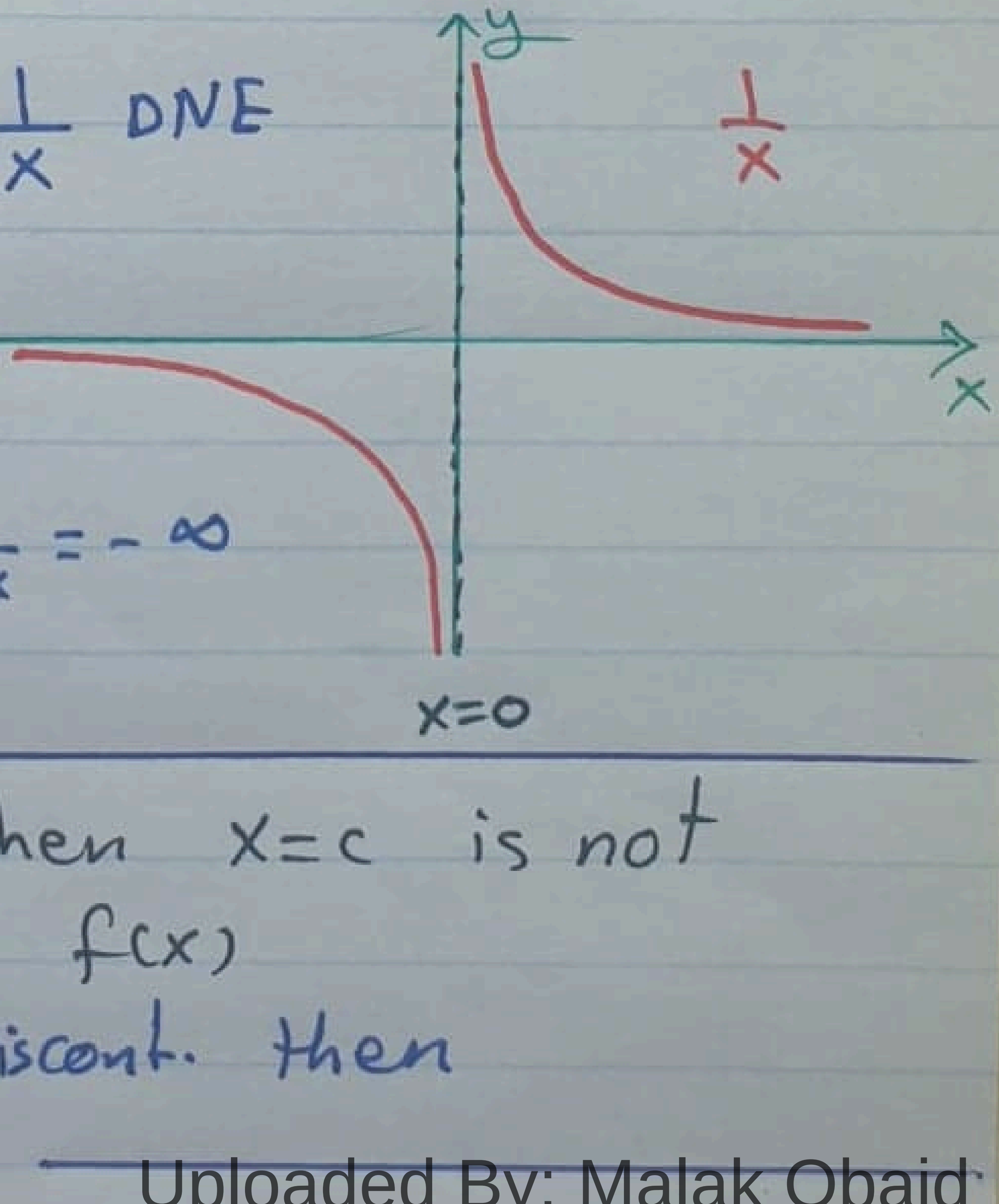
- $F(2) = \lim\limits_{x \to 2} F(x) = \dfrac{-1}{2}$ so $F(x)$ is cont. at $x = 2$
- $x = 2$ is removable dis cont. since $f(2) = \dfrac{0}{0}$

Exp Is $x = 0$ removable discont. for $f(x) = \dfrac{1}{x}$?

No since $f(0) = \dfrac{1}{0}$ or $\lim\limits_{x \to 0} \dfrac{1}{x}$ DNE

$x = 0$ is vertical Asymptote

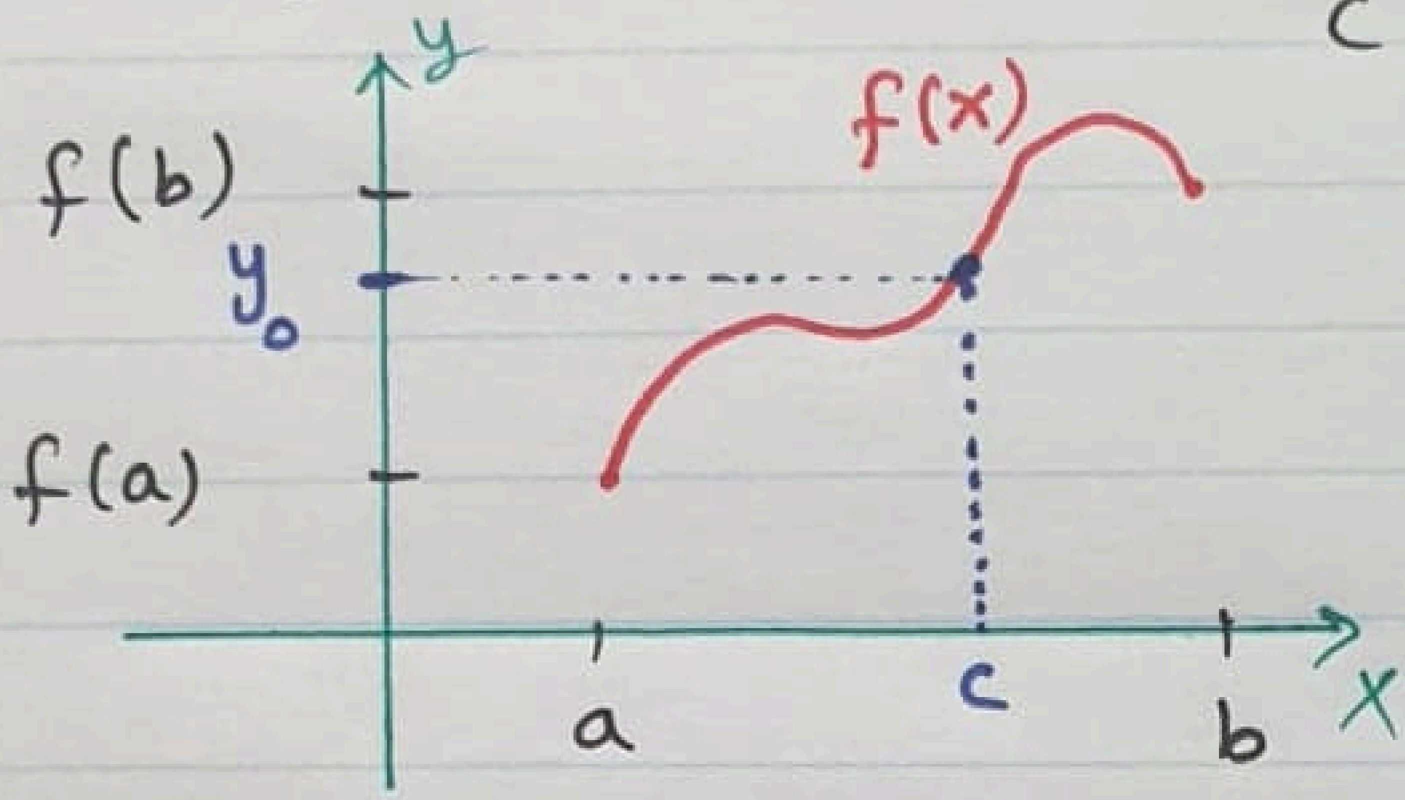
Recall $\lim\limits_{x \to 0^+} \dfrac{1}{x} = \infty$ and $\lim\limits_{x \to 0^-} \dfrac{1}{x} = -\infty$

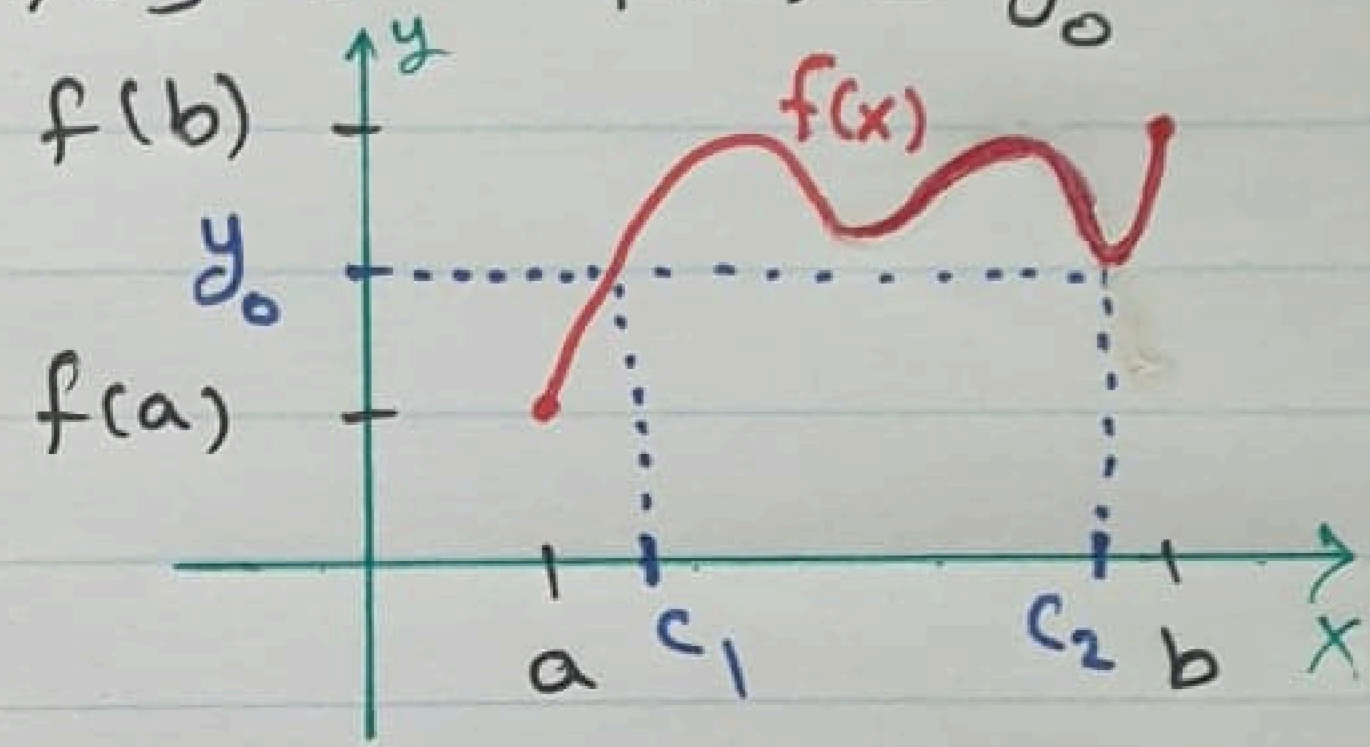


Remark
- If $x = c$ is V. Asy. then $x = c$ is not removable dis cont. for $f(x)$
- If $x = c$ is removable discont. then $x = c$ is not V. Asy.

Th (Intermediate Value Theorem – IVT)

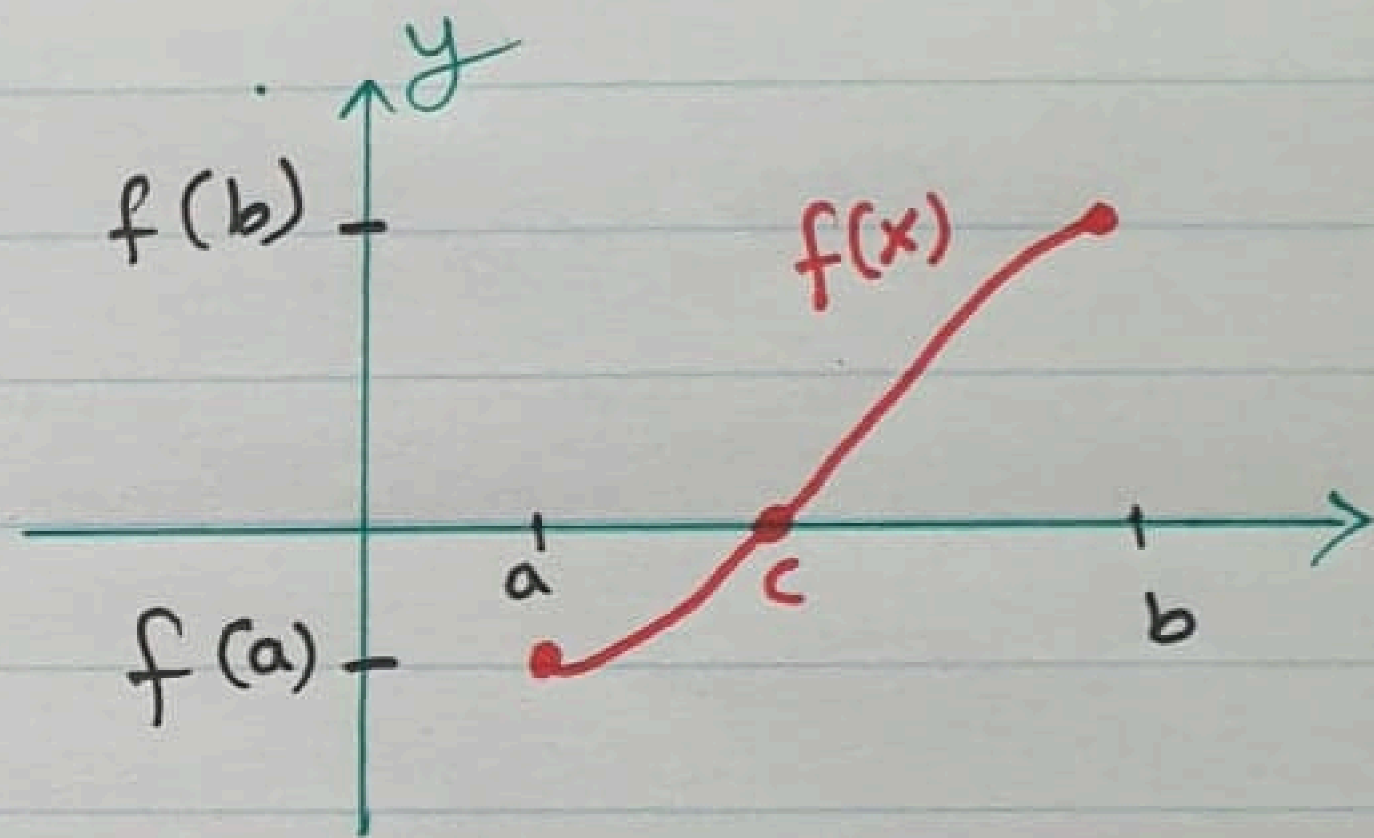
- Assume $f(x)$ is cont. function on closed interval $[a,b]$.
- If $y_0 \in (f(a), f(b))$, then $\exists$ at least one number $c \in [a,b]$ s.t $f(c) = y_0$





Th (Bolzano Th)

- Assume f is cont. on $[a,b]$
- If $f(a)\, f(b) < 0$, then $\exists\, c \in [a,b]$ s.t $f(c) = 0$



c is called zero of $f(x)$
or root of $f(x)$
or x-intercept
or the intersection of $f(x)$ with x-axis.
or solution to $f(x) = 0$

Exp Is there any root for the equation $x^5 - 2x^3 - 2 = 0$ between $x = 1$ and $x = 2$?

- $f(x) = x^5 - 2x^3 - 2$ cont. on $[1,2]$
- $$\left.\begin{array}{l} f(1) = 1 - 2 - 2 < 0 \\ f(2) = 32 - 16 - 2 > 0 \end{array}\right\} \Rightarrow f(1)\, f(2) < 0 \Rightarrow \text{By Bolzano}$$
  Th. $\exists\, c \in [1,2]$ s.t $f(c) = 0$

Or $f(x) = x^5 - 2x^3 - 2$ cont. on $[1,2]$ and $y_0 = 0 \in (f(a), f(b))$
$\Rightarrow$ By IVT $\exists\, c \in [1,2]$ s.t $f(c) = y_0 = 0$

<u>Question</u>: How to sketch rational function

$$f(x) = \frac{g(x)}{h(x)} \quad ?$$

g,h poly.

Answer: We use the help of Limits and Asymptotes.

Asymptotes (lines)

Horizontal Asymptote (H. Asy) $\boxed{1}$

Vertical Asymptote (V. Asy) $\boxed{2}$

Oblique Asy. (O. Asy) $\boxed{3}$

$\boxed{1}$ H. Asy : $y = b$ is H. Asy for the graph of $f(x)$

if $\lim\limits_{x \to \infty} f(x) = b$ or $\lim\limits_{x \to -\infty} f(x) = b$

$\boxed{2}$ V. Asy : $x = a$ is V. Asy. for the graph of $f(x)$

if $\lim\limits_{x \to a^+} f(x) = \infty$ (or $-\infty$) <u>or</u> $\lim\limits_{x \to a^-} f(x) = \infty$ (or $-\infty$)

Check zeros of $h(x)$

$\boxed{3}$ O. Asy : $y = ax + b$ is O. Asy for the graph of $f(x)$
if the degree of $g(x)$ is one larger than the degree of $h(x)$

$g(x)$: numerator
$h(x)$: denomerator

Remark:
- If $f(x)$ has H. Asy then $f(x)$ has no O. Asy.
- If $f(x)$ has O. Asy then $f(x)$ has no H. Asy.

Exp Find Asy. of $f(x) = \frac{1}{x}$ and sketch f

H. Asy. $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x} = 0 \Rightarrow y = 0$ is H. Asy.

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x} = 0$

V. Asy. check zeros of denominators
check $x = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \frac{1}{\text{small}+} = \infty \Rightarrow x = 0$ is V. Asy

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = \frac{1}{\text{small}-} = -\infty$

O. Asy. since $\exists$ H. Asy $\Rightarrow \nexists$ O. Asy.