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ENEE 2307:
STAT:-

Ch 1: Fundamental of probability:-

* Experement:-
① can be repeated (infinite times). like

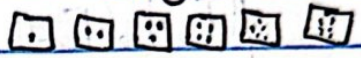
- flipping a coin

(H) head (T) tail

$S = \{H, T\}$

sample outcome — Sample outcome

- tossing a dice



sample Space: $S = \{1, 2, 3, 4, 5, 6\}$ — Sample outcome

Exp: A: even num.
$A = \{2, 4, 6\}$
B: div by 3
$B = \{3, 6\}$
C: Less than 1
$C = \emptyset$
D: Less than 10
$D = S = \{1, 2, 3, 4, 5, 6\}$

② has a well-known outcomes:-
* Sample Space (S)
* Sample outcomes
* Event

Exp:- Consider the experiment of flipping a green coin and red coin.
[1] Write the Sample Space!

$S = \{\underset{RG}{HH}, \underset{RG}{HT}, \underset{RG}{TH}, \underset{RG}{TT}\}$

[2] A: a match
$A = \{HH, TT\}$

[3] B: the red coin is Head.
$B = \{\underset{RG}{HH}, \underset{RG}{HT}\}$

Exp2:- Consider the experiment of flipping two similar coins at the same time in the same place.

[1] write the Sample Space:-

$S = \{HH, HT, TH, TT\}$.

Exp3:- Consider the experiment of flipping a coin and tossing a dice.

[1] write the Sample Space:

$S = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$

[2] A: Head is observed in the coin and an even number is observed on the dice.

$A = \{(H,2), (H,4), (H,6)\}$

[3] B: odd numbers are observed on the die.

$B = \{(H,1), (H,3), (H,5), (T,1), (T,3), (T,5)\}$

Exp 1:- Let us consider the experiment of tossing two dice.

[1] A: the sum of the numbers observed is 6.

$A = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$.

[2] B: the sum of the numbers observed is less than 5.

$B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$.

[3] C: a match is observed.

$C = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

* Algebra of the Sets :-

⇒ Intersection "∩" — "تقاطع" والعناصر المشتركة بين مجموعتين

B

4 D: the sum of the two numbers is less than 5 and the two numbers match.

C

فإن $D = B \cap BC = \{(1,1), (2,2)\}$

⇒ Union "∪" — "اتحاد" وتجمع العناصر في مجموعتها ولكن بدون تكرار

A

5 E: the sum of the numbers observed is 6 or there is a match

فإن $E = A \cup C = \{(1,5), (2,4), (3,3), (4,2), (5,1), (1,1), (2,2), (4,4), (5,5), (6,6)\}$

⇒ Complement ✓ — المتمم (تكملة المجموعة) وهي كل هذه العناصر التي ليست موجودة في المجموعة

$\bar{C} = \{(1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5)\}$

$C = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$ ← هذه هي

فمتممها يتكون من كل العناصر ماعدا هذه العناصر الموجودة في C.

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* Demorgan's Laws.

1 $\overline{A \cup B} = \bar{A} \cap \bar{B}$

2 $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Exp2) Let the sample space $S = \{1,2,3,4,5,6\}$ and the event $A = \{1,3,5\}$, $B = \{2,3,4\}$

1 $\overline{A \cup B} = \overline{\{1,3,5\} \cup \{2,3,4\}} = \{1,2,3,4,5\} \rightarrow = \{6\}$

or

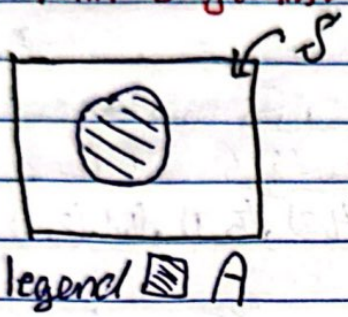
$\bar{A} \cap \bar{B} \rightarrow \bar{A} = \{2,4,6\}$, $\bar{B} = \{1,5,6\} \rightarrow \bar{A} \cap \bar{B} = \{6\}$

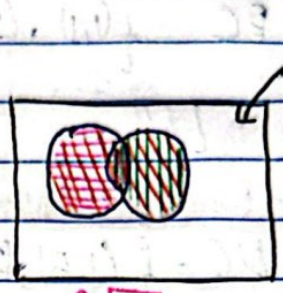
2. $\overline{A \cap B} = \overline{\{1,3,5\} \cap \{2,3,4\}} = \overline{\{3\}} \rightarrow = \{1,2,4,5,6\}$

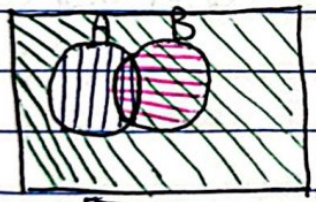
or

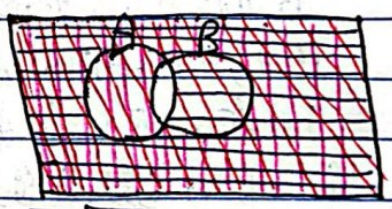
$\overline{A} \cup \overline{B} = \{2,4,6\} \cup \{1,5,6\} \rightarrow \{1,2,4,5,6\}$

* Venn Diagrams:-









* $S = \{1,2,3,4,5,6\}$

$A = \{1,2\}$

$B = \{2,4\}$

$C = A \cap B = \{2\}$, $\overline{C} = \overline{A \cap B} = \{1,3,4,5,6\}$

Exp 3: Let us consider the sample space. $S = \{4 \leq x \leq 10\}$

$A = \{5 \leq x \leq 10\}$, $B = \{4 < x \leq 7\}$

1. $\bar{A} = \{4 \leq x < 5\}$
2. $A \cap B = \{5 \leq x \leq 7\}$
3. $\bar{B} = \{4, 7 < x \leq 10\}$

* Probability :- ← Classical (a priori) قبلي

⇒ 1. $P(A) = \frac{\text{number of sample outcomes in } A}{\text{number of sample outcomes in } S}$ ... (1)

Exp 4) Let $S = \{1, 2, 3, 5, 6\}$, $A = \{1, 5, 6\}$, $B = \{2, 3, 5, 6\}$

1. $P(A) = \frac{3}{5}$ 2. $P(B) = \frac{4}{5}$
3. $P(A \cap B) = A \cap B = \{5, 6\} \rightarrow P(A \cap B) = \frac{2}{5}$
4. $P(\overline{A \cap B}) = \overline{A \cap B} = \bar{A} \cup \bar{B} = \{1, 2, 3\} \rightarrow P(\overline{A \cap B}) = \frac{3}{5}$

$\bar{A} = \{2, 3\}$

$B = \{1\}$

بعدي "a post priori"

⇒ 2. Relative frequency

$P(A) = \lim_{n \to \infty} \frac{\text{number of times } A \text{ occurs}}{\text{number of trails } (n)}$ ... (2)

Exp4) A coin is flipped for 10" times, assume head is observed in 4x10" times, what is the probability of observing head after flipping the coin?

$$P(H) = \frac{4\times 10''}{10\times 10''} = \frac{4}{10} = 0.4$$

- Fair coin → $P(H) = 0.5$, $P(T) = 0.5$
- Biased coin → $P(H) \neq P(T)$.

⇒ [3] Subjective.

احتمال شخصي

Rev.

→ Definitions of probability :-

[1] Classical → $P(A) = \frac{\text{num. of outcomes in } A}{\text{in } S}$

[2] Relative frequency → $P(A) = \lim_{n\to\infty} \frac{\text{number of times } a \text{ occurs}}{\text{number of trails } (n)}$

[3] Subjective

→ [4] Axiomatic

[1] $P(A) \geq 0$

[2] $P(S) = 1$

} $0 \leq P(A) \leq 1$

[3] [a] if events A and B are disjoint (mutually exclusive) يعني

then $P(A \cup B) = P(A) + P(B)$ يعني

[b] if three events are disjoints (mutually exclusive)

then $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

* Notes:-

(1) two events A and B are said to be disjoint (mutually exclusive) if $A \cap B = \emptyset$

(2) three events A, B and C are said disjoint (" ")

if [a] $A \cap B = \emptyset$ [c] $B \cap C = \emptyset$

[b] $A \cap C = \emptyset$ [d] $A \cap B \cap C = \emptyset$

* لازم الاربعة يتحققوا، لو واحد ما تحقق يكون

* disjoint

Exp 1> Let $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2\}$, $B = \{3, 4\}$, $C = \{5\}$, $D = \{1, 6\}$.

1. Are A and B disjoint?
   $A \cap B \overset{??}{=} \emptyset$
   $\{1, 2\} \cap \{3, 4\} = \emptyset$ → Yes, they are disjoint.
2. Are A and D disjoint?
   $A \cap D = \{1, 2\} \cap \{1, 6\} = \{1\} \neq \emptyset$ → No, they aren't disjoint
3. Are A, B and D disjoint?
   → $A \cap B = \emptyset$
   ✗ $A \cap D \neq \emptyset$ → So A, B and D are not disjoint
4. Are A, B and C are disjoint?
   → $A \cap B = \{1, 2\} \cap \{3, 4\} = \emptyset$
   → $A \cap C = \{1, 2\} \cap \{5\} = \emptyset$
   → $B \cap C = \{3, 4\} \cap \{5\} = \emptyset$
   → $A \cap B \cap C = \{1, 2\} \cap \{3, 4\} \cap \{5\} = \emptyset$
   } So, A, B and C are disjoint.

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5. $P(A \cup B) = ?$
   ① $P(A \cup B) = P(\{1, 2, 3, 4\}) = 4/6$
   or
   ② $P(A \cup B) = P(A) + P(B) = 2/6 + 2/6 = 4/6$
   (disjoint)
6. $P(A \cap D) = P(\{1\}) = 1/6$
7. $P(A \cup B \cup C) = P(\{1, 2\} \cup \{3, 4\} \cup \{5, 6\})$
   $= P(\{1, 2, 3, 4, 6\}) = 5/6$
8. $P(A \cap B \cap C) = P(\emptyset) = 0$

Notes:

1. $A \cap A = A$
2. $A \cup A = A$
3. $A \cap \bar{A} = \emptyset$
4. $A \cup \bar{A} = S$
5. $S^c = \emptyset$ (c complement)
6. $\emptyset^c = S$
7. $S \cap \emptyset = \emptyset$
8. $S \cup \emptyset = S$
9. $S \cap A = A$
10. $S \cup A = S$
11. $\emptyset \cap A = \emptyset$
12. $\emptyset \cup A = A$

Ex2) Use Venn diagram to prove $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

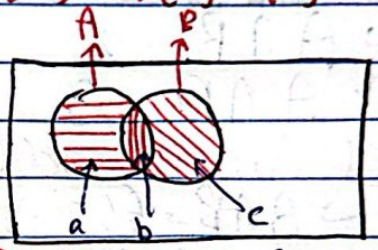
$P(A \cup B) = P(a \cup b \cup c)$ (a, b disjoint; b, c disjoint)

$= P(a) + P(b) + P(c)$

$P(a) + P(b) + P(c) + P(b) - P(b)$

$P(a \cup b) + P(c \cup b) - P(b)$

$\rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ✓



a ∩ b = ∅
a ∩ c = ∅
b ∩ c = ∅
a ∩ B ∩ c = ∅
→ a, b, c disjoint

Ex3) Let $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 3\}$, $B = \{2, 4\}$, $C = \{3, 5\}$

① $P(A \cup B) \overset{①}{=} P(\{1, 2, 3, 4\}) = 4/6$

(A, B disjoint) ② $= P(A) + P(B) = 2/6 + 2/6 = 4/6$

② $P(A \cup C) \overset{①}{=} P(\{1, 3, 5\}) = 3/5$

② $= P(A) + P(C) - P(A \cap C)$

(not disjoint) $= 2/6 + 2/6 - 1/6 = 3/6$

Exp 4) Let the event A, B and C be disjoint and defined over the sample space S, Assume P(A) = 0.25, P(B) = 0.75 and A ∪ B ∪ C = S Determine P(C)?

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$

(A ∪ B are disjoint)

$P(S) = 0.25 + 0.75 + P(C)$

$1 = 1 + P(C) \longrightarrow P(C) = 0$

Exp 5) Let A and B be two events defined over the sample space S Use the axioms of probability to prove the following :-

① $P(\bar{A}) = 1 - P(A)$

$P(A \cup \bar{A}) = P(S)$

(A ∪ Ā are disjoint)

$A \cap \bar{A} = \emptyset$

$P(A) + P(\bar{A}) = 1 \longrightarrow P(\bar{A}) = 1 - P(A)$

② $P(\emptyset) = 0$

$P(S \cup \emptyset) = P(S)$

(S ∪ ∅ are disjoint)

$P(S) + P(\emptyset) = P(S) \longrightarrow P(\emptyset) = 0$

Exp 6) Let the event A and B be defined over S. P(A) = 0.25 and P(B) = 0.4 Assume A and B are disjoint.

(a) determine $P(\bar{A} \cap \bar{B})$

$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

(A ∪ B are disjoint)

$= 1 - [P(A) + P(B)] = 1 - [0.25 + 0.4]$

$1 - 0.65 = \boxed{0.35}$

(b) $P(\bar{A} \cup \bar{B}) = ?$

$P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B})$

$= 1 - P(A \cap B)$

$= 1 - 0 = \boxed{1}$

* if A, B, and C are three events, then:

$$P(A \cup B \cup C) = P(A)+P(B)+P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Exp(2.5): One integer is chosen at random from the numbers $\{1, 2, \ldots, 50\}$. What is the probability that the chosen number is divisible by 6? Assume all 50 outcomes are equally likely.

$\frac{1}{50}$ ... 50 ... ...

$$\Rightarrow S = \{1, 2, 3, 4, \ldots, 50\}$$
$$A = \{6, 12, 18, 24, 30, 36, 42, 48\}$$

$$P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } S} = \frac{8}{50}$$

Exp(2.6): if the probability of occurrence of an even number is twice as likely as that of an odd number in Exp(2.5). Find P(A): A is defined above.

↳ P(even) = 2P(odd), odd P, even 2P

$$P(S) = P(\text{even}) + P(\text{odd}) = 1 \Rightarrow P(\{1\} \cup \{2\} \cup \{3\} \cup \cdots \cup \{50\})$$
$$= P(\{1\}) + P(\{2\}) + P(\{3\}) + \cdots + P(\{50\})$$
$$= P + 2P + P + \cdots + 2P = 25P + 25(2P)$$
$$\rightarrow 1 = 75P \rightarrow P = 1/75$$

$$\therefore P(A) = 8 \times (2P) = \frac{16}{75}$$

... 8 ... 2P

Let (P) be the probability of occurrence of an odd number, then (2P) will be the ... of an even number.

$$P(S) = 25(P) + 25(2P) \Rightarrow 1 = 75P \rightarrow P = \frac{1}{75}$$

odd, 25 ... even, 25

$$P(A) = 8 \rightarrow 2P = 8 \times \frac{2}{75} = \frac{16}{75}$$

Exp(2.7):

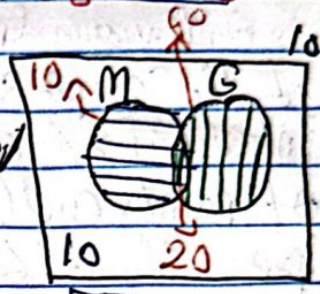
Suppose that a company has 100 employees who are classified according to their marital status and according to whether they are college graduates or not, it is known that 30% of the employees are married and the percent of graduate is 80%. Moreover, 10 employees are neither married nor graduates. What proportion of married employees are graduates? → $M \cap G$

$M \cup G = 90$ → 100 − 10 = neither M nor G

$P(M \cup G) = 0.9$

$P(M) = 0.3$

$P(G) = 0.8$



$P(M \cup G) = P(M) + P(G) - P(M \cap G)$

$90 = 0.3 + 0.8 - P(M \cap G)$

$\rightarrow P(M \cap G) = 20$

= 30

= 80

= ?? → $M \cap G$

⇒ two third of the married employees in the company are graduates.

$M \cap G = \frac{20}{30} = \frac{2}{3}$

Exp(2.8): An experiment has two possible outcomes, the first occurs with probability (P), the second with probability (P^2)

find P?

$P(S) = 1 \rightarrow P(S) = P(A \cup B)$

$= P(A) + P(B)$

$\rightarrow P + P^2 = 1$

$P^2 + P - 1 = 0 \rightarrow P = \frac{-1 + \sqrt{5}}{2}$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ → $a=1, b=1, c=-1$ → $\frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$

Exp(2.9), Exp(2.10)

* Discrete probability function :-
if the Sample Space generated by an experiment contains either a finite or a countable infinite number of outcomes then its is called a discrete Sample Space.
Ex: "S" = {1, 2, 3, 4, 5, 6} is countable finite
Ex: "S" = {1, 2, 3, 4, ...} is countable infinite

* Continuous Probability function :-
if the Sample Space associated with an experiment is an interval of real number then (S) an uncountable infinite number of points and (S) is said to be continuous.
Ex: "S" = {1 ≤ x ≤ 2} is uncountable infinite

* Conditional Probabilities and Statistical Independence :-
Def :-
Given two events A and B with P(A) and P(B) > 0,
we define the conditional probability of A given B has occurred as
→ $P(A/B) = \frac{P(A \cap B)}{P(B)}$
and the probability of B given A has occurred as :-
→ $P(B/A) = \frac{P(A \cap B)}{P(A)}$

Exp: $S=\{1,2,3,4,5,6\}$, $A=\{2,4,6\}$, $B=\{3,6\}$, $C=\{1,2,3,4\}$

a) $P(A)$, $P(B)$, $P(C)$?

$P(A)=3/6$, $P(B)=2/6$, $P(C)=4/6$.

b) What is the probability of observing an even number (A) divisible by 3 (B)?

$P(A \cap B) = P(\{6\}) = 1/6$.

c) Assume an even number (A) is observed, what is the probability that is divisible by 3 (B)?

① $S_{new}=\{2,4,6\}$.

$B_{new}=\{6\} \rightarrow P(B_{new})=1/3$.

② Conditional probability:-

$P(B/A)$ ⟶ probability of B given A (A occurred)

$$P(B/A)=\frac{P(B \cap A)}{P(A)}=\frac{1/6}{3/6}=1/3$$

d) Assume a number divisible by 3 (B) is observed, what is the probability that is an even number (A)

$$P(A/B)=\frac{P(A \cap B)}{P(B)}=\frac{1/6}{2/6}=1/2$$

e) Assume the number is less than 5 (C), what is the probability that is an even number (A)

$$P(A/C)=\frac{P(A \cap C)}{P(C)}=\frac{2/6}{4/6}=\frac{2}{4}=1/2$$

Ex01: Let us consider an experiment with the sample space $S=\{1,2,3,4,5,6\}$. Let us assume that the probability of observing a sample is proportional to its value.

1) A: even numbers. Find $P(A)$.

$A=\{2,4,6\}$, $P(A)=P(\{2,4,6\})=??$

$\rightarrow \frac{2}{21}+\frac{4}{21}+\frac{6}{21}=\boxed{12/21}$ ($2k+4k+6k$)

$\rightarrow P(S)=1$

$P(\{1,2,3,4,5,6\})=1$ (disjoint)

$P(\{1\}\cup\{2\}\cup\{3\}\cup\{4\}\cup\{5\}\cup\{6\})=1$

$P(\{1\})+P(\{2\})+P(\{3\})+P(\{4\})+P(\{5\})+P(\{6\})=1$

$k+2k+3k+4k+5k+6k=1 \rightarrow \boxed{k=1/21}$

2) B: number less than 5 is observed.

$P(B)=P(\{1,2,3,4\})=1k+2k+3k+4k$

$=1/21+2/21+3/21+4/21=\boxed{10/21}$

3) C: an even number less than 5 is observed. (A ∩ B)

$C=A\cap B=\{2,4\} \rightarrow P(C)=(2+4)k=\boxed{6/21}$

4) D: number divisible by 3.

$D=\{3,6\} \rightarrow P(D)=(3+6)k=\boxed{9/21}$

5) $P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{6/21}{10/21}=\boxed{6/10}$

↳ Assume a number less than 5 is observed, what is the probability that it is even?

6) $P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{6/21}{k/21}=\boxed{6/2}$

Note:

→ Two events A and B are said to be **Statistically independent** if $P(A \cap B) = P(A)\,P(B)$.

⇒ Three events (A, B, and C) are said to be **Statistically independent** if:

① $P(A \cap B) = P(A)\,P(B)$ ② $P(B \cap C) = P(B)\,P(C)$

③ $P(A \cap C) = P(A)\,P(C)$ ④ $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

Stat. ind. لو لم يحقق الشرط الرابع يحقق و اذا لم يحقق الشروط الثلاثة الأولى لا يحقق الرابع

Ex1: Are A and B statistically independent?

$P(A \cap B) \overset{??}{=} P(A)\,P(B)$

$6/21 \overset{??}{=} 12/21 \cdot 10/21 \rightarrow 6/21 \neq \frac{120}{441}$

→ So A and B are not Stat. ind.

Ex2: Consider the experiment of tossing a dice, A: even number is observed, B: number less than 5 is observed

(a) $P(A) = ?$

$S = \{1,2,3,4,5,6\}$, $A = \{2,4,6\} \rightarrow P(A) = 3/6$

(b) $P(B) = ?$

$B = \{1,2,3,4\} \rightarrow P(B) = 4/6$

(c) $P(A/B) = ?$

$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{2/6}{4/6} = 2/4 = 1/2$

(d) $P(B/A) = ?$

$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{2/6}{3/6} = 2/3$

(e) Are A and B statistically independent?

$P(A \cap B) \overset{??}{=} P(A)\,P(B)$

$2/6 \overset{?}{=} 3/6 \cdot 4/6 \rightarrow 2/6 = 2/6$

$1/3 \quad 2/3$

→ Yes, so A B are Stat. indep.

[f] C: number is divisible by 3, Are A and C Stat. indep.?

$C = \{3, 6\}$

$P(A \cap C) \overset{??}{=} P(A).P(C)$

$1/6 \overset{?}{=} 3/6 . 2/6 \rightarrow 1/6 = 1/6$

Yes, they are Stat. indep.

[g] ~~The~~ Are B and C stat. indep?

$P(B \cap C) \overset{??}{=} P(B).P(C)$

$1/6 \overset{?}{=} 4/6 . 3/6 \rightarrow 1/6 \neq 2/9$

(2/3, 1/3)

So B, C are not Stat. indep

[h] Are B and C disjoint?

$B \cap C = \{3\} \neq \emptyset \rightarrow$ So they are not disjoint.

Exp3: $S = \{1, 2, 3, 4\}$, $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 4\}$, are A, B, and C stat. indep.?

$P(A) = 1/2$
$P(B) = 1/2$
$P(C) = 1/2$

① $P(A \cap B) \overset{??}{=} P(A)(P(B)$

$1/4 \overset{??}{=} 1/2 . 1/2$

$1/4 = 1/4$

② $P(A \cap C) \overset{??}{=} P(A)P(C)$

$1/4 \overset{?}{=} 1/2 . 1/2 \rightarrow 1/4 = 1/4$

③ $P(B \cap C) \overset{??}{=} P(B)\,P(C)$

$1/4 \overset{?}{=} 1/2 . 1/2 \rightarrow 1/4 = 1/4$

④ $P(A \cap B \cap C) \overset{?}{=} P(A) . P(B) . P(C)$

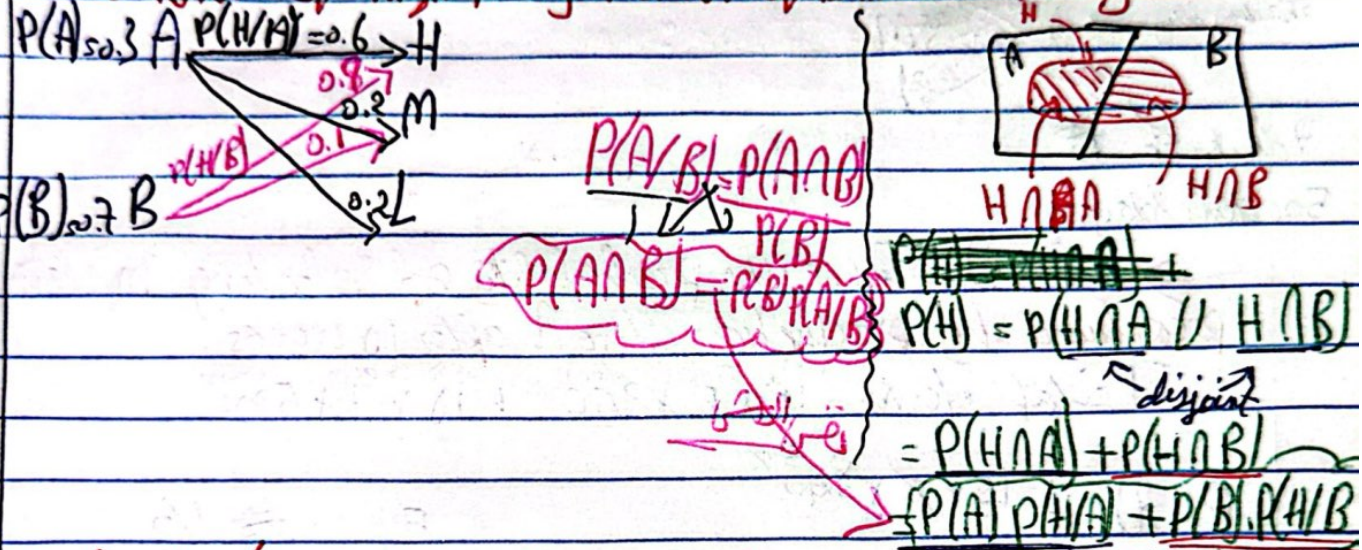
$1/4 \overset{??}{=} 1/2 . 1/2 . 1/2$

$1/4 \neq 1/8$

So they are not stat. indep.

Total probability theorem :-

Exp#1: A factory consist of 2 production lines A and B. Production line A produces 30% of the products out of which 60% are of high quality, 20% of medium quality and 20% of low quality. Production line B produces the rest of the products and 90% are of high quality and 10% of medium quality.



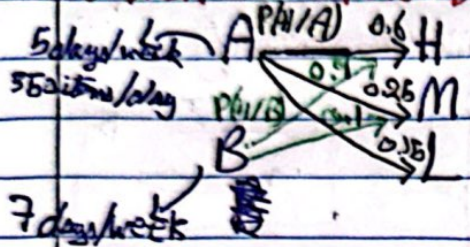
[a] An item / product is select randomly, what is the probability that it is of high quality ?

$$P(H) = P(A)\,P(H/A) + P(B)\,P(H/B)$$
$$= 0.7 \cdot 0.6 + 0.7 \cdot 0.9 = \boxed{0.81}$$

[b] An item is selected at random, what is the probability that it is of low quality ?

$$P(L) = P(A)\,P(L/A) + P(B)\,P(L/B)$$
$$= 0.3 \cdot 0.2 + 0.7 \cdot 0$$
$$= \boxed{0.06}$$

Exp:- A factory has two productions lines A and B. Production Line A work 5 days a week and produces 350 items per day out of which 60% of high quality and 25% of medum quality and 15% of low quality, Production line B work 7 days a week and produces 500 items per day out which 90% of high quality and 10% of medum quality:-



→ [a] Determine the probability of selecting an item produced by production line A after 10 weeks.

$$\text{Total items} = \underbrace{10 \times 5 \times 350}_{\text{weeks}\ \text{A days}\ \text{items}} + \underbrace{10 \times 7 \times 500}_{\text{weeks}\ \text{B days}\ \text{items}}$$

$$P(A) = \frac{10 \times 5 \times 350}{10 \times 5 \times 350 + 10 \times 7 \times 500} = \frac{5}{15} = 1/3$$

[b] $P(B) = 1 - P(A) = 2/3$

or $\dfrac{10 \times 7 \times 500}{10 \times 5 \times 350 + 10 \times 7 \times 500} = 2/3$

[c] An item is selected at random, what is the probability that it is of high quality?

$$P(H) = P(A)P(H/A) + P(B)P(H/B)$$
$$= 1/2 * 0.6 + 2/3 * 0.9 = 0.8$$

H given

[d] if a high quality item is selected, what is the probability that it was produced by A?

$$P(A/H) = \frac{P(A \cap H)}{P(H)} = \frac{P(A)\,P(H/A)}{P(H)} \rightarrow \frac{1/3 * 0.6}{0.8} = \frac{0.2}{0.8} = \boxed{\frac{1}{4}}$$

* $P(A/H) = \frac{P(A \cap H)}{P(H)}$

$P(H/A) = \frac{P(A \cap H)}{P(A)}$

[e] An item is selected and found to be of medium quality, what is the probability that it was produced by A?

$$P(M) = P(A)\,P(M/A) + P(B)\,P(M/B)$$
$$= 1/3 * 0.25 + 2/3 * 0.1 = 0.15$$

$$\rightarrow P(A/M) = \frac{P(A \cap M)}{P(M)} = \frac{P(A)\,P(M/A)}{P(M)} = \frac{1/3 * 0.25}{0.15} = \boxed{\frac{5}{9}}$$

[f] An item is selected at random and found to be of low quality what is the probability that it was produced by A?

$$P(L) = P(A)\,P(L/A) + P(B)\,P(L/B)$$
$$= 1/3 * 0.15 + 2/3 * 0 \rightarrow = \boxed{0.05}$$

$$P(A/L) = \frac{P(A \cap L)}{P(L)} = \frac{P(A)\,P(L/A)}{P(L)} = \frac{1/3 * 0.15}{0.05} = \boxed{1}$$

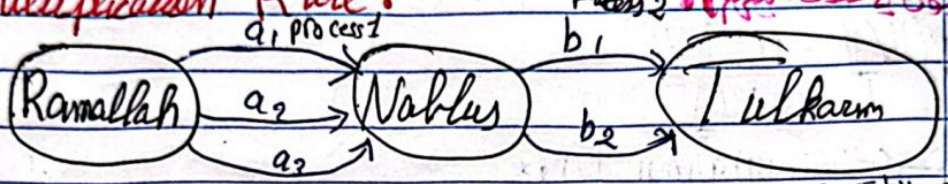
[9] what is the probability of selected a medum quality item produced by A?

$P(A \cap M) = P(A)\, P(M/A)$

$= 1/3 * 0.25 = \frac{0.25}{3} = \boxed{1/12}$

Counting Techniques :-

→ Multiplication Rule:



عدد الطرق من رام الله لطولكرم = عدد الطرق من رام الله لنابلس × عدد الطرق من نابلس لطولكرم

$= 3 \times 2 = 6$

$S = \{ a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2, a_3 b_1, a_3 b_2 \}$

التكرار ممنوع

→ Permutation → without replacement (repetition is not allowed)

→ order is important

الترتيب مهم

تباديل

عدد الاختيارات الكلية

$P_k^n = \frac{n!}{(n-k)!}$

عدد الاختيارات التي نختارها

مثال: كم عدد الطرق التي يمكن أن يجلس بها 5 أشخاص على 4 كراسي؟

$P_4^5 = \frac{5!}{(5-4)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} = 120$

5 أشخاص ← 4 كراسي

→ Combination → without replacement (repetition is not allowed)

order is not important

الترتيب غير مهم

توافيق

عدد الاختيارات الكلية

$C_k^n = \frac{n!}{k!(n-k)!}$

عدد الاختيارات التي نختارها

مثال: كم طريقة يمكن أن يجلس بها 5 أشخاص على 4 كراسي إذا كان الترتيب غير مهم؟

$C_4^5 = \frac{5!}{4!(5-4)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5$

Exp: A person type a 5 letter computer password, what is the probability that no letter is repeated? (#of letter is 26).
A

# of Sample outcomes is $S = 26*26*26*26*26 = 26^5$

# of Sample outcomes in A $= 26*25*24*23*22$

$\rightarrow P^{n=26}_{k=5} = \frac{26!}{(26-5)!} = \frac{26*25*24*23*22*21}{21!}$

$P(A) = \frac{26*25*24*23*22}{26^5} = 0.6643674$

→ Sample Space:

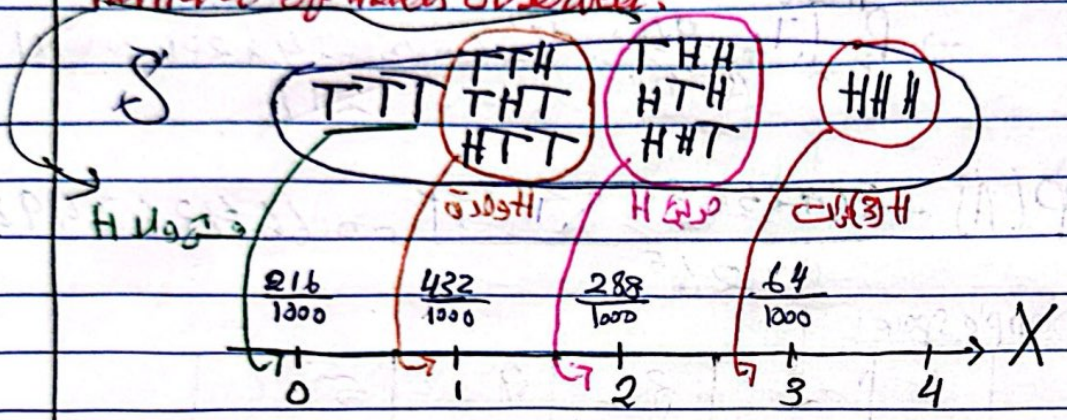
	1	2	3	4	5
1	1,1	1,2	1,3	1,4	1,5
2	2,1	2,2	2,3	2,4	2,5
3	3,1	3,2	3,3	3,4	3,5
4	4,1	4,2	4,3	4,4	4,5
5	5,1	5,2	5,3	5,4	5,5

- اذا كان التكرار ممنوع (تباديل) ← 20
- اذا كان الترتيب مش مهم (ممنوع التكرار توافيق) ← 10
يعني مثلاً (1,2) و (2,1) ⇒ يعتبرهم خيار واحد

Chapter 2: Single Random Variables and Probability Distributions

Exp 1: let us consider the experiment of flipping three coins. Assume the $P(H) = 4/10$ and $P(T) = 6/10$ for each of the coins.

→ let us define the Random Variable x such that it represent the number of heads observed.



[a] Determine the probability of observing exactly one head.

$P(TTH, THT, HTT) = P(TTH) + P(THT) + P(HTT)$

$\rightarrow P(TTH) = P(T)\,P(T)\,P(H) =$

$= \frac{6}{10} * \frac{6}{10} * \frac{4}{10} = \frac{144}{1000}$

$P(THT) = P(T)\,P(H)\,P(T) = \frac{6}{10} * \frac{4}{10} * \frac{6}{10} = \frac{144}{1000}$ +

$P(THT) = \frac{144}{1000}$ +

$\rightarrow P(\{HTT, THT, TTH\}) = \frac{432}{1000}$

$= 3 * \frac{4}{10} * (\frac{6}{10})^2 = \frac{432}{1000}$

[b] Determine the probability of observing exactly two heads.

$P(THH, HTH, HHT) = 3 * P(T)\,[P(H)]^2$

$= 3 * \frac{6}{10} * (\frac{4}{10})^2 = \frac{288}{1000}$

(c) Determine the probability of observing three heads?

$P(HHH) = \left(\frac{4}{10}\right)^3 = \frac{64}{1000}$

(a) Determine $P(X=1)$?

$P(X=1) = P(\{HTT, THT, TTH\}) = \frac{432}{1000}$

(b) $P(X=0) = P(\{TTT\}) = \left(\frac{6}{10}\right)^3 = \frac{216}{1000}$

(f) Determine the Probability Mass function of X?

$P(X=2) = \frac{288}{1000}$, $P(X=4) = 0$

$P(X=3) = \frac{64}{1000}$, $P(X=-1) = 0$

$$\rightarrow P(X=x) = \begin{cases} \frac{216}{1000} & x=0 \\ \frac{432}{1000} & x=1 \\ \frac{288}{1000} & x=2 \\ \frac{64}{1000} & x=3 \\ 0 & \text{otherwise} \end{cases}$$

Properties of P.M.F :-

* $P(X=x) \geq 0$
* $\sum_{-\infty}^{\infty} P(X=x) = 1$

* Comulative Distribution Function :-

$$F_X(x) = P(X \leq x)$$

g) $F_X(0.5) = P(X \leq 0.5) = P(x=0) = \frac{216}{1000}$

h) $F_X(1.5) = P(X \leq 1.5)$
$= P(x=0) + P(x=1) = \frac{216}{1000} + \frac{432}{1000} = \frac{648}{1000}$

i) $F_X(2.7) = P(X \leq 2.7)$
$= P(x=0) + P(x=1) + P(x=2) = \frac{936}{1000}$

j) $F_X(2) = P(X \leq 2) = P(x=0) + P(x=1) + P(x=2) = \frac{936}{1000}$

k) $F_X(2^-) = P(X < 2) = P(x=0) + P(x=1) = \frac{648}{1000}$

l) $F_X(2^+) = P(X \leq 2) + P(x>2) + P(x \geq 2) = \frac{936}{1000}$

Note!

Properties of CDF :-

1. $F_X(-\infty) = P(X \leq -\infty) = 0$
2. $F_X(\infty) = P(x \leq \infty) = 1$
3. $P(X=x) = F_X(x) - F_X(x^-)$

Exp 2:- Let X be a R.V with the following PMF:-

$$P_X(X=x) = \begin{cases} 0.2 & x=-1 \\ 0.3 & x=1 \\ G & x=2 \\ 0.4 & x=3 \\ 0 & \text{otherwise} \end{cases}$$

1. Determine the value of constant G?

$$\sum_{-\infty}^{\infty} P(X=x) = 1 \rightarrow 0.2 + 0.3 + G + 0.4 = 1 \rightarrow \boxed{G = 0.1}$$

2. $F_X(1) = P(X \leq 1) = P(1) + P(-1) = 0.3 + 0.2 = \boxed{0.5}$

3. $F_X(-\infty) = ??$

$= P(X \leq -\infty) = 0$

4. $F_X(\infty) = ??$

$P(X \leq \infty) = P(x=1) + P(x=-1) + P(x=2) + P(x=3) = 1$

5. $F_X(2^-) = P(X < 2) = P(-1) + P(1) = 0.5$

6. $P(X > 2)$? $= P(X=3) = 0.4$

or

$1 - P(X \leq 2)$

$1 - F_X(2) = 1 - 0.6 = 0.4$

Ex+P3) Let X be a R.V. with the following CDF.

$$F_X(x) = \begin{cases} A^0 & x \le -2 \\ 0.2 & -2 \le x < 0 \\ 0.5 & 0 \le x < 3 \\ k & 3 \le x \end{cases}$$

(1) Determine the value of k, A?

Note: حسب

$F_X(-\infty) = A = 0$, $F_X(\infty) = k = 1$

(2) $F_X(1) = ?$ $= 0.5$

(3) $P(X=0) = ?$

$P(X=0) = P(X \le 0) - P(X < 0)$

$= F_X(0) - F_X(0^-)$

$= 0.5 - 0.2 = 0.3$

(4) $P(X=3) = ?$

$P(X=3) = P(X \le 3) - P(X < 3)$

$= F_X(3) - F_X(3^-)$

$1 - 0.5 = 0.5$

(5) $F_X(7) = 1$

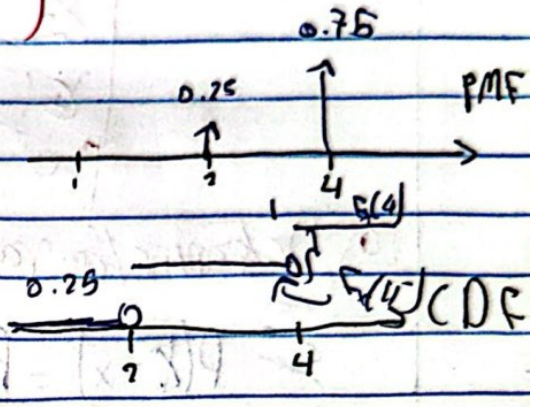
(6) $P(X \ge 3) = ?$ $\rightarrow P(X \ge 3) = 1 - P(X < 3)$

$= 1 - F_X(3^-) = 1 - 0.5 = 0.5$

Ex#4) let x be a R.V with the following PMF :-

$$P(X=x) = \begin{cases} 0.25 & x=2 \\ 0.75 & x=4 \\ 0 & \text{otherwise} \end{cases}$$

1. $F_x(1) = P(x \le 1) = 0$
2. $F_x(5) = P(x \le 5) = 1$
3. $F_x(7.5) = P(x \le 7.5) = 1$



$$F_x(x) = \begin{cases} 0 & x < 2 \\ 0.25 & 2 \le x < 4 \\ 1 & 4 \le x \end{cases}$$

* Properties PMF :-

1. $P(X=x) \ge 0$
2. $\sum_{x=-\infty}^{\infty} P(X=x) = 1$

* Properties of CDF :-

1. $F_x(-\infty) = 0$
2. $F_x(\infty) = 1$
3. $F_x(x) = F_x(x^+)$
4. $x_2 > x_1 \rightarrow F(x_2) \ge F(x_1)$

Exp 1) Let X be a R.V with the following PMF:-

$$P(X=x) = \begin{cases} 1/4 & x=-3 \\ 1/8 & x=-1 \\ 1/2 & x=1 \\ B & x=2 \\ 0 & O.W \end{cases}$$

[a] Determine the value of the constant B.

$$\sum_{x=-\infty}^{\infty} P(X=x) = P(x=-3)+P(x=-1)+P(x=1)+P(x=2)$$
$$= 1/4 + 1/8 + 1/2 + B = 1$$
$$\rightarrow \boxed{B = 1/8}$$

[b] $F_X(-\infty) = ??$

$$F_X(-\infty) = P(X \leq -\infty) = 0$$

[c] $F_X(\infty) = ??$

$$F_X(\infty) = P(X \leq \infty) = 1$$

[d] $F_X(1^-) = ??$

$$P(X \leq 1^-) = P(X < 1) = 3/8$$

[e] $F_X(1^+) = ??$

$$P(X \leq 1^+) = P(X \leq 1) = 7/8$$

[f] $F_X(1) = ?$

$$F_X(1) = F_X(1^+) = \frac{7}{8}$$

$P(X<1)$ $\quad$ $P(X \leq 1)$.

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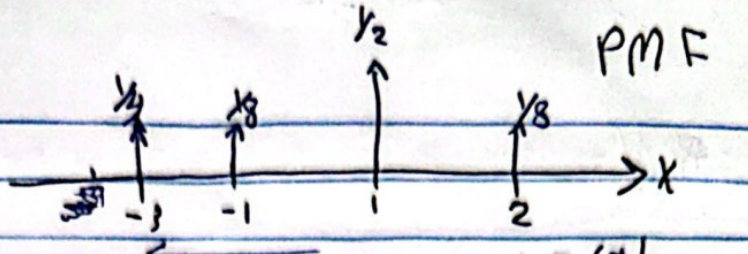
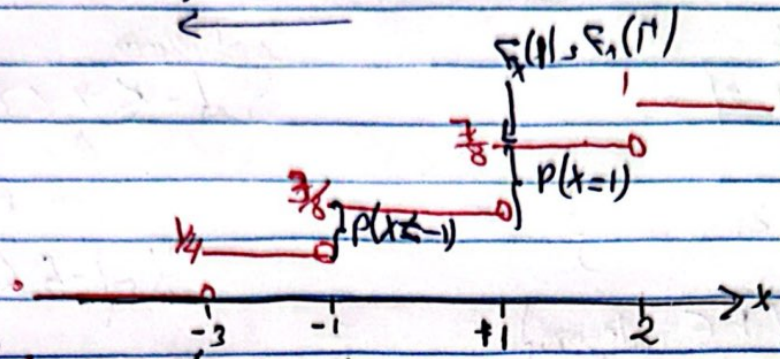




g) $F_X(1.5) = P(X \leq 1.5) = 7/8$

h) $F_X(2.25) = P(X \leq 2.25) = 1$

i) Write the CDF of X.

$$F_X(x) = \begin{cases} 0 & x < -3 \\ 1/4 & -3 \leq x < -1 \\ 3/8 & -1 \leq x < 1 \\ 7/8 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

j) $P(-1 \leq X \leq 2) = ?$ → $P(x=-1) + P(x=1) + P(x=2) = 6/8 = 3/4$

or → $P(X \leq 2) - P(X < -1)$

$= F_X(2) - F_X(-1) = 1 - 1/4 = 3/4$

k) $P(-1 < X \leq 2)$ → $P(1) + P(2) = 5/8$

or → $P(X \leq 2) - P(X \leq -1)$

$= F_X(2) - F_X(-1)$

$= 1 - 3/8 = 5/8$

l) $P(-1 \leq X < 2) = P(X < 2) - P(X < -1) = 7/8 - 1/4 = 5/8$

m) $P(-1 < X < 2) = P(X < 2) - P(X \leq -1) = 7/8 - 3/8 = 4/8 = 1/2$

R.Vs

Discrete

$S = \{-1, 1, 2\}$

or $S = \{-1, 1, 2, \dots\}$

$= \{x \in \text{integers}\}$

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 2, 4, 5\}$

$P(A) = 4/6$

← يكون فيها الاعداد بدون اعشار او كسور

Continuous

$S = \{-\infty < x < \infty\}$

↑ real numbers

$S = \{-5 < x\}$

$S = \{-1 \leq x \leq 3\} \rightarrow 4$ (real)

$A = \{-1 \leq x \leq 2\} \rightarrow 3$

$P(A) = ?$

$P(x=0) = 0$

← عشان كده بنحولها لفترات

* Probability density Function (PDF)

$$f_x(x) = \begin{cases} 1/4 & 1 \leq x \leq 5 \\ 0 & \text{o.w} \end{cases}$$

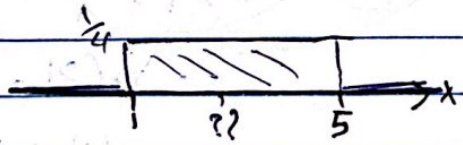
Valid PDF?

$\int_{-\infty}^{\infty} f_x(x)\,dx$

$= \int_{-\infty}^{1} 0\,dx + \int_{1}^{5} \frac{1}{4}\,dx + \int_{5}^{\infty} 0\,dx$

$= \frac{1}{4}x \Big|_1^5$

$\frac{1}{4}(5-1) = 1$



$$g(x) = \begin{cases} \frac{2x}{15} & 1 \leq x \leq 4 \\ 0 & \text{o.w} \end{cases}$$

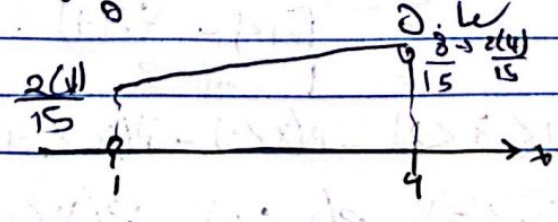
Valid PDF?

$\int_{-\infty}^{\infty} f_x\,dx \overset{??}{=} 1$

$\int_{1}^{4} \frac{2x}{15}\,dx = \frac{2x^2}{30} \Big|_1^4$

$\frac{2(16-1)}{30}$

$= 1$



* Properties of PDF?

① $f_x(x) \geq 0$

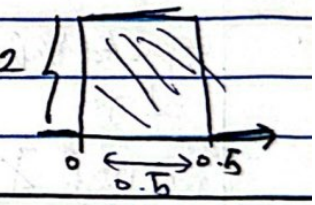
② $\int_{-\infty}^{\infty} f_x(x)\,dx = 1$

Exp 2):- Let x be a R.V with the following PDF?:-

$$f_x(x) = \begin{cases} k^2 & 0 \leq x \leq 0.5 \\ 0 & \text{O.W} \end{cases}$$

[a] Determine the value of the constant k?

$$\int_{-\infty}^{\infty} f_x(x)\,dx = 1 \rightarrow \int_{0}^{0.5} k\,dx = kx\Big|_0^{0.5} \Rightarrow k(0.5) = 1 \rightarrow \boxed{k = 2}$$



Exp 3):- Let x be a R.V with the following PDF:-

$$f_x(x) = \begin{cases} kx^2 & 0 < x < 2 \\ 0 & \text{O.W} \end{cases}$$

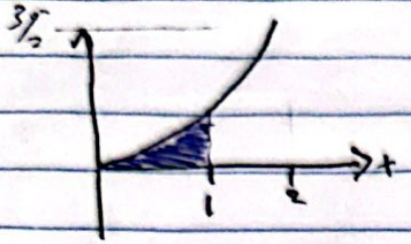
[1] Determine the value of the constant k:-

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{2} kx^2\,dx \quad \frac{kx^3}{3}\Big|_0^2 \rightarrow \frac{8k}{3} = 1 \rightarrow \boxed{k = \frac{3}{8}}$$

② $F_x(1) = ?$ $= P(x \le 1)$

$$\int_{-\infty}^{1} f(x)\,dx = \int_{0}^{1} \frac{3}{8}x^2\,dx$$

$$= \frac{x^3}{8}\Big|_0^1 = \boxed{\frac{1}{8}}$$



③ $P(0.5 \le x \le 1.5) = ?$

$$= \int_{0.5}^{1.5} \frac{3}{8}x^2\,dx = \frac{x^3}{8}\Big|_{0.5}^{1.5}$$

$$\frac{3}{8}\left[(1.5-0.5)^3\right] = \boxed{??}$$

Mean or Expected Value :-

- $E\{x\} = \mu_x \overset{D}{=} \sum_{x=-\infty}^{\infty} x\,P(X=x)$

  → expected Value of X or mean of X

  $\overset{C}{=} \int_{-\infty}^{\infty} x\,f_x(x)\,dx$

- $E\{g(x)\} \overset{D}{=} \sum g(x)\,P(X=x)$

  $\overset{C}{=} \int_{-\infty}^{\infty} g(x)\,f_x(x)\,dx$

Variance of X

$$\sigma_x^2 = Var\{x\} = E\{(x-\mu_x)^2\}$$

$$\overset{D}{=} \sum_{x=-\infty}^{\infty} (x-\mu_x)^2 P(X=x)$$

$$\overset{C}{=} \int_{-\infty}^{\infty} (x-\mu_x)^2 f_x(x)\,dx$$

* Standard deviation of X :

$\sigma_X = \sqrt{\sigma_X^2}$

Exp1: Let X be a R.V with the following:

$$P(X=x) = \begin{cases} 1/4 & x=2 \\ 1/4 & x=-1 \\ 1/4 & x=1 \\ 1/4 & x=2 \\ 0 & \text{O.W} \end{cases}$$

Determine the :-

1. $\mu_X = \sum_{x=-\infty}^{\infty} x\,p(X=x) = (-2)(1/4) + (-1)(1/4) + 1(1/4) + 2(1/4) = [\,]$

2. $\sigma_X^2 = E\{(X-\mu_X)^2\} = E\{(X-0)^2\} = E\{X^2\} = \sum_{x=-\infty}^{\infty} x^2 p(X=x)$

$= (-2)^2 \cdot (1/4) + (-1)^2 \cdot (1/4) + 1^2 (1/4) + 2^2 (1/4) = \frac{10}{4} = \boxed{2.5}$

3. $\sigma_X = \sqrt{\sigma_X^2} = \sqrt{2.5} = 1.58114$

Exp2: $P(X=x) = \begin{cases} 1/4 & x=-4 \\ 1/4 & x=-3 \\ 1/4 & x=-1 \\ 1/4 & x=0 \\ 0 & \text{O.W} \end{cases}$

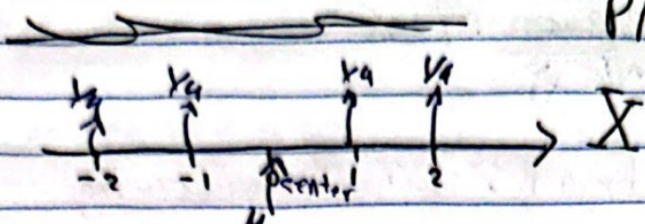
a. $\mu_X = \sum_{x=-\infty}^{\infty} x\,P(X=x) = -4 \cdot \frac{1}{4} + -3 \cdot \frac{1}{4} + -1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \boxed{-2}$

b. $\sigma_X^2 = E\{(X-\mu_X)^2\} = E\{(X+2)^2\} = \sum_{x=-\infty}^{\infty} (x+2)^2 P(X=x)$

$= \underset{-4+2}{(-2)^2} \cdot \frac{1}{4} + \underset{-3+2}{(-1)^2} \cdot \frac{1}{4} + \underset{-1+2}{(1)^2} \cdot \frac{1}{4} + \underset{0+2}{(2)^2} \cdot \frac{1}{4} = 2.5$

c. $\sigma_X = \sqrt{\sigma_X^2} = \sqrt{2.5} = 1.58114$

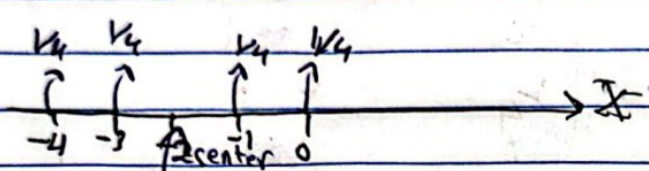
Exp1



$\mu_x = 0$

$\sigma_x^2 = 2.5$

Exp2



$\mu_x = -2$

$\sigma_x^2 = 2.5$

Exp3 $P(X=x) = \begin{cases} \frac{1}{4} & x=-4 \\ \frac{1}{4} & x=-2 \\ \frac{1}{4} & x=2 \\ \frac{1}{4} & x=4 \\ 0 & o.w \end{cases}$

a) $\mu_x = \sum_{x=-\infty}^{\infty} x P(X=x) = (-4)\frac{1}{4} + (-2)\frac{1}{4} + (2)\frac{1}{4} + (4)\frac{1}{4} = 0$

b) σ_x^2

$\mu_x = 0$

$\rightarrow E\{(x-\mu_x)^2\} = E\{x^2\} = \sum_{x=-\infty}^{\infty} x^2 P(X=x)$

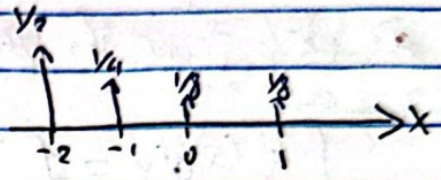
$= (-4)^2 \cdot \frac{1}{4} + (-2)^2 \cdot \frac{1}{4} + (2)^2 \cdot \frac{1}{4} + (4)^2 \cdot \frac{1}{4} = \frac{40}{4}$

$= 10$

c) $\sigma_x = \sqrt{\sigma_x^2} = \sqrt{10} = 3.16228$

* لما تكون σ_x^2 قليلة يعني الناتج قريبة على بعض
اما اذا كان على يعني الناتج بعيدة عن بعض (متشتتة / متوزعة)
كلما زاد التشتت زاد الـ Variance

Ex) $P(X=x) = \begin{cases} 1/2 & x=-2 \\ 1/4 & x=-1 \\ 1/8 & x=0 \\ 1/8 & x=1 \\ 0 & O.W \end{cases}$



1) $\mu_x = \sum_{x=-\infty}^{\infty} x\,P(X=x)$

$= -2\cdot\frac{1}{2} + -1\cdot\frac{1}{4} + 0\cdot\frac{1}{8} + 1\cdot\frac{1}{8} = -\frac{9}{8}$

2) $\sigma_x^2 = E\{(x-(-\frac{9}{8}))^2\} = E\{(\frac{9}{8}+x)^2\} = \sum_{-\infty}^{\infty} (\frac{9}{8}+x)^2 \cdot P(X=x)$

$= (\frac{9}{8}-2)^2\cdot\frac{1}{2} + (\frac{9}{8}-1)^2\cdot\frac{1}{4} + (\frac{9}{8}+0)^2\cdot\frac{1}{8} + (\frac{9}{8}+1)^2\cdot\frac{1}{8}$

$= (-\frac{7}{8})^2\cdot\frac{1}{2} + (\frac{1}{8})^2\cdot\frac{1}{4} + (\frac{9}{8})^2\cdot\frac{1}{8} + (\frac{17}{8})^2\cdot\frac{1}{8}$

$= 1.109375 \approx 1.10938$

3) $\sigma_x = \sqrt{\sigma_x^2} = \sqrt{1.10938} = 1.05327$

Exp 5: Let x be R.V with the following PDF

$$f_x(t) = \begin{cases} k(1-t^2) & -1 < t < 1 \\ 0 & \text{O.W} \end{cases}$$

a] Determine the value of k?

$$\int_{-\infty}^{\infty} f_x(t)\,dt = 1 \rightarrow \int_{-1}^{1} k(1-t^2)\,dt = 1 \;,\; k\left\{t - \frac{t^3}{3}\right\}\Big|_{-1}^{1} = 1$$

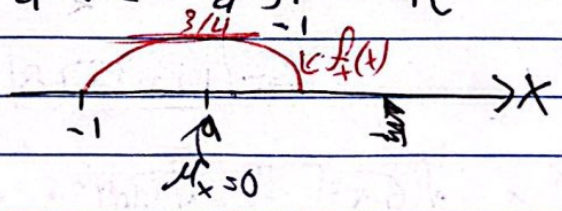
$$= k\left\{\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right\} = 1$$

$$4/3\,k = 1 \rightarrow \boxed{k = 3/4}$$

b] Determine the expected value of x:

$$\mu_x = \int_{-\infty}^{\infty} x\, f_x(t)\,dt = \int_{-1}^{1} x \cdot \frac{3}{4}(1-x^2)\,dx = \int_{-1}^{1} \frac{3}{4}(x - x^3)\,dx$$

$$= \frac{3}{4}\left\{\frac{t^2}{2} - \frac{t^4}{4}\right\}\Big|_{-1}^{1} = \frac{3}{4}\left\{\left(\frac{1}{2} - \frac{1}{4}\right) - \left(\frac{1}{2} - \frac{1}{4}\right)\right\} = \boxed{0}$$



b] σ_t^2 ?

$$\sigma_t^2 = \Sigma\{t^2\} = \int_{-\infty}^{\infty} t^2 f_t(t)\,dt = \int_{-1}^{1} t^2 \cdot \tfrac{3}{4}(1-t^2)\,dt$$

$$\left(\Sigma\{(t-0)^2\}\right)$$

$$= \frac{3}{4}\int_{-1}^{1}(t^2 - t^4)\,dt = \frac{3}{4}\left\{\frac{t^3}{3} - \frac{t^5}{5}\right\}\Big|_{-1}^{1}$$

$$= \frac{3}{4}\left\{\left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{-1}{3} + \frac{1}{5}\right)\right\} = \frac{3}{4}\left\{\frac{2}{3} - \frac{2}{5}\right\} = \boxed{0.2}$$

c] σ_t ?

$$\sigma_t = \sqrt{\sigma_t^2} = \sqrt{0.2} = 0.44721$$

Exp 6) $f_x(x) = \begin{cases} k(4-x^2) & -2 \le x \le 2 \\ 0 & O.W \end{cases}$

μ_x ? , σ_x^2 ? , σ_x ?

Find k → $\int_{-\infty}^{\infty} f(x)\,dx = 1 \rightarrow k\int_{-2}^{2}(4-x^2)\,dx = k\left[\int_{-2}^{2} 4\,dx - \int_{-2}^{2} x^2\,dx\right] = 1$

$k\left[4(2+2) - \left(\frac{x^3}{3}\Big|_{-2}^{2}\right)\right] = 1$

$k\left[16 - \left(\frac{8}{3} - \frac{-8}{3}\right)\right] = 1$

$\rightarrow k\left[16 - \frac{16}{3}\right] = 1 \rightarrow \frac{32}{3}k = 1 \rightarrow \boxed{k = \frac{3}{32}}$

$\rightarrow \mu_x = \int_{-\infty}^{\infty} x\, f_x(x)\,dx \rightarrow \int_{-2}^{2} \frac{3}{32}(4-x^2).x\,dx \rightarrow \frac{3}{32}\int_{-2}^{2} 4x - x^3\,dx$

$= \frac{3}{32}\left[2x^2 - \frac{x^4}{4}\Big|_{-2}^{2}\right] \rightarrow \frac{2}{32}\left[\left(8 - \frac{8}{3}\right) - \left(8 - \frac{8}{3}\right)\right]$

$\sigma_x^2 = E(x^2 - \mu_x)^2 = E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_{-2}^{2} x^2 \cdot \frac{3}{32}(4-x^2)\,dx \rightarrow \frac{3}{32}\int_{-2}^{2}(4x^2 - x^4)\,dx = \frac{3}{32}\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]$

$= \frac{3}{32}\left[\left(\frac{32}{3} - \frac{32}{5}\right)\right] \rightarrow \frac{1}{32}\left[\frac{64}{5}\right] = 2/5 = \boxed{0.4}$

$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{0.4} = \boxed{0.63246}$

Exp7) $f_x(t) = \begin{cases} H^{\frac{1}{4}} & -2 \le x \le 2 \\ 0 & \text{o.w} \end{cases}$

$\int_{-\infty}^{\infty} f(t)dt = 1 \rightarrow \int_{-2}^{2} H\,dt = 1 \rightarrow H\{2-(-2)\} = 1 \rightarrow 4H = 1$

$H = \frac{1}{4}$

$\mu_x = \int_{-2}^{2} x f_x(t)dt = \int_{-2}^{2} \frac{1}{4} x\,dx = \frac{1}{4}\frac{t^2}{2}\Big|_{-2}^{2}$

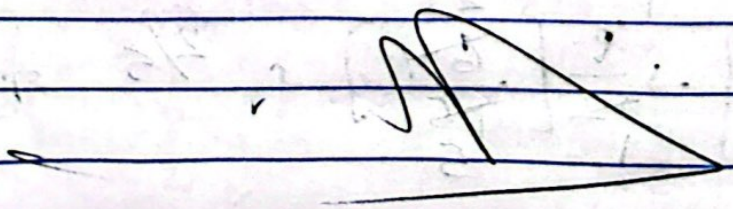
$= \frac{1}{8}\{4-4\} = 0$

$\sigma_x^2 = E\{(x-\mu_x)^2\} = E\{t^2\} = \int_{-\infty}^{\infty} t^2 \cdot f_x(t)dt$

$= \int_{-2}^{2} \frac{1}{4} t^2\,dt = \frac{1}{4}\frac{t^3}{3}\Big|_{-2}^{2}$

$= \frac{1}{4}\left\{\frac{8}{3} - \left(-\frac{8}{3}\right)\right\} = \frac{1}{4}\left(\frac{16}{3}\right) = \frac{4}{3} = 1.33$

$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{1.33} = 1.1547$



Ex 1: Let x be a R.V with the following pdf:-

$$f_x(x) = \begin{cases} k(x^2+x) & 0 \le x \le 2 \\ 0 & O.W \end{cases}$$

(1) Determine the value of the constant k?

$$\int_{-\infty}^{\infty} f_x(x)\,dx = \int_0^2 k(x^2+x)\,dx = 1 \rightarrow k\left[\frac{x^3}{3} + \frac{x^2}{2}\Big|_0^2\right] = 1$$

$$\rightarrow k\left\{\left(\frac{8}{3} + \frac{4}{2}\right) - 0\right\} = 1 \rightarrow k\left(\frac{14}{3}\right) = 1$$

$$\boxed{k = 3/14}$$

(2) $P(X \ge 1) = ?$

$$= \int_1^{\infty} f_x(x)\,dx = \int_1^2 \frac{3}{14}(x^2+x)\,dx = \frac{3}{14}\left[\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_1^2\right]$$

$$\rightarrow \frac{3}{14}\left[\left(\frac{8}{3} + \frac{4}{2}\right) - \left(\frac{1}{3} + \frac{1}{2}\right)\right] \rightarrow \frac{3}{14}\left[\left(\frac{7}{3} + \frac{3}{2}\right)\right] \rightarrow \frac{1}{14} \cdot \frac{23}{2} = \boxed{\frac{23}{28}}$$

(3) $P(X \ge 1.5 / X > 1) = \dfrac{P(X \ge 1.5 \cap X > 1)}{P(X>1)}$ (A: $X \ge 1.5$, B: given $X > 1$)

$$= \frac{P(X \ge 1.5)}{P(X>1)} = \frac{\int_{1.5}^{2} f_x(x)\,dx}{\int_1^2 f_x(x)\,dx} = \frac{\int_{1.5}^{2} \frac{3}{14}(x^2+x)\,dx}{\int_1^2 \frac{3}{14}(x^2+x)\,dx} = \frac{\frac{3}{14}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_{1.5}^{2}}{\frac{3}{14}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_1^2}$$

$$= \frac{\frac{3}{14}\left(\frac{8}{3} + \frac{4}{2}\right) - \left(\frac{(1.5)^3}{3} + \frac{(1.5)^2}{2}\right)}{23/28} = \frac{\frac{3}{14}\left(\frac{14}{3} - \frac{9}{4}\right)}{23/28} = \frac{0.51785}{23/28} = 0.63043$$

(4) $P(X \le 1.4 / X > 1) = \dfrac{P(X \le 1.4 \cap X > 1)}{P(X>1)} = \dfrac{P(1 < X \le 1.4)}{P(X>1)}$

$$= \frac{\int_1^{1.4} \frac{3}{14}(x^2+x)\,dx}{\frac{23}{28}} = \frac{\frac{3}{14}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_1^{1.4}}{21/28} = \frac{\frac{3}{14}\left[\frac{(1.4)^3}{3} + \frac{(1.4)^2}{2}\right] - \left(\frac{1}{3} + \frac{1}{2}\right)}{23/28} = \frac{0.22742}{23/28}$$

$$= 0.2769$$

5) $P(X>1.5 \mid X<3) = \dfrac{P(X>1.5 \cap X<3)}{P(X<3)} = \dfrac{P(1.5<X<3)}{P(X<3)}$

$$= \frac{\int_{1.5}^{2} \frac{3}{14}(x^2+x)\,dx}{\int_{0}^{2} \frac{3}{14}(x^2+x)\,dx} = \frac{\frac{3}{14}\left(\frac{x^3}{3}+\frac{x^2}{2}\Big|_{1.5}^{2}\right)}{1} = \frac{0.5179}{1} = \boxed{0.5179}$$

6) Determine the mean of x:-

$$\mu_x = \int_{-\infty}^{\infty} x\, f_x(x)\,dx = \int_{0}^{2} x\cdot\frac{3}{14}(x^2+x) = \frac{3}{14}\int_{0}^{2}(x^3+x^2)\,dx$$

$$= \frac{3}{14}\left[\frac{x^4}{4}+\frac{x^3}{3}\Big|_0^2\right] = \frac{3}{14}\left[\left(\frac{16}{4}+\frac{8}{3}\right)-0\right] = \frac{3}{14}\left[\frac{20}{3}\right] = \frac{20}{14} = \boxed{\frac{10}{7}}$$

7) Determine the variance of x:-

$$\sigma_x = E(x-\mu_x)^2 = \int_{-\infty}^{\infty}(x-\tfrac{10}{7})^2 f_x(x)\,dx = \int_{0}^{2}(x-\tfrac{10}{7})^2\cdot\frac{3}{14}(x^2+x)\,dx$$

$$= \frac{3}{14}\int_{0}^{2}(x-\tfrac{10}{7})^2\cdot(x^2+x)\,dx = \boxed{\frac{46}{245}}$$

* Note :-

* Mode of the distribution

$$x_{mode} : f_x(x_{mode}) = \max[f_x(x)]$$

8) Determine the mode of $f_x(x)$:-

$$f_x' = 0 \rightarrow \frac{df_x(x)}{dx} = 0 \rightarrow \frac{d\left[\frac{3}{14}(x^2+x)\right]}{dx} = 0 \quad (\text{constant} = 0) \quad \frac{3}{14}(2x+1) = 0$$

$$\rightarrow 2x+1 = 0 \rightarrow 2x = -1 \rightarrow \boxed{x = -\frac{1}{2}}$$

check of the ends of the interval $[0,2]$

$f_x(0) = 0$, $f_x(2) = \frac{3}{14}(2^2+2) = \boxed{\frac{18}{14}}$, $\frac{18}{14} > 0$

So, the mode of $f_x(x)$ is at $x=2$

Ex2) let x be a R.V with the following p.d.f:-

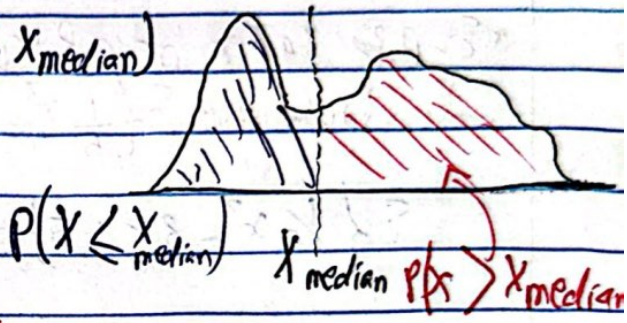
$$f_x(t) = \begin{cases} \frac{2}{9}x & 0 \le x \le 3 \\ 0 & o.w \end{cases}$$

1) Determine the mode of $f_x(t)$?

$\frac{df_x}{dx} = 0 \rightarrow \frac{2}{9} \neq 0$, $f_x(0) = 0$, $f_x(3) = 2/3$ ⟹ So, the mode of $f_x(t)$ is 3

* Note: • Median of distribution

x_{median} : $P(X \le x)_{median} = P(X > x_{median})$



2) Determine the median of $f_x(x)$:-

$$P(X \le x_0) = \int_0^{x_0} \frac{2}{9}x\,dx = \frac{x^2}{9}\Big|_0^{x_0} = \left(\frac{x_0^2}{9} - 0\right)$$

(median)

$$= \frac{x_0^2}{9} = \frac{1}{2} \rightarrow t^2 = \frac{9}{2} \quad \cancel{x_0 = +x}$$

$x_0 = \pm\frac{3}{\sqrt{2}}$

x_2

$x_3 = \frac{3}{\sqrt{2}}$

* Notes:-

1) $E\{ax\} = \int_{-\infty}^{\infty} a \cdot x\, f_x(t)\,dx = a\int_{-\infty}^{\infty} f_x(t)\,dx = aE(x)$

(a: constant)

2) $E\{b\} = \int_{-\infty}^{\infty} b\, f_x(x)\,dx = b\int_{-\infty}^{\infty} f_x(t)\,dx = b$

(b: constant)

3) $E\{ax+b\} = ?? = \int_{-\infty}^{\infty} [ax+b]\, f_x(x)\,dx = a\int_{-\infty}^{\infty} f_x(x)\,dx + b\int f_x(x)\,dx$

$= aE(x) + b$

4) $E\{x^2\} = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx \neq (E\{x\})^2$

5) $\sigma_x^2 = E\{(x-\mu_x)^2\} = E\{x^2 - 2\mu_x x + \mu_x^2\}$

$= E\{x^2\} - E\{2\mu_x x\} + E\{\mu_x^2\}$

$= E\{x^2\} - 2\mu_x E\{x\} + \mu_x^2$

$= E\{x^2\} - 2\mu_x^2 + \mu_x^2$

$\boxed{\sigma_x^2 = E\{x^2\} - \mu_x^2}$

Exp3) let x be a R.V with $\mu_x = 3$ and $\sigma_x^2 = 6$

Determine $E\{\frac{1}{5}x^2 - 3x + 5\}$?

$E\{\frac{1}{5}x^2 - 3x + 5\} = \frac{1}{5}E\{x^2\} - 3\overbrace{E\{x\}}^{3} + 5$

$\rightarrow \frac{1}{5}(15) - 9 + 5 = \boxed{-1}$

$\sigma_x^2 = E\{x^2\} - \mu_x^2$

$6 = E\{x^2\} - 9$

$\boxed{E\{x^2\} = 15}$

---

* Common Discrete Distribution:-

1) Binomial Distribution:

Experiment repeated for n times (trails)

* Only two outcomes "S", "f"
* P(S) and P(f) do not change
* Trails are independent

X: number of success in the n trails.

$$P(X=x) = \begin{cases} \binom{n}{x} P(S)^x [1-P(S)]^{n-x} & x = 0, 1, 2, \ldots, n \\ 0 & o.w \end{cases}$$

$\mu_x = n\,P(S)$, $\sigma_x^2 = n\,P(S)[1-P(S)]$.

2 Geometric Distribution:

Experiment is repeated
- Only two outcomes "S" and "F".
- P(S) and P(F) do not change.
- trails are independent

X: number of trails to the first success.

$$P(X=x) = P(F)^{x-1} P(S)$$

$$\mu_X = \frac{1}{P(S)}, \quad \sigma_X^2 = \frac{1-P(S)}{P(S)^2}$$

Exp4:- let us consider the experiment of tossing the dice.

[a] what is the probability of getting a number divisible by 3 in 4 trails assuming the dice is tossed for 10 times.

n=10, S: divisible by 3, $P("S") = \frac{2}{6} = \frac{1}{3}$, $P(F) = \frac{2}{3}$

X: number of success in the 10 trails.

$$P(X=x) = \begin{cases} \binom{10}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{10-x} & x = 0,1,2,\dots,10 \\ 0 & o.w \end{cases}$$

$$P(X=4) = \binom{10}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^6 = \frac{10!}{(10-4)!4!}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^6 = \boxed{0.22760}$$

[b] what is the probability of getting a number divisible by 3 the first time in the fourth trial?

X: Success in the 4th trail

FFFS $\rightarrow$

$$P(X=x) = \begin{cases} P(F)^{x-1}P(S) & x = 0,1,2,\dots \\ 0 & o.w \end{cases}$$

$$P(X=4) = \left(\frac{2}{3}\right)^{4-1}\left(\frac{1}{3}\right) = \left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) = 0.098765.$$

[C] Assume the dice is tossed for 8 trials, what is the average number of trials at which a number less than 5 is observed

S: Success: number less than 5 = {1, 2, 3, 4}

$P(S) = 4/6 = 2/3$

X: number of success in 8 trails

$\mu_x = n P(S) = 8 \cdot \frac{2}{3} = \boxed{16/3}$

Rev: Common Distribution :-

[1] Binomial Distribution.

[2] Geometric Distribution:

→ [3] Hyper-geometric Distribution.

$$P(X=x) = \begin{cases} \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} & , x = 0, 1, 2, \dots, \min(k, n) \\ 0 & , \text{O.W} \end{cases}$$

→ $P = \frac{k}{N}$, $\mu_x = nP$, $\sigma_x^2 = np(1-p)\left(\frac{N-n}{N-1}\right)$.

Type 1: k / Type 2: N-k

Type 1: x / Type 2: n-x

$n \leq N$

Exp:- let us consider an experiment of picking a ball from a box, The box contains 5 red balls and 3 blue balls.

[1] Assume three balls will be picked, what is the probability that only one of them is red?

$$P(X=x) = \begin{cases} \frac{\binom{5}{x}\binom{3}{3-x}}{\binom{8}{3}} & , x = 0, 1, 2, 3 = \min(3, 5) \\ 0 & , \text{O.W} \end{cases}$$

Type I: k = 5 (red) / Type II: N-k = 3 (blue) , N = 8

x / n-x , $n \leq N$

$8 = 3 + 5$

$3 = x + 3 - x$

$$P(X=1) = \frac{\binom{5}{1}\binom{3}{3-1}}{\binom{8}{3}} = \frac{\frac{5!}{4!1!} \cdot \frac{3!}{1!2!}}{\frac{8!}{5!3!}} = \boxed{\frac{15}{56}}$$

X: number of red balls in the 3 selected balls.

② Assume 5 balls to be selected, what is the probability that at least 2 balls are blue?

X: number of blue balls.

real / blue: $N-b=5$ / $b=3$, $N=8$

$5-x$ / x, $n=5$

$$P(X=x)=\begin{cases}\dfrac{\binom{3}{x}\binom{5}{5-x}}{\binom{8}{5}} & x=0,1,2,\dots,\min(5,3)\\ 0 & \text{o.w}\end{cases}$$

$$P(X\geq 2)=P(x=2)+P(x=3)+P(x=4)+\dots$$

$$=\frac{\binom{3}{2}\binom{5}{3}}{\binom{8}{5}}+\frac{\binom{3}{3}\binom{5}{2}}{\binom{8}{5}}+0+0 \longrightarrow \binom{3}{4}=0 \text{ رياضيا}$$

④ Poisson Distribution:-

$$P(X=x)=\begin{cases}\dfrac{e^{-b}b^{x}}{x!} & x=0,1,2-- \quad b>0\\ 0 & \text{o.w}\end{cases}$$

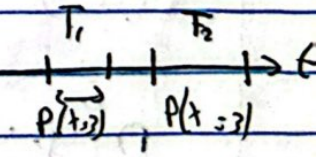
$$,\ M_x=\sigma_x^2=b$$

Example:- messages arrive to a computer server according to a poisson distribution with a rate of 10 messages/hour

@ What is the Probability that 3 messages arrive in 2 hours??

$$b=\underbrace{\lambda}_{\text{rate}} * \underbrace{t}_{\text{period}} = 10\ \frac{\text{messages}}{\text{hour}} * 2\ \text{hours} = 20\ \text{messages}$$

$$\rightarrow P(X=x)=\begin{cases}\dfrac{e^{-20}(20)^{x}}{x!} & x=0,1,2-\dots\\ 0 & \text{ow}\end{cases}$$



$$P(X=3)=\frac{e^{-20}(20)^{3}}{3!}=2.74820\times 10^{-6}$$

b) What is the Probability that at least 2 messages arrive in 30 minutes??

$\lambda t = 10 \frac{\text{messages}}{\text{hour}} \cdot 30\,\text{min} \cdot \frac{1\,\text{hour}}{60\,\text{min}} = \frac{10}{2}$

$= 5$ messages

$$P(X=x) = \begin{cases} \frac{e^{-5}(5)^x}{x!} & x = 0, 1, \dots \\ 0 & \text{o.w} \end{cases}$$

$P(X \geq 2) = P(X=2) + P(X=3) + \dots$

$= 1 - P(X < 2)$

$= 1 - P(X=0) - P(X=1) = 1 - \frac{e^{-5}(5)^0}{0!} - \frac{e^{-5}(5)^1}{1!}$

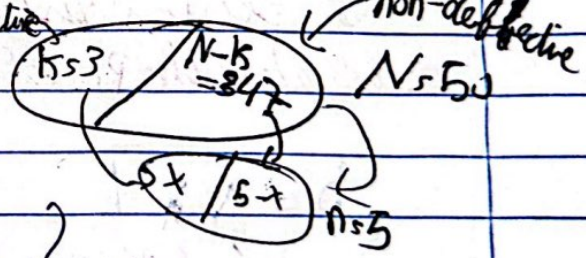
$= 0.933$

Exp (3-19) P36: حل بالعربي

50 electric motors ← there are 3 defective motors.

5 motors are selected for inspection

X: number of defective motors in the 5 selected



$$P(X=x) = \begin{cases} \frac{\binom{3}{x}\binom{47}{5-x}}{\binom{50}{5}} & x = 0, 1, 2, 3 \\ 0 & \text{o.w} \end{cases}$$

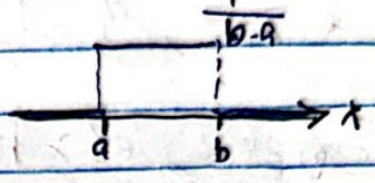
$P(X \geq 1) = 1 - P(X=0)$

$= 1 - \frac{\binom{3}{0}\binom{47}{5}}{\binom{50}{5}} = 0.28$

Common Continuous Distributions :-

1 Uniform Distribution :-

$$f_x(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & o.w \end{cases}$$



$$\mu_x = \frac{a+b}{2} \qquad \sigma_x^2 = \frac{(b-a)^2}{12}$$

Exp: let x to be a R.V that follow the uniform distribution over the interval [-2, 4].

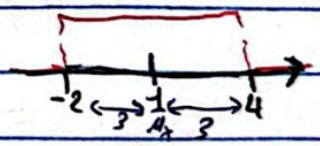
a Write the Pdf of x :-

$$f_x(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0 & o.w \end{cases} = \begin{cases} \frac{1}{6}, & -2 \le x \le 4 \\ 0 & o.w \end{cases}$$

b Determine $E\{2x^2+3x-5\}$?

$$\rightarrow E\{2x^2+3x-5\} = 2E\{x^2\} + 3E\{x\} - 5$$

$$E\{x\} = \mu_x = \frac{a+b}{2} = \frac{-2+4}{2} = \frac{2}{2} = 1$$



$$\sigma_x^2 = E\{x^2\} - \mu_x$$

$$\frac{(b-a)^2}{12} = E\{x^2\} - 1 \rightarrow E\{x^2\} = \frac{(4-(-2))^2}{12} + 1 = \frac{36}{12} + 1 = 4$$

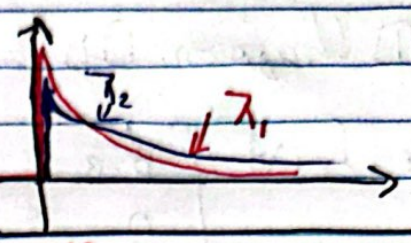
$$\Rightarrow 2E\{x^2\} + 3E\{x\} - 5 \rightarrow 2(4) + 3(1) - 5 = 6$$

c $F_x(1.5) = ?$

$$F_x(1.5) = P(x \le 1.5) = \int_{-\infty}^{-2} 0\,dx + \int_{-2}^{1.5} \frac{1}{6}\,dx = \frac{x}{6}\Big|_{-2}^{1.5} = \frac{1.5 - (-2)}{6} = \frac{3.5}{6}$$

2 Exponential Distribution :-

$$f_x(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & o.w \end{cases}$$



* $\lambda = 0$, $\mu_x = \frac{1}{\lambda}$, $\sigma^2 = \frac{1}{\lambda^2}$

Example 2) let x be a R.V that follows the exponential distribution with a mean of 0.5

a) Write the pdf of x?

$\mu_x = \frac{1}{\lambda} \rightarrow 0.5 = \frac{1}{\lambda} \rightarrow \boxed{\lambda = 2}$

$$\rightarrow f_x(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & o.w. \end{cases}$$

b) $P(X > 1)$?

$$= \int_1^\infty f_x(x)\,dx = \int_1^\infty 2e^{-2x}\,dx$$

$$= -e^{-2x}\Big|_1^\infty = -e^{-\infty} - (-e^{-2}) = e^{-2} = \frac{1}{e^2}$$

c) $P(1 \leq x / x \leq 3) = ?$

$$= \frac{P(1 \leq x \cap x \leq 3)}{P(x \leq 3)} = \frac{P(1 \leq x \leq 3)}{P(x \leq 3)}$$

$$= \frac{\int_1^3 2e^{-2x}\,dx}{\int_0^3 2e^{-2x}\,dx} = \frac{-e^{-2x}\Big|_1^3}{-e^{-2x}\Big|_0^3} = \frac{-e^{-6} + e^{-2}}{-e^{-6} + e^{0}} = \frac{-e^{-6} + e^{-2}}{-e^{-6} + 1}$$

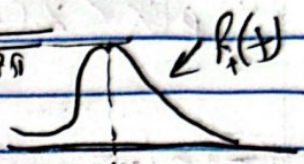
$$= \frac{\frac{1}{e^2} - \frac{1}{e^6}}{1 - \frac{1}{e^6}} = \frac{\frac{e^4}{e^6} - \frac{1}{e^6}}{\frac{e^6}{e^6} - \frac{1}{e^6}} = \frac{\frac{e^4 - 1}{e^6}}{\frac{e^6 - 1}{e^6}} = \frac{e^4 - 1}{e^6 - 1}$$

(3) Normal (Gaussian) Distribution :-

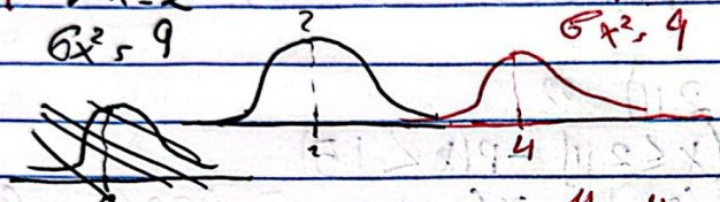
$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}}, \quad -\infty < x < \infty$$

Exp3) : let x be a Gaussian with mean of 4 and standard deviation of 3. write the pdf of x. $(\sigma_x)^2 = (3)^2 \rightarrow \sigma_x^2 = 9$

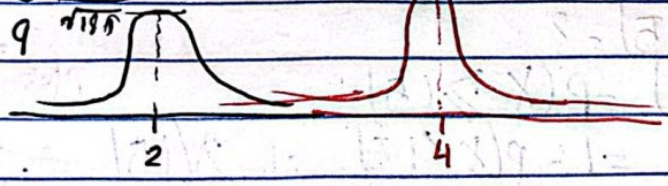
$$f_x(x) = \frac{1}{\sqrt{2\pi \cdot 9}} e^{\frac{(x-4)^2}{2 \cdot 9}} \quad -\infty < x < \infty$$



* $\mu_x = 2$, $\sigma_x^2 = 9$ — $\mu_x = 4$, $\sigma_x^2 = 4$



* $\mu_x = 2$, $\sigma_x^2 = 9$ — $\frac{1}{\sqrt{10\pi}}$, $\mu_x = 4$, $\sigma_x^2 = 5$



Note!

* For Gaussian Distribution **mean = mode = median**

* Standard Gaussian (Normal) Distribution is a gaussian (normal) distribution ($\mu_x = 0$) and unity var. ($\sigma_x^2 = 1$)

Exp4) let x be a R.V that follows the Normal distribution with $\mu_x = 0$, and $\sigma_x^2 = 1$. Determain the folbwing :-

1) $P(x < 0) = ?$ $= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = \phi(0) = 0.5$

من الجدول

* حالات لا نضع الجدول لانه يكون standard والتكامل

② $P(X \leq 3.12) = ?$

$= \int_{-\infty}^{3.12} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \Phi(3.12) = 0.9991$

↑ standard Gaussian

③ $P(X \leq 2.56) = ?$

$= \Phi(2.56) = 0.9948$

④ $P(X < 2.12)$

$= P(X \leq 2.119999\ldots 9) = \Phi(2.12) = 0.9830$

⑤ $P(X > 2.5) = 1 - P(X \leq 2.5) = 1 - \Phi(2.5) = 1 - 0.9938 = 0.0062$

⑥ $P(1.5 \leq X \leq 2.11) = ??$

$= P(X \leq 2.11) - P(X < 1.5)$

$= \Phi(2.11) - \Phi(1.5) \quad = 0.9826 - 0.9332$

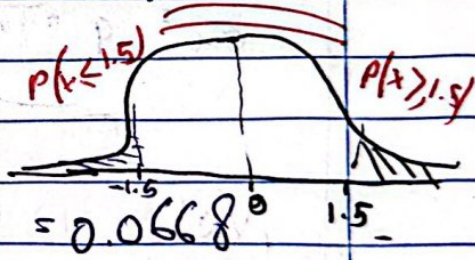
$= 0.0494$ Ans

⑦ $P(X \leq -1.5) = ?$

$P(X \leq -1.5) = P(X \geq 1.5)$

$= 1 - P(X < 1.5) \quad = 1 - \Phi(1.5)$

$= 1 - 0.9332 = 0.0668$



Note

$\Phi(-a) = 1 - \Phi(a)$

Exp 5: let X be a Gaussian R.V with mean of 3 and variance of 64, Determine the following?

① $P(X \leq 7)$?

$$f_X(x) = \frac{1}{\sqrt{2\pi * 64}} e^{\frac{-(x-3)^2}{2*64}} \qquad -\infty < x < \infty$$

not Standard.

$\mu_X = 3$, $\sigma_X^2 = 64$

$$P(X \leq 7) = \int_{-\infty}^{7} \frac{1}{\sqrt{2\pi * 64}} e^{\frac{-(x-3)^2}{2*64}} dx$$

$$z = \frac{x-3}{8}$$

$$dz = \frac{dx}{8} \rightarrow dx = 8\,dz$$

$$x = 7 \rightarrow z = \frac{7-3}{8}$$

$$x = -\infty \rightarrow z = -\infty$$

$$\Rightarrow \Phi = \int_{-\infty}^{\frac{7-3}{8}} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz = \Phi\left(\frac{7-3}{8}\right)$$

$\mu_z = 0$, $\sigma_z^2 = 1$

$$= \Phi\left(\frac{4}{8}\right) = \Phi(0.5) = 0.6915$$

* Z = Standard Gaussian

② $P(X \leq 5) = ?$

$\mu_X = ?$

$\sigma_X^2 = 64$

$$= \Phi\left(\frac{5-3}{\sqrt{64}}\right) = \Phi\left(\frac{2}{8}\right) = \Phi(0.25) = 0.5987$$

③ $P(X>11) = ?$

$= 1 - P(X \leq 11)$

$= 1 - \phi\left(\frac{11-3}{8}\right) = 1 - \phi\left(\frac{8}{8}\right) = 1 - \phi(1)$

$= 1 - 0.8413$

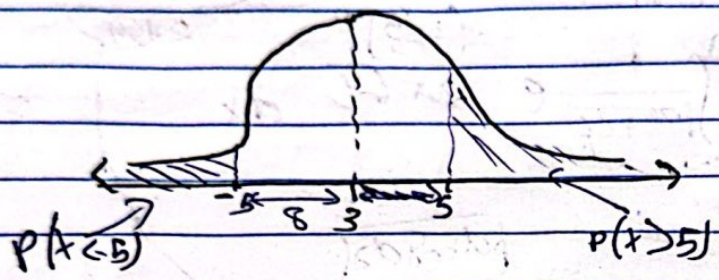
$= 0.1587$

④ $P(X<-5) = ??$

$P(X<-5) = \phi\left(\frac{-5-3}{8}\right) = \phi(-1) = 1 - \phi(1)$

$= 1 - 0.8413$

$= 0.1587$



⑤ $P(X \leq -5 / X < 11) = ? \dfrac{P(X<-5 \cap X<11)}{P(X<11)}$

$= \dfrac{P(X<-5)}{P(X<11)} = \dfrac{\phi\left(\frac{-5-3}{8}\right)}{\phi\left(\frac{11-3}{8}\right)} = \dfrac{\phi(-1)}{\phi(1)} = \dfrac{1-0.8413}{1-0.8413} = 1$

~~Diff~~ Continuous Discrete

PDF CDF PDF

$f_X(t) = \{$ $F_X(x) = \begin{cases} 0 & x<0 \\ 0.4 & 0 \leq x < 2 \\ 0.6 & 2 \leq x < 4 \\ 1 & 4 < x \end{cases}$ $P(X=x) = \begin{cases} \frac{3}{4} & x=1 \\ \frac{1}{4} & x=2 \\ \{ & x=5 \\ 0 & o.w \end{cases}$

$f_X(t) = \dfrac{d F_X(t)}{dt}$, $F_X(t) = \int_{-\infty}^{t} f_X(t)\,dt$

$F_X(t=5) = P(X \leq 5) = \int_{-\infty}^{5} f_X(t)\,dt$

* Approximation :-

1. Hyper-geometric ⟶ Binomial

$$P(X=x) = \begin{cases} \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} & , x = 0,1,2,\dots,\min(n,k) \\ 0 & \text{o.w} \end{cases} \xrightarrow{p=\frac{k}{N}} P(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & , x = 0,1,\dots \\ 0 & \text{o.w} \end{cases}$$

$\frac{n}{N} < 0.1$

2. Binomial ⟶ Poisson

$$P(X=x) = \lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x} \longrightarrow P(X=x) = \frac{e^{-b} b^x}{x!}$$

$b = np(S)$

Ex (3-23) P.140 :

1000 bits — لاحظ

$p(\text{error}) = \frac{1}{365}$ — $\text{error} \to X = 10$

$$P(X=10) = \binom{1000}{10} \left(\frac{1}{365}\right)^{10} \left(\frac{364}{365}\right)^{1000-10}$$

→ approximation using Poisson

$\Rightarrow b = np = \frac{1000}{365}$

$\mu_{\text{Poisson}} = \mu_{\text{Binomial}}$

$$P(X=10) \simeq e^{-\frac{1000}{365}} \frac{\left(\frac{1000}{365}\right)^{10}}{10!} = \boxed{\quad}$$

$= 6.566 \times 10^{-3}$.

⟶

[3] Binomial ⟶ Gaussian

$$F_x(x) = P(x \le x) = \sum_{x=0}^{x} \binom{n}{x} p^x (1-p)^{n-x} \longrightarrow P(X \le t) = \Phi\left(\frac{t - \mu_x}{\sqrt{\sigma_x^2}}\right)$$

$\mu_x = np(s)$

$\sigma_x^2 = np(s)(1-p(s))$

Ex P(3-28) P49 : Lecture

Consider a Binomial experiment with $n = 1000$ and $P(s) = 0.2$

[a] Determine the $P(x \le 240)$?

$$P(X = x) = \begin{cases} \binom{1000}{x} (0.2)^x (0.8)^{1000-x} & x = 0, 1, \ldots, 1000 \\ 0 & \text{O.W} \end{cases}$$

$$P(x \le 240) = P(x=0) + P(x=1) + \cdots + P(x=240) = \sum_{x=0}^{240} \binom{1000}{x} (0.2)^x (0.8)^{1000-x}$$

[b] Use the Gaussian approximation to determine $P(x \le 240)$?

$\mu_x = np(s) = 1000 \cdot 0.2 = 200$, $\sigma_x^2 = np(1-p) = 1000 \cdot 0.2 \cdot 0.8 = 160$

$$P(X \le 240) = \Phi\left(\frac{240 - \mu_x}{\sqrt{\sigma_x^2}}\right) = \Phi\left(\frac{240 - 200}{\sqrt{160}}\right) = \Phi(3.16) = 0.999216$$

[4] Poisson ⟶ Gaussian

$\mu_x = \sigma_x^2 = b$

$$P(X \le x) = \sum_{x=0}^{x} \frac{e^{-b} b^x}{x!} \longrightarrow P(X \le t) = \Phi\left(\frac{x - b}{\sqrt{b}}\right)$$

Exp (3-29) P 50 :-

Poisson , $\mu_x = 1000\ cm^3$

(a) $P(x=950)$?

$T = 1\ cm^3$, $b = \lambda T$, $1000 = \lambda(1) \rightarrow \lambda = 1000$

$$P(X=950) = \frac{e^{-1000}(1000)^{950}}{950!}$$

(b) $p(x=950$ in $3cm^3)$:-

$\mu_x = b = \lambda T = 1000 = \lambda \cdot 1 \rightarrow \lambda = 1000$ (T = 1 cm^3)

$\hookrightarrow b = \lambda T = 1000 \times 3 = 3000 = \mu_x = b$ (T = 3 cm^3)

$$p(x=950 \text{ in } 3cm^3) = \frac{e^{-3000}(3000)^{950}}{950!}$$

(c) Use the Gaussian approximation to estimate the number of probability of having a most particle in $1cm^3$.

$$p(X \le 950) = \Phi\left(\frac{950-1000}{\sqrt{1000}}\right) = \quad , \mu_x = \sigma_x^2 = b = 1000$$

$= 1000$

Ex:- Assume the number of viruses particles in a $1cm^3$ of water follow a poisson distribution with a mean of 5000 (μ_x) viruses in 10 cm (T) of water if a $3cm^3$ of water are analyzed;

(a) what is the probability that it contains 950 viruses?

$b = \lambda T$ (T = $3cm^3$) , From question $\mu_x = b = \lambda T$

$5000 = \lambda 10 \rightarrow \lambda = 500$ viruses

$= 5000$ viruses / cm^3 , $3cm^3 = 1500$ viruses

$$p(X=950) = \frac{e^{-1500}(1500)^{950}}{950!}$$

(b) What is the probability that $2cm^3$ of water contains at least 3500 viruses, Use the gaussian to approximate the answer?

$b = \lambda \cdot T = \frac{5000 vir}{1 m^3} \cdot 2cm^3 = 1000 vir$

$\mu_x = \sigma_x^2 = b = 1000$

$P(x \geq 3500) = 1 - P(x < 3500)$

$= 1 - \Phi\left(\frac{3500 - 1000}{\sqrt{1000}}\right) = \Phi($

---

* Transformation of R.V:-

→ Discrete :-

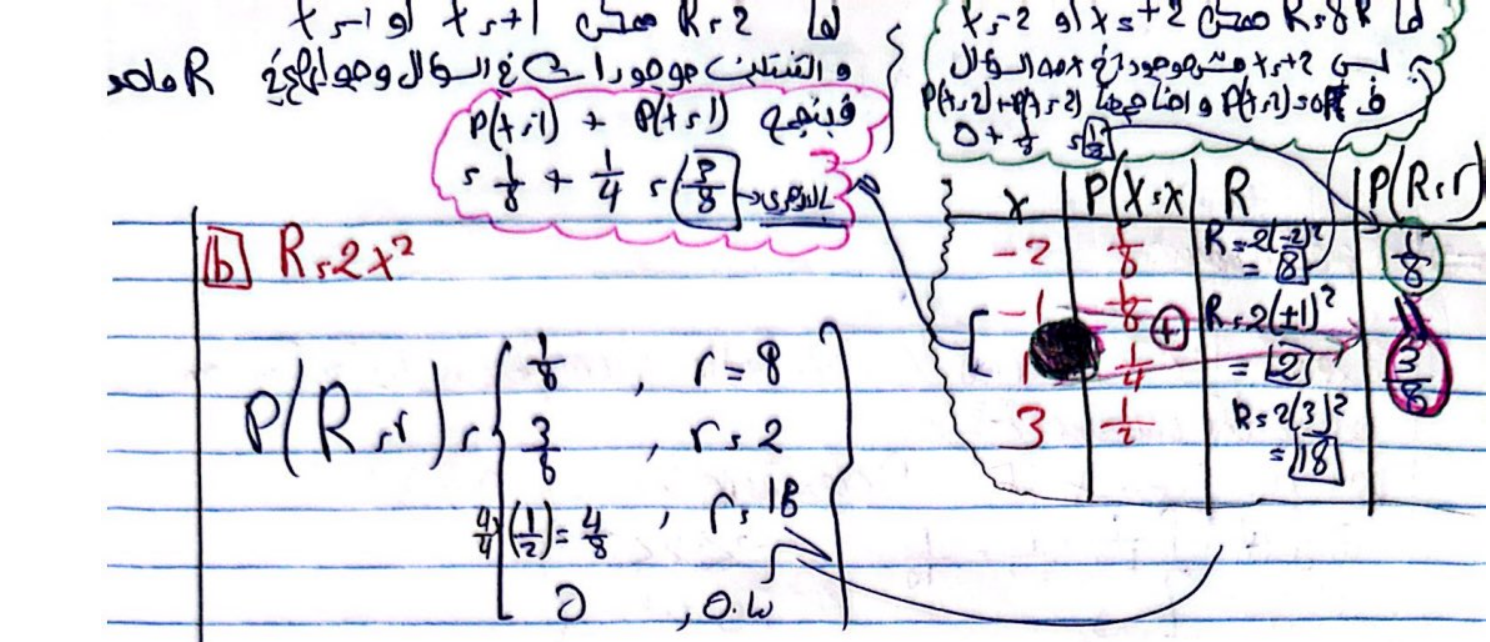
Ex1) let x be a R.V with the following P.M.F:-

$$P(X=x) = \begin{cases} \frac{1}{8} & x = -2 \\ \frac{1}{8} & x = -1 \\ \frac{1}{4} & x = 1 \\ \frac{1}{2} & x = 3 \\ 0 & O.W \end{cases}$$

(a) let $y = 2x + 5$, Determine the PMF of y.

$$P(Y=y) = \begin{cases} 1/8, & y = 1 \\ 1/8, & y = 3 \\ 1/4, & y = 7 \\ 1/2, & y = 11 \\ 0, & O.W \end{cases}$$

x	$P(X=x)$	$y = 2x + 5$	$P(Y=y)$
-2	$\frac{1}{8}$	$y = 2(-2)+5 = 1$	$\frac{1}{8}$
-1	$\frac{1}{8}$	$y = 2(-1)+5 = 3$	$\frac{1}{8}$
1	$\frac{1}{4}$	$y = 2(1)+5 = 7$	$\frac{1}{4}$
3	$\frac{1}{2}$	$y = 2(3)+5 = 11$	$\frac{1}{2}$



x	P(X=x)	R	P(R=r)
-2	1/8	$R=2(-2)^2=8$	1/8
-1	1/8	$R=2(-1)^2=2$	3/8
1	1/4		
3	1/2	$R=2(3)^2=18$	1/2

$$P(R=r)=\begin{cases}\frac{1}{8}, & r=8\\ \frac{3}{8}, & r=2\\ \frac{1}{2}, & r=18\\ 0, & \text{O.W}\end{cases}$$

Ex2> let x be a R.V that follows Binomial Distribution with $n=2$, $P(S)=\frac{1}{4}$. let a new R.V $y=2x+1$ Determine the PMF of y.

$$P(X=x)=\begin{cases}\binom{n}{x}P(S)^x(1-P(S))^{n-x}, & x=0,1,2\\ 0, & \text{O.W}\end{cases}$$

$$P(x=0)=\binom{2}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^2=\frac{2!}{(2-0)!0!}\cdot\frac{9}{16}=\frac{9}{16}$$

$$P(x=1)=\binom{2}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^1=\frac{2!}{(2-1)!1!}\cdot\frac{1}{4}\cdot\frac{3}{4}=\frac{6}{16}$$

$$P(x=2)=\binom{2}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^0=\frac{2!}{(2-2)!2!}\left(\frac{1}{4}\right)^2(1)=\frac{1}{16}$$

x	P(x=x)	y=2x+1	P(y)
0	9/16	y=2(0)+1=1	9/16
1	6/16	y=2(1)+1=3	6/16
2	1/16	y=2(2)+1=5	1/16

$$P(y)=\begin{cases}\frac{9}{16}, & x=1\\ \frac{6}{16}, & x=3\\ \frac{1}{16}, & x=5\\ 0 & \text{O.W}\end{cases}$$

Continuous:-

Exp: let x be a R.V that follows uniform distribution over the interval $[-3, 7]$

[a] let $y = 2x - 5$. Determine the PDF of Y.

① $f_x(x) = \begin{cases} \frac{1}{b-a} = \frac{1}{7-(-3)} = \frac{1}{10} & , -3 \le x \le 7 \\ 0 & , o.w \end{cases}$

② $y = 2x - 5$

$y + 5 = 2x$

$x = \frac{y+5}{2}$

③ $\frac{dy}{dx} = 2$

④ $f_y(y) = \left| \frac{f_x(x)}{\frac{dy}{dx}} \right|_{x = g(y)}$

4a $x < -3 \rightarrow y < -11$

$2x < -6$ (×2)

$2x - 5 < -11$ (−5)

$\Rightarrow f_y(y) = \frac{0}{2} = 0$

4b $-3 < x < 7 \rightarrow -11 \le y \le 9$

$-6 \le 2x \le 14$ (×2)

$-11 \le 2x - 5 \le 9$ (−5)

4c $7 < x \rightarrow 9 < y$

$f_y(y) = 0$

$f_y(y) = \frac{\frac{1}{10}}{2} = \frac{1}{20}$

$f_y(y) = \begin{cases} 1/20 & , -11 \le x \le 9 \\ 0 & , o.w \end{cases}$

Uniform(a, b) $\rightarrow \frac{1}{b-a} = \frac{1}{9-(-11)} = \frac{1}{20}$

(b) let $Y = X^2$. Determine the pdf of Y?

① $f_X(x) = \begin{cases} \frac{1}{10}, \frac{1}{2\cdot 3}, \frac{1}{10}, -3 < x < 7 \\ 0 \quad \text{o.w} \end{cases}$ $\{-3, 3\}$

② $Y = x^2 \rightarrow x = \pm\sqrt{y}$

③ $\frac{dy}{dx} = 2x$

④ $f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\Big|_{x=\sqrt{y}} + \frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\Big|_{x=-\sqrt{y}}$

(4a) $x < -7$ and $x > 7 \rightarrow y > 49$

$f_Y(y) = 0 + 0 = 0$

(4b) $-7 < x < -3$ and $3 < x < 7 \rightarrow 9 < y < 49$

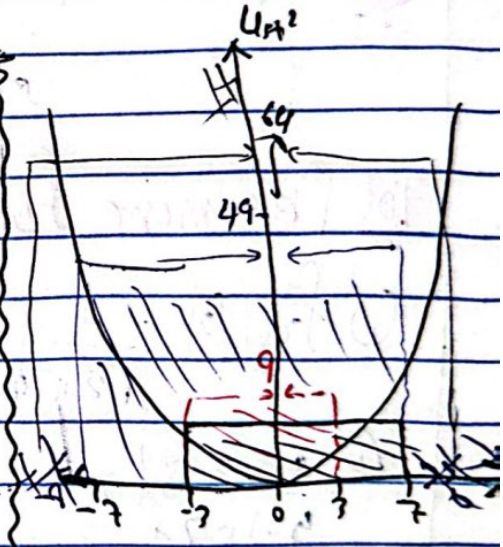
$f_Y(y) = \frac{\frac{1}{10}}{|2x|}\Big|_{x=\sqrt{y}} + 0 = \frac{1}{20\sqrt{y}}$

$y = x^2 \rightarrow x = +3, \sqrt{y}$, $x = -3, -\sqrt{y}$

(4c) $-3 < x < 3$, $0 < y < 9$

$f_Y(y) = \frac{\frac{1}{10}}{|2x|}\Big|_{x=\sqrt{y}} + \frac{\frac{1}{10}}{|2x|}\Big|_{x=-\sqrt{y}}$

$= \frac{1}{20\sqrt{y}} + \frac{1}{20\sqrt{y}} = \frac{2}{20\sqrt{y}}$



$f_Y(y) = \begin{cases} \frac{2}{20\sqrt{y}} & 0 < y < 9 \\ \frac{1}{20\sqrt{y}} & 9 < y < 49 \\ 0 & \text{o.w} \end{cases}$

Exp 4) let x bea Gausian R.V with $\mu_x = 2, \sigma_x^2 = 9$
let $y = 2x + 1$ Determine

[a] μ_y, σ_y^2 ?

$\mu_y = E\{y\} = E\{2x+1\} = 2\mu_x + 1 = 2(2)+1 = 5$

$\sigma_y^2 = 4\sigma_x^2 = 4 * 9 = 36$

Note: $y = ax + b$

$E\{y\} = a E\{x\} + b, \; a\mu_x + b$

$Var\{y\} = E\{(y-\mu_y)^2\} = E\{(ax+b-a\mu_x-b)^2\} = E\{(ax-a\mu_x)^2\}$

$= a^2 E\{(x-\mu_x)^2\} = a^2\sigma_x^2$

[b] Determine $f_y(y)$?

① $f_x(x) = \frac{1}{\sqrt{2\pi \cdot 9}} e^{\frac{-(x-2)^2}{2*9}} \quad -\infty < x < \infty$

② $y = 2x+1$

$y-1 = 2x$

$x = \frac{y-1}{2}$

③ $\frac{dy}{dx} = 2$

④ $f_y(y) = f(y) = \left| \frac{f_x(x)}{\frac{dy}{dx}} \right|_{x = \frac{y-1}{2}}$

$= \frac{1}{\sqrt{2\pi * 9}} e^{\frac{-\left(\frac{y-1}{2} - 2\right)^2}{2*9}} \cdot \frac{1}{2}$

$= \frac{1}{\sqrt{2\pi * 9 * 4}} e^{\frac{-(y-5)^2}{2*9*4}} \quad -\infty < y < \infty$

$= \frac{1}{\sqrt{2\pi * 36}} e^{\frac{-(y-5)^2}{2*36}} \quad -\infty < y < \infty$

Chapter 3:- Probability Distribution For more than One Random variable.

$P(X=x, Y=y, R=r, \dots)$

$p(X=x, Y=y)$

$\Rightarrow$ joint-PMF

Exp: let x and y be two R.V. with the following Joint-PMF:

$$P(X=x) = \begin{cases} \frac{1}{8} & , x=-1,\ y=0 \\ \frac{1}{8} & , x=-1,\ y=1 \\ \frac{1}{4} & , x=0,\ y=0 \\ \frac{1}{8} & , x=1,\ y=0 \\ k\ \frac{3}{8} & , x=1,\ y=2 \\ 0 & , \text{o.w} \end{cases}$$

[a] Determine the value of constant k?

$$\sum_{y=-\infty}^{\infty} \sum_{x=-\infty}^{\infty} P(X=x, Y=y) = 1$$

$$P(x=-1, y=0) + P(x=-1, y=1) + \dots = 1$$

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} + k = 1 \rightarrow \frac{5}{8} + k = 1 \rightarrow k = 1 - \frac{5}{8}$$

$$k = \frac{3}{8}$$

[b] $P(X \le 0, Y \le 0) = ?$

$= P(X=-1, Y=0) + P(X=0, Y=0)$

$= \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

[C] $P(X \le 0, Y \ge 1) = ?$

$P((X=-1), (Y=1)) = \frac{1}{8}$

x \ y	0	1	2
-1	$\frac{1}{8}$	$\frac{1}{8}$	0
0	$\frac{1}{4}$	0	0
1	$\frac{1}{8}$	0	$\frac{3}{8}$

$x \le 0$; $y \le 0$

x \ y	0	1	2
-1	$\frac{1}{8}$	$\frac{1}{8}$	0
0	$\frac{1}{4}$	0	0
1	$\frac{1}{8}$	0	$\frac{3}{8}$

$x \le 0$; $y \ge 1$

* properties of Joint-PMF

1. $P(X=x, Y=y) \geq 0$
2. $\sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} P(X=x, Y=y) = 1$

x \ y	0	1	2
-1	$\frac{1}{8}$	$\frac{1}{8}$	0
0	$\frac{1}{4}$	0	0
1	$\frac{1}{8}$	0	$\frac{1}{2}$

$\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{2}$

- a
- e
- f
- g

d) $P(X \leq 0 / Y \geq 1) = ?$

$$P(X \leq 0 / Y \geq 1) = \frac{P(X \leq 0, Y \geq 1)}{P(Y \geq 1)} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4}$$

e) $P(X > 0) = ?$

$$\frac{1}{8} + \frac{3}{8} = \frac{4}{8}$$

f) $F_{XY}(0,1) = ?$ (joint CDF of X, Y)

$$= P(X \leq 0, Y \leq 1) = \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \frac{4}{8} = \frac{1}{2}$$

g) $F_{YX}(0,1)$

$$= P(Y \leq 0, X \leq 1) = \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$$

H) $F_{X,Y}(3,-2) = ?$

$$P(X \leq 3, Y \leq -2) = 0$$

(X ≤ 3: all values of X in the table; Y ≤ -2: no values of Y in the table)

I) $F_{X,Y}(3,3) = ?$

$$P(X \leq 3, Y \leq 3) = 1$$

(all values of X; all values of Y; all of the table)

J) $F_{X,Y}(-3,-2)$

$$P(X \leq -3, Y \leq -2) = 0$$

[k] $F_{X,Y}(-\infty,-\infty)$

$= P(X \le -\infty, Y \le -\infty) = \boxed{0}$

[L] $F_{X,Y}(\infty,\infty)$

$= P(X < \infty, Y \le \infty) = \boxed{1}$

[M] $F_{X,Y}(\infty,-\infty) = ?$

$F_{X,Y}(+\infty,-\infty) = P(X < \infty, Y \le -\infty) = 0$

[n] Determine the marginal PMF of X?

$\rightarrow P(X=-1) = P(X=-1, -\infty < Y < -\infty)$

$= \sum_{y=-\infty}^{\infty} P(X=-1, Y=y)$

$= P(X=-1, Y=0) + P(X=-1, Y=1)$

$= \frac{1}{8} + \frac{1}{8} = \boxed{\frac{1}{4}}$

$\rightarrow P(X=0) = \sum_{y=-\infty}^{\infty} P(X=0, Y=y)$

$= \boxed{\frac{1}{4}}$

$\rightarrow P(X=1) = \sum_{y=-\infty}^{\infty} P(X=1, Y=y)$

$= \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \boxed{\frac{1}{2}}$

So $P(X=x) = \begin{cases} \frac{1}{4} & x=-1 \\ \frac{1}{4} & x=0 \\ \frac{1}{2} & x=1 \\ 0 & 0.W \end{cases}$

[o] PMF of Y?

$P(Y=3) = \sum_{x=-\infty}^{\infty} P(Y=3, X=x) = 0$

~~$P(Y=2)$~~

$P(Y=2) = \sum_{x=-\infty}^{\infty} P(Y=2, X=x) = \frac{3}{8}$

$P(Y=1) = \sum_{x=-\infty}^{\infty} P(Y=1, X=x) = \frac{1}{8}$

$P(Y=0) = \sum_{x=-\infty}^{\infty} (P(Y=0, X=x) = \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{4}{8}$

So $P(Y=y) = \begin{cases} 4/8, & y=0 \\ 1/8, & y=1 \\ 3/8, & y=2 \\ 0, & y>2 \\ 0 & o.w \end{cases}$

* Note: X and Y are said to be statis. Independent if: $P(X=x, Y=y) = P(X=x)P(Y=y)$.

[p] Are x and y S. Independent??

- $P(X=-1, Y=0) \stackrel{?}{=} P(x=-1)P(Y=0)$

$\frac{1}{8} \stackrel{?}{=} \frac{1}{4} * \frac{4}{8} = \frac{1}{8}$

- $P(X=-1, Y=1) \stackrel{?}{=} P(x=-1)\cdot P(Y=1)$

$\frac{1}{8} \stackrel{?}{=} \frac{1}{4} \cdot \frac{1}{8}$

$\frac{1}{8} \neq \frac{1}{32}$

So x, y are not Statistically Independent.

Q $P(X=-1 / Y=+1) = ??$

$$= \frac{P(X=-1, Y=1)}{P(Y=1)} = \frac{1/8}{1/8} = 1$$

R $P(X \geq 0 / Y \leq 0, X \leq 0) = ??$ (شرط ١ — شرط ٢)

→ $\frac{P(X \geq 0, X \leq 0, Y \leq 0)}{P(Y \leq 0, X \leq 0)} = \frac{P(X=0, Y \leq 0)}{P(Y \leq 0, X \leq 0)} = \frac{1/4}{1/8 + 1/4} = \frac{1/4}{3/8}$

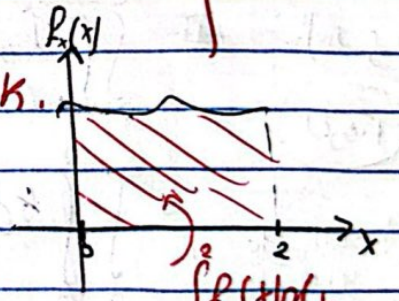
$= \frac{2/3}{3/8}$ ∴ $\boxed{2/3}$

Ex:- let x and Y be two R.Vs with the following Joint-Pd

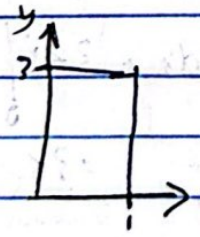
$$f_{x,y}(x,y) = \begin{cases} K & 0 \leq x \leq 1, 0 < y \leq 3 \\ 0 & o.w \end{cases}$$

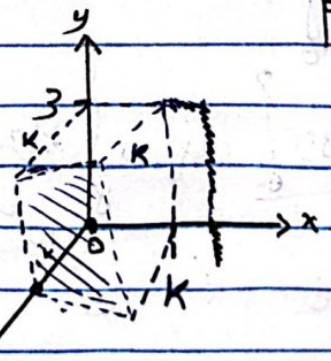
a Determine the value of the constant K.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{x,y}(x,y)\,dy\,dx = 1$$

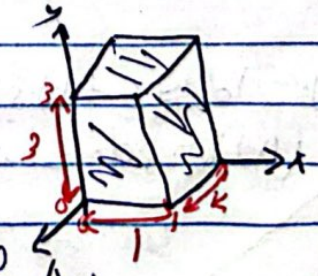


$\int f_x(t)dt$ هو المساحة تحت الدالة لـ R.V. واحد





$f_{x,y}(x,y)$



$f_{x,y}(x,y)$

$V_{\text{مستطيل}} = 1 \times 3 \times K$

$= 3K$

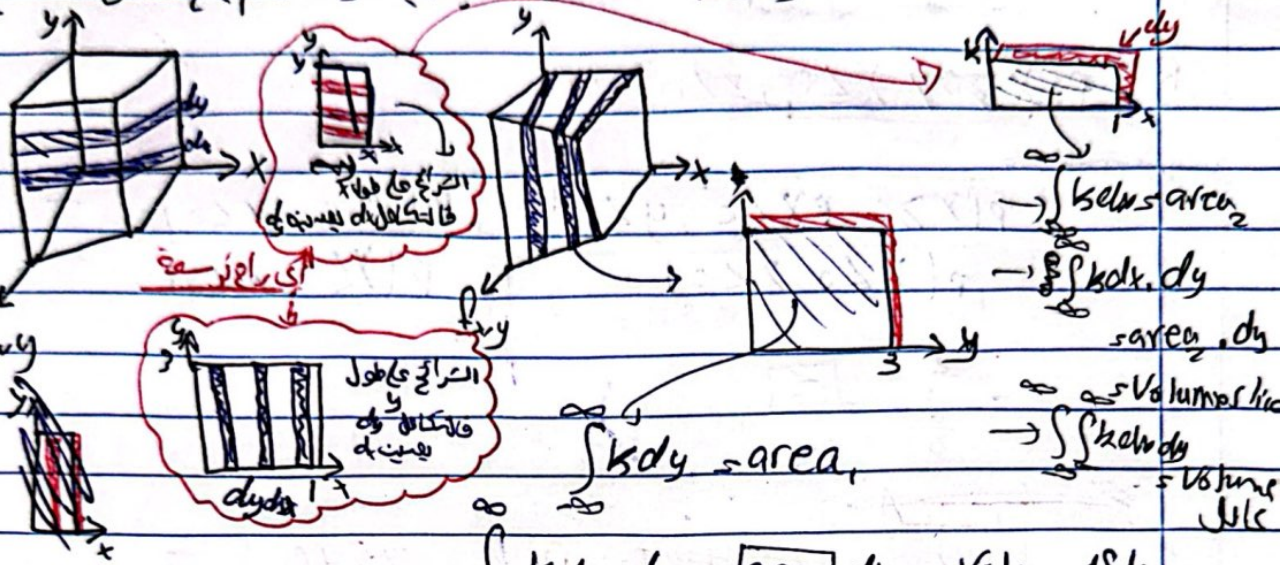
$1 = 3K \rightarrow \boxed{K = 1/3}$

$\therefore V = 1$

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_{x,y}(x,y)\,dy\,dx\right] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_{x,y}(x,y)\,dx\,dy\right]$$

التكامل الاول — التكامل الاول

لو كاملت dy بعدين dx او dx بعدين dy لازم يكاملو نفس الشكل



$\int k\,dy = area_1$

$\int k\,dx = area_2$

$\int\int k\,dx\,dy = area_2 \cdot dy$

$Volume\,slice = \int\int k\,dx\,dy$

$\int k\,dy\,dx = area\,dx = Volume\,Slice$

$\int\int k\,dy\,dx = Volume$ كامل

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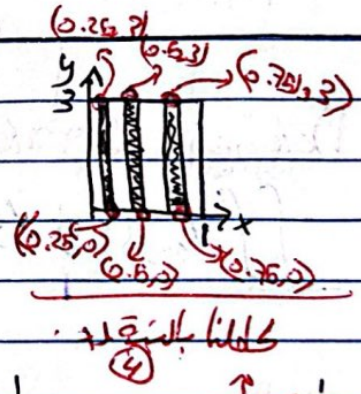
1 حل بدون ال

$$\int_{0}^{1}\int_{0}^{3} f_{x,y}(x,y)\,dy\,dx$$

الطريقة 1



كاملنا بالنسبة لـ y ④

$$= \int_{0}^{1}\int_{0}^{3} k\,dy\,dx$$

بشتغل بالتكامل بالنسبة لـ y بعدين بـ x

$$\int_{0}^{1}\left[ky\right]_{0}^{3}dx \rightarrow \int_{0}^{1} k[3-0]\,dx = \int_{0}^{1} 3k\,dx = \left[3kx\right]_{0}^{1} = 3k[1-0]$$

① تكامل بالنسبة لـ y

② تعويض الحدود لـ y

③ عوضنا الحدود والحيت لازم نكامل بالنسبة لـ x

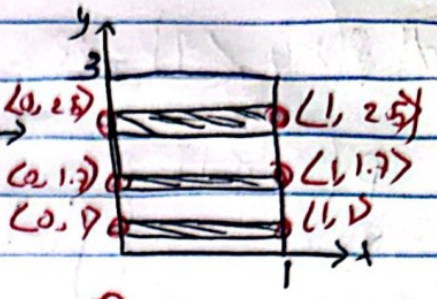
⑤ بعد تعويض منورد نكامل

$3k = 1 \rightarrow k = \frac{1}{3}$

* لما بعمل slice عمودي بيكون طول الـ slice هو dy (تكامل y) بعدين بـ dx (تكامل x)

Ex 2 (pdf)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{x,y}(x,y)\,dx\,dy$$



$$\int_0^3\int_0^1 k\,dx\,dy$$

بياخد التكامل بالنسبة لـ x الأول

$$\int_0^3 \Big[kx\Big]_0^1 dy \rightarrow \int_0^3 k[1-0]\,dy = \int_0^3 1k\,dy = \Big[ky\Big]_0^3 = k[3-0]$$

① تكامل بالنسبة لـ x

② عوضنا حدود x

③ عوضنا الحدود اللي جوا التكامل بالنسبة لـ x

④ كاملنا بالنسبة لـ y

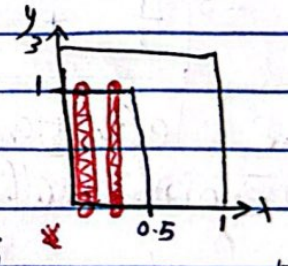
⑤ عوضنا حدود y

$= 3k = 1$

$k = \frac{1}{3}$

* Slicer ال (التكامل الأول) يعني أول تكامل بالنسبة لـ x (dx) يعني بعدين dy (التكامل الثاني)

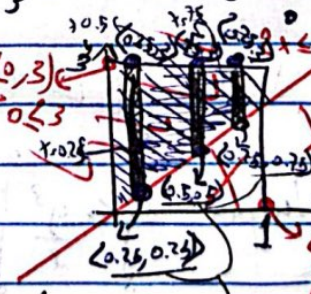
b $P(0 \le x \le 0.5,\ 0 \le y \le 1) = ?$



$$= \int_0^{0.5}\int_0^1 \frac{1}{3}\,dy\,dx = \int_0^{0.5} \frac{1}{3}y\Big|_0^1\,dx$$

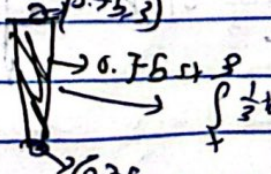
$$= \int_0^{0.5} \frac{1}{3}y\,dx = \frac{1}{3}x\Big|_0^{0.5} = \frac{1}{3}\cdot 0.5 = \frac{1}{6}$$

c $P(x \le y) = ?$



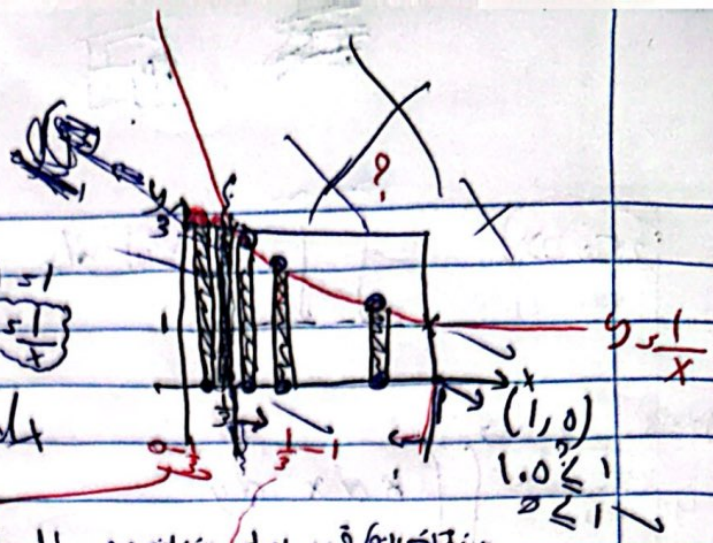
$x \le y$ ؟

يرسم خط $y=x$ وبعدين لازم أعرف أي جزء من المربع يحقق الشرط عن طريق نقاط ويشوف مين يحقق الشرط



$$= \int_0^1\int_x^1 \frac{1}{3}\,dy\,dx = \int_0^1 \frac{1}{3}y\Big|_x^1\,dx = \int_0^1 \frac{1}{3}(1-x)\,dx$$

$$= \int_0^1 \frac{1}{3}\left(x - \frac{x^2}{2}\right)\Big|_0^1 = \frac{1}{3}\left[\left(1-\frac{1}{2}\right)-(0-0)\right] = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$



[d] $P(xy \le 1) = ??$ $\quad xy = 1 \Rightarrow y = \frac{1}{x}$

$$P(xy \le 1) = \iint \frac{1}{3}\, dy\, dx$$

$$= \int_0^{\frac{1}{3}} \int_0^{3} \frac{1}{3}\, dy\, dx + \int_{\frac{1}{3}}^{1} \int_0^{\frac{1}{x}} \frac{1}{3}\, dy\, dx$$

$$\int_0^{\frac{1}{3}} \frac{1}{3} y \Big|_0^3 \, dx \qquad \int_{\frac{1}{3}}^{1} \frac{1}{3} y \Big|_0^{\frac{1}{x}} \, dx$$

$$\int_0^{\frac{1}{3}} 1\, dx \qquad \int_{\frac{1}{3}}^{1} \frac{1}{3x}\, dx = \frac{1}{3} \ln x \Big|_{\frac{1}{3}}^{1}$$

$$= \frac{1}{3} \left[\ln x \Big|_{\frac{1}{3}}^{1} \right]$$

$$= \frac{1}{3} \ln 3$$

$$= \frac{1}{3}$$

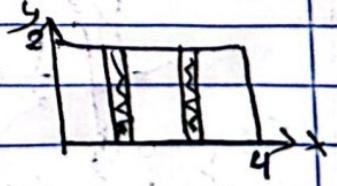
$$\rightarrow \boxed{\frac{1}{3} + \frac{1}{3} \ln 3}$$

Exp: Let x and y be two R.Vs with the following Joint pdf:-

$$f_{xy}(x,y) = \begin{cases} kxy^2 & 0 < x < 4,\ 0 < y < 2 \\ 0 & \text{O.W} \end{cases}$$

[a] Determine the value of the constant k?

$$\int_0^4 \left[\int_0^2 kxy^2\, dy \right] dx = \int_0^4 kx \frac{y^3}{3} \Big|_0^2 \, dx$$

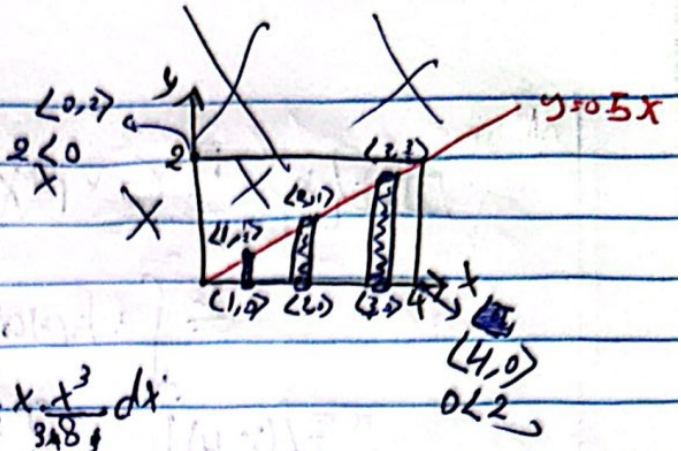


$$= \int_0^4 kx \left[\frac{8}{3} - 0 \right] dx = \int_0^4 \frac{8}{3} kx\, dx = \left[\frac{8}{3} \frac{kx^2}{2} \right]_0^4$$

$$= \frac{8}{3} k \left[\frac{16}{2} - 0 \right] \rightarrow \frac{8}{3} k \cdot 8 = 1 \rightarrow \frac{64}{3} k = 1$$

$$\boxed{k = 3/64}$$

b $P(Y < 0.5X) = ??$

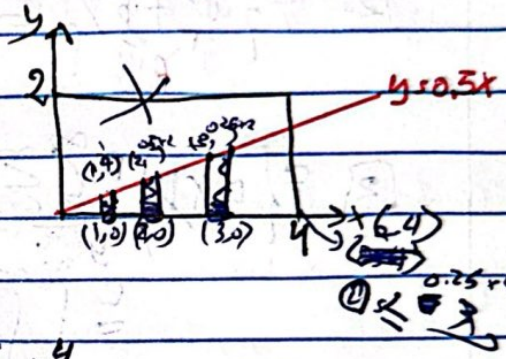


$$\int_0^4 \int_0^{0.5x} \frac{3}{64} x y^2 \, dy \, dx$$

$$= \int_0^4 \frac{3}{64} x \frac{y^3}{3} \Big|_0^{0.5x} dx = \int_0^4 \frac{3}{64} x \frac{x^3}{3 \cdot 8} dx$$

$$= \int_0^4 \frac{3}{64} \frac{x^4}{8 \cdot 3} dx = \int_0^4 \frac{x^4}{64 \cdot 8} dx = \frac{x^5}{64 \cdot 8 \cdot 5} \Big|_0^4 = \frac{4^5}{64 \cdot 8 \cdot 5} = \boxed{\frac{2}{5}}$$

c $P(Y \leq 0.25X) = ??$

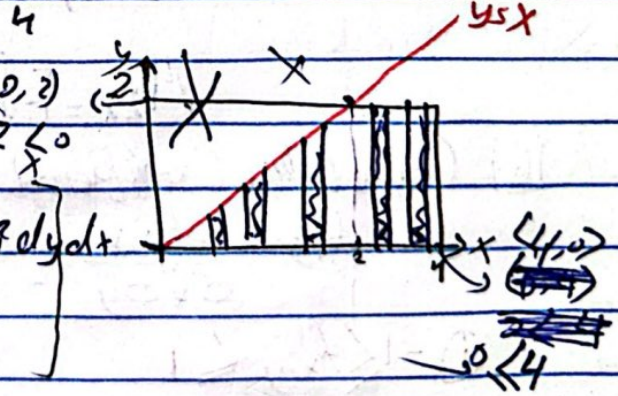


$$\int_0^4 \int_0^{0.25x} \frac{3}{64} x y^2 \, dy \, dx$$

$$\int_0^4 \frac{3}{64} x \frac{y^3}{3} \Big|_0^{0.25x} dx \to \int_0^4 \frac{x \left(\frac{x}{4}\right)^3}{64} dx = \int_0^4 \frac{x^4}{64 \cdot 64} dx$$

$$= \frac{x^5}{64 \cdot 64 \cdot 5} \Big|_0^4 = \frac{4^5}{64 \cdot 64 \cdot 5} = \frac{1}{20}$$

d $P(Y < X) = ?$



$$\int_0^2 \int_0^x \frac{3}{64} x y^2 \, dy \, dx + \int_2^4 \int_0^2 \frac{3}{64} x y^2 \, dy \, dx$$

$$= \frac{1}{20} + \frac{3}{4} \cdot \frac{2}{5} = \frac{1}{20} + \frac{15}{20} = \boxed{\frac{17}{20}}$$

Notes:-

1. $E\{x\} \overset{D}{=} \sum_{x=-\infty}^{\infty} x\, P(X=x) \overset{C}{=} \int_{-\infty}^{\infty} x\, f_x(x)\,dx$

   $E\{g(x)\} \overset{D}{=} \sum_{-\infty}^{\infty} g(x)\, P(X=x) \overset{C}{=} \int_{-\infty}^{\infty} g(x)\, f_x(x)\,dx$

2. $E\{(x,y)\} \overset{D}{=} \sum_{y=-\infty}^{\infty}\sum_{x=-\infty}^{\infty} g(x,y)\, P(X=x, Y=y)$

   $\overset{C}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{xy}(x,y)\,dy\,dx$

3. $E\{aX+bY\} = aE\{X\} + bE\{Y\}$ (X, Y: RVs)

4. $E\{aXY\} \overset{D}{=} \sum_{y=-\infty}^{\infty}\sum_{x=-\infty}^{\infty} a\,x\,y\, P(X=x, Y=y)$

   $\overset{C}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a\,x\,y\, f_{xy}(x,y)\,dy\,dx$

5. if x and y are statistically independent $\rightarrow E\{aXY\} = aE\{X\}E\{Y\}$

6. Covariance M_{xy}

   $\rightarrow M_{xy} = E\{(x-\mu_x)(y-\mu_y)\}$

   $M_{xx} = E\{(x-\mu_x)(x-\mu_x)\} = E\{(x-\mu_x)^2\} = \sigma_x^2$

7. Correlation Coefficient

   $\rho_{xy} = \frac{M_{xy}}{\sigma_x \sigma_y}$

   ① $-1 \le \rho_{xy} \le 1$

   ② $\rho_{xy} = 0 \longleftarrow$ x and y are uncorrelated

   ③ $\rho_{xy} = \pm 1 \longleftarrow$ x and y are fully correlated

[7] * if x and y are statistically independent
then x and y are uncorrelated

* if x and y are uncorrelated then they may or may not be S. independent.

[8] $\mu_{xy} = E\{(x-\mu_x)(y-\mu_y)\} = E\{xy\} - \mu_x \mu_y$

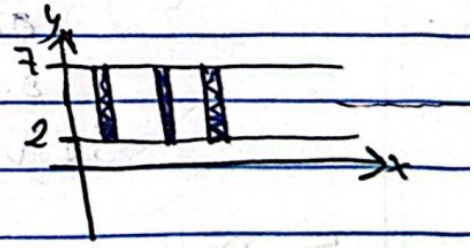
[9] * marginal pdf of x $\rightarrow f_x(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dy$

* marginal pdf of y $\rightarrow f_y(y) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dx$

[10] x and y will be statistically independent if $f_{x,y}(x,y) = f_x(x) f_y(y)$

Exp: let x and y be two R.Vs with the following joint-pdf:

$$f_{x,y}(x,y) = \begin{cases} ke^{-0.2x} & 0 \le x,\ 2 \le y \le 7 \\ 0 & o.w \end{cases}$$



[a] find k:-

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{x,y}(x,y)\, dy\, dx = 1$$

$$\rightarrow \int_0^{\infty}\left[\int_2^7 ke^{-0.2x}\, dy\right] dx = \int_0^{\infty} ke^{-0.2x}\, y\Big|_2^7\, dx = 5k\int_0^{\infty} e^{-0.2x}\, dx$$

$$= \frac{5ke^{-0.2x}}{-0.2}\Big|_0^{\infty} = \frac{5k}{+0.2}\left[\underbrace{e^{\infty}}_{0} - \underbrace{e^{0}}_{+1}\right] = 25k(-1) \rightarrow k = \frac{1}{25} = \boxed{0.04}$$

b Are x and y statistically independent??

$$f_{x,y}(x,y) = f_x(x) f_y(y)$$

$$\rightarrow f_x(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\,dy = \int_{2}^{7} 0.04 e^{-0.2x}\,dy = 0.04 e^{-0.2x} y \Big|_2^7 = \boxed{0.2 e^{-0.2x}}$$

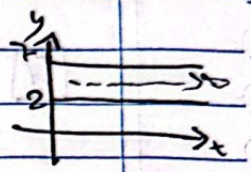
exponential

$\lambda = 0.2$

$\hookrightarrow M_x = \frac{1}{\lambda} = \frac{1}{0.2} = 5$

$\sigma_x^2 = \frac{1}{\lambda^2} = \frac{1}{(0.2)^2} = 25$

$$f_x(x) = \begin{cases} 0.2 e^{-0.2x} & 0 < x \\ 0 & \text{o.w} \end{cases}$$



$$\rightarrow f_y(y) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\,dx = \int_{0}^{\infty} 0.04 e^{-0.2x}\,dx = \frac{0.04 e^{-0.2x}}{-0.2} \Big|_0^\infty$$

$e^{\infty} \to 0$, $e^{0} \to 1$

$$= -0.2[0 - 1] = \boxed{0.2}$$

uniform $a = 2$, $b = 7$

$\rightarrow \frac{1}{b-a} = \frac{1}{5} = 0.2$

$\hookrightarrow M_y = \frac{a+b}{2} = \frac{9}{2} = 4.5$

$\sigma_y^2 = \frac{(b-a)^2}{12} = \frac{25}{12}$

$$f_y(y) = \begin{cases} 0.2 & 2 \le y \le 7 \\ 0 & \text{o.w} \end{cases}$$

$$\Rightarrow f_{x,y}(x,y) \stackrel{?}{=} f_x(x) f_y(y)$$

$$0.4 e^{-0.2x} \stackrel{?}{=} 0.2 e^{-0.2x} \cdot 0.2$$

$$0.4 e^{-0.2x} = 0.4 e^{-0.2x}$$

So yes, x and y are S. independent.

c Determine the correlation coefficient between x and y.

$$\rho_{xy} = \frac{\mu_{x,y}}{\sigma_x \sigma_y} = 0$$ because x and y are S. independent

لأنهم مستقلين إحصائياً

d $E\{3xy\}$

$$= 3 E\{x\} E\{y\} = 3 * 5 * 4.5 = 67.5$$

because x and y are S. Indep.

Note: $R = ax + by$

$\rightarrow \mu_R = E\{R\} = a\mu_x + b\mu_y$

$\rightarrow \sigma_R^2 = Var\{R\} = a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_x\sigma_y\rho_{x,y}$

$= \boxed{a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\,\mu_{x,y}}$

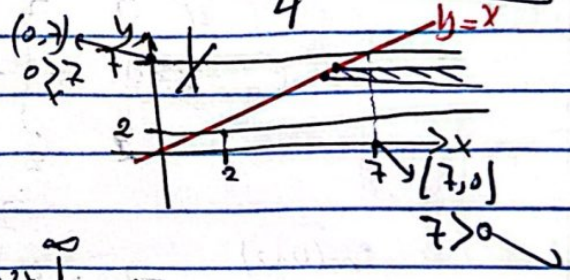
(e) $R = 2x - 3y$, Determine $Var\{R\}$??

$\sigma_R^2 = (2)^2 * \sigma_x^2 + (-3)^2\sigma_y^2 + 2(+2)(-3)\sigma_x\sigma_y\rho_{x,y}$ (0)

0, a, bare SI

$= 4 * 25 + 9 * \frac{25}{12}$ ↳ $= 100 + \frac{75}{4} = \boxed{118.75}$

(f) $P(x > y)$



$$\int_2^7 \int_0^{\infty} 0.04e^{-0.2x}\,dx\,dy$$

$$= \int_2^7 \left[\frac{0.04}{-0.2}e^{-0.2x}\right]_y^{\infty} dy$$

$$= \int_2^7 +0.2\left[0 + e^{-0.2y}\right]dy \rightarrow \int_2^7 0.2e^{-0.2y}\,dy = \frac{0.2e^{-0.2y}}{-0.2}\Big|_2^7$$

$$= -e^{-1.4} - (-e^{-0.4}) = e^{-0.4} - e^{-1.4} = \boxed{0.4237}$$

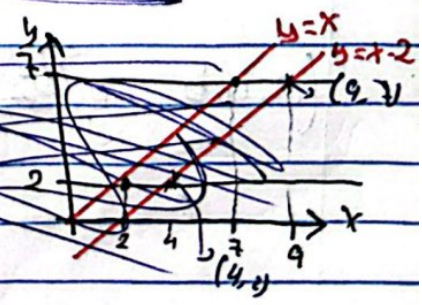
(g) $P(y < x < y+2)$??

$x = y+2$

$y = x-2$

$x = 9 \rightarrow y = 7$

$x = 4 \rightarrow y = 2$



$$\int_2^7 \int_y^{y+2} 0.4e^{-0.2x}\,dx\,dy$$

$\boxed{1.4}$

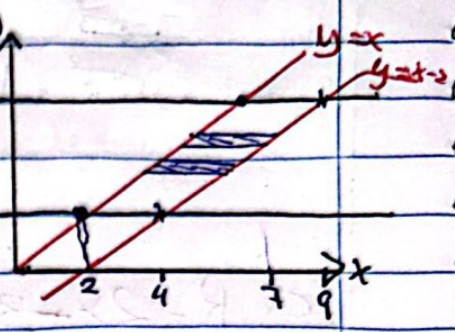
[9] $P(Y < X \leq Y+2) = ??$ $\quad X = Y+2$

$Y = X-2$

$X = 9 \rightarrow Y = 7$

$X = 4 \rightarrow Y = 2$

$X = 2 \rightarrow Y = 0$



$$\int_{2}^{7}\int_{y}^{y+2} 0.04 e^{-0.2x}\,dx\,dy$$

$$= \int_{2}^{7} \frac{0.04}{-0.2} e^{-0.2x}\Big|_{y}^{y+2}\,dy = \int_{2}^{7} -0.2\left[e^{-0.2(y+2)} - e^{-0.2y}\right]dy = \int_{2}^{7} -0.2e^{-0.2y-0.4} + 0.2e^{-0.2y}\,dy$$

$$= \frac{-0.2e^{-0.2(y+2)}}{-0.2} - \frac{0.2e^{-0.2y}}{-0.2}\Big|_{2}^{7} = e^{-0.2(y+2)} - e^{-0.2y}\Big|_{2}^{7}$$

$$= \left(e^{-0.2(9)} - e^{-0.2(7)}\right) - \left(e^{-0.2(4)} - e^{-0.2(4)}\right)$$

$$(0.165 + 0.247) - (0.450 + 0.670)$$

$$0.$$

Transformation :-

Ex:- Let X and Y be two R.Vs with the following Joint-PMF

X \ Y	-1	1	2
-1	1/4	1/8	0
1	0	1/2	1/8

→ let $R = X + 2Y^2$. determine the PMF of R ?

x	X	$P(X=x, Y=y)$	$r = x + 2y^2$	$P(R=r)$
-1	-1	1/4	$r = (-1) + 2(-1)^2 = 1$	$\frac{1}{4} + \frac{1}{8} = \frac{3}{8}$
-1	1	1/8	$r = (-1) + 2(1)^2 = 1$	
1	1	1/2	$r = 1 + 2(1) = 3$	1/2
1	2	1/8	$r = 1 + 2(2) = 5$	1/8

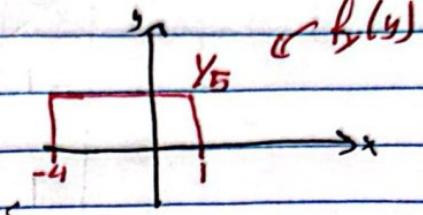
So $P(R=r) = \begin{cases} 3/8 & r=1 \\ 1/2 = 4/8 & r=3 \\ 1/8 & r=5 \\ 0 & O.W \end{cases}$

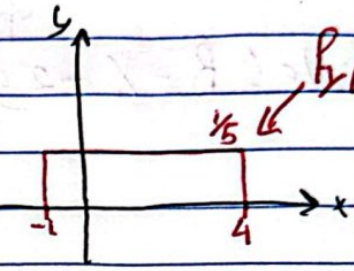
لما عوض X=-1
مع Y=1 و Y=-1
طلع جواب R
نفس الرقم لذا
فيجمع الاحتمالين

Exp:- let X and Y be two statistically independent R.V.
X has a uniform distribution over the interval [-2, 6]
Y has also a uniform distribution over the interval [-4, 1].
let $Z = X + Y$. determine the pdf of Z?

$$f_X(x) = \begin{cases} \frac{1}{8} & -2 \le x \le 6 \\ 0 & o.w \end{cases}$$

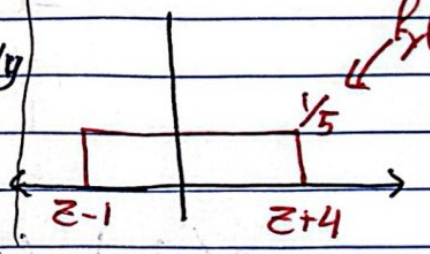
$$f_Y(y) = \begin{cases} \frac{1}{5} & -4 \le x \le 1 \\ 0 & o.w \end{cases}$$





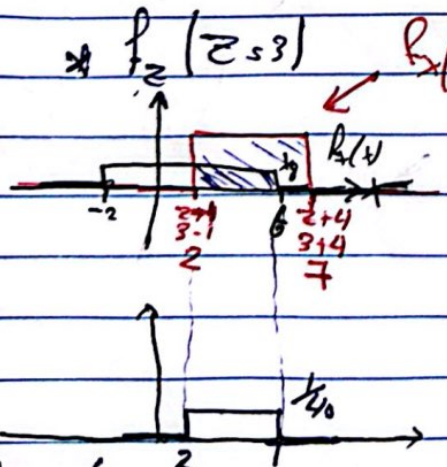
Note: if x and y are two S. I continuous R.V. and $Z = X + Y$, then the Pdf of Z can be determined using the convolution Integral

$$\rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx = \int_{-\infty}^{\infty} f_Y(y) f_X(z - y)\, dy$$

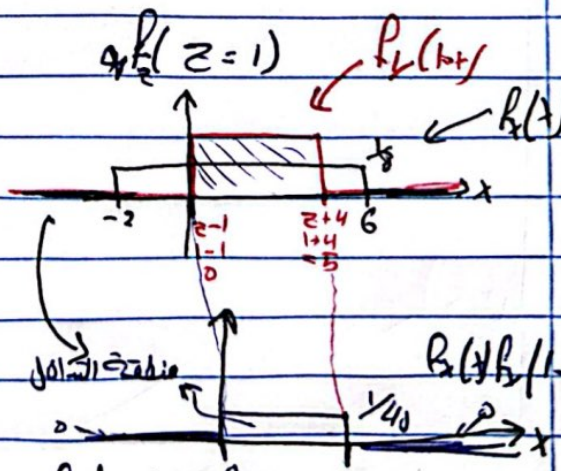


$$f_Y(z - x) = \begin{cases} \frac{1}{5} & -4 \le z - x \le -1 \\ 0 & o.w \end{cases}$$

$$= \begin{cases} \frac{1}{5} & \underset{o.w}{\overset{x-1}{(-z-4 \le -x \le -z+1)}} \longrightarrow \begin{cases} \frac{1}{5} & z-1 \le x \le z+4 \\ 0 & o.w \end{cases} \end{cases}$$



$$f_Z(z) = \int_2^6 \frac{1}{40}\, dx = \frac{x}{40}\Big|_2^6 = \frac{z-2}{40} = \boxed{\frac{4}{40}}$$

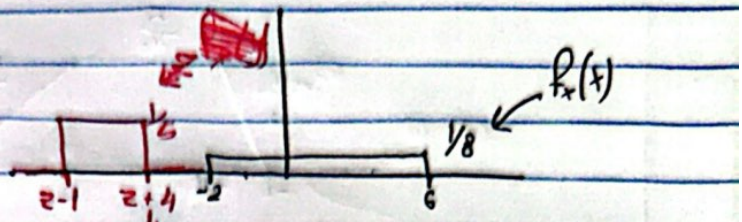


$$f_Z(z=1) = \int_0^5 \frac{1}{40}\, dx = \frac{x}{40}\Big|_0^5 = \boxed{\frac{5}{40}}$$

1) $z+4 < -2$

$z < -6$

$f_z(z) = 0$



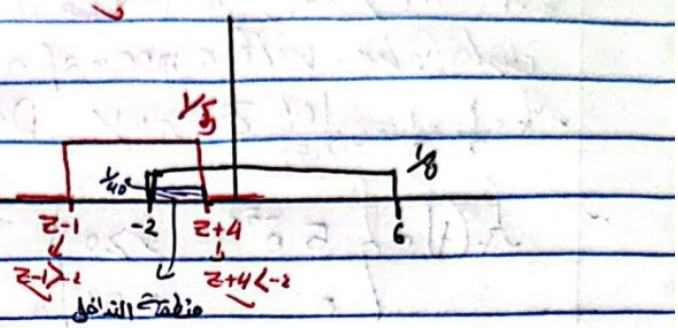
فتحة تداخل

2) $z+4 > -2,\ z-1 < -2$

$z > -6,\ z < -1$

$$f_z(z) = \int_{-2}^{z+4} \frac{1}{40}\,dx = \frac{x}{40}\Big|_{-2}^{z+4} = \frac{z+4+2}{40}$$

$$\boxed{\frac{z+6}{40}}$$



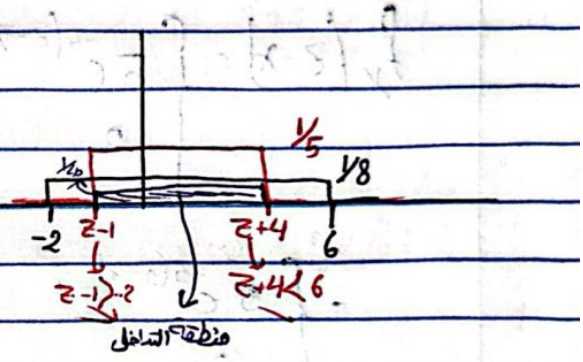
منطقة التداخل

3) $z-1 > -2,\ z+4 < 6$

$z > -1,\ z < 2$

$$f_z(z) = \int_{z-1}^{z+4} \frac{1}{40}\,dx = \frac{x}{40}\Big|_{z-1}^{z+4} = \frac{z+4-z+1}{40}$$

$$= \boxed{\frac{5}{40}}$$

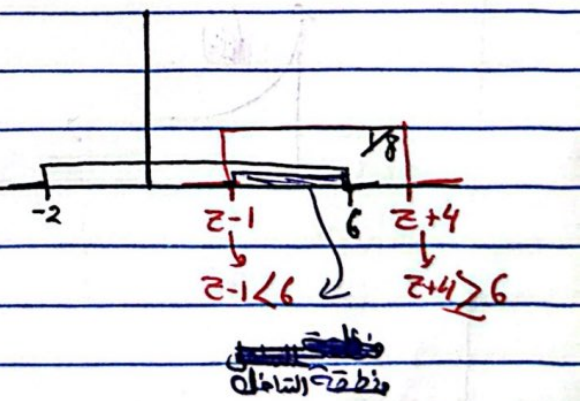


منطقة التداخل

4) $-1 < z < 6,\ z+4 > 6$

$z < 7 \qquad z > 2$

$$f_z(z) = \int_{z-1}^{6} \frac{1}{40}\,dx = \frac{x}{40}\Big|_{z-1}^{6} = \frac{6-z+1}{40} \quad \boxed{\frac{7-z}{40}}$$

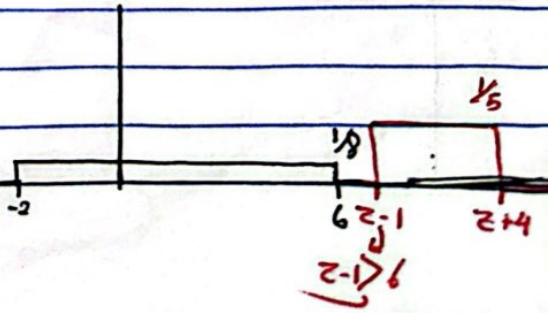


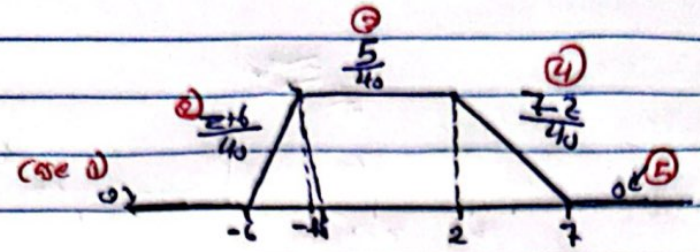
منطقة التداخل

5) $z-1 > 6$,

$z > 7$

$f_z(z) = 0$ فتحة تداخل





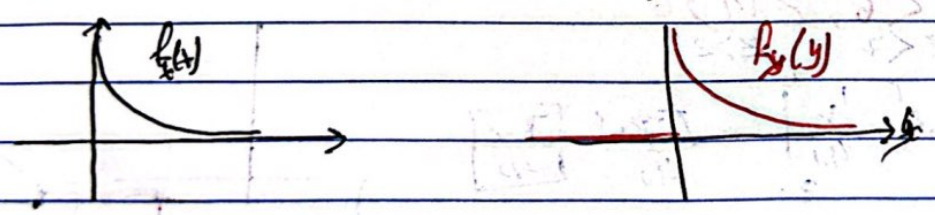
Exp:- let X and Y be two S. Indep. R.Vs with an exponential distribution with a mean of 0.2.

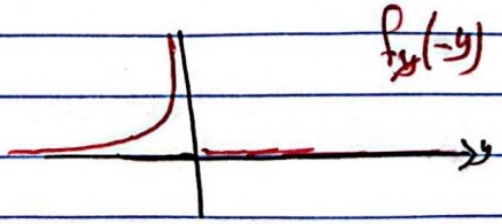
$\mu = \frac{1}{\lambda} = \frac{1}{5} = 0.2 \rightarrow \lambda = 5$ let $Z = X + Y$. Determine the pdf of Z

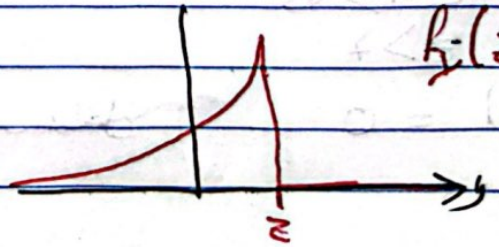
$$f_X(x) = \begin{cases} 5e^{-5x} & x \geq 0 \\ 0 & o.w \end{cases}, \quad f_Y(y) = \begin{cases} 5e^{-5y} & y \geq 0 \\ 0 & o.w \end{cases}$$

$$f_Y(z-x) = \begin{cases} 5e^{-5(z-x)} & z-x > 0 \\ 0 & o.w \end{cases}$$

$$= \begin{cases} 5e^{-5(z-x)} & z > x \\ 0 & o.w \end{cases}$$

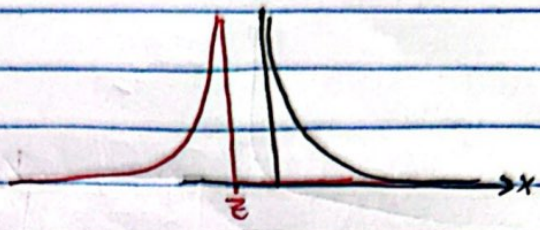






① $z<0$

$\therefore f_z(z) = 0$ في نطاق الحل

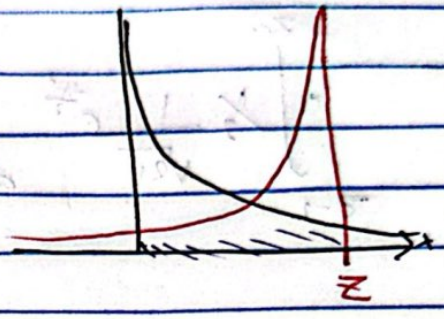


② $z>0$

$$f_z(z) = \int_0^z 5e^{-5x} * 5e^{-5(z-x)}\,dx$$

$$= \int_0^z 25e^{-5x-5z+5x}\,dx = \int_0^z 25e^{-5z}\,dx$$

$$= 25e^{-5z} \cdot x \Big|_0^z = \boxed{25ze^{-5z}}$$

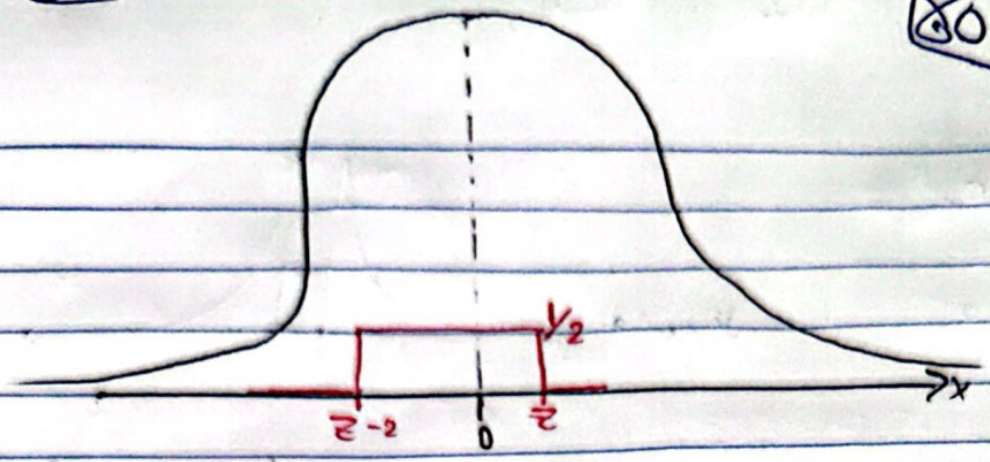


Ex: let x be a Gaussian R.V with zero mean and unity variance. y is another R.V with uniform distribution over the interval $[0,2]$. Assume x and y are SI and $z = x+y$. Determine $f_z(0)$.?

$$f_x(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} & -\infty < x < \infty \end{cases}$$

$$f_y(y) = \begin{cases} \frac{1}{2} & 0 \le y \le 2 \\ 0 & \text{o.w} \end{cases} \rightarrow f_y(z-x) = \begin{cases} \frac{1}{2} & 0 \le z+x \le 2 \\ 0 & \text{o.w} \end{cases}$$

$$= \begin{cases} \frac{1}{2} & -z \le -x \le 2-z \\ 0 & \text{o.w} \end{cases} \rightarrow = \begin{cases} \frac{1}{2} & z-2 \le x \le z \\ 0 & \text{o.w} \end{cases}$$



$$= \int_{z-2}^{z} \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \frac{1}{2} \int_{z-2}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

$$= \frac{1}{2}\left[\Phi(z) - \Phi(z-2)\right]$$

$$f_z(0) = \frac{1}{2}\left[\Phi(0) - \Phi(-2)\right] =$$

$$\Phi\left(\frac{0-\mu}{\sigma}\right) - \Phi\left(\frac{-2-\mu}{\sigma}\right)$$

($\mu = 0$, $\sigma^2 = 1$)

* $t \rightarrow$ Standard gaussian

$$P(1 \le t \le 5) = \int_{1}^{5} \frac{1}{\sqrt{2\pi}} e^{\frac{t^2}{2}} dt$$

$$\Phi\left(\frac{5-\mu}{\sigma}\right) - \Phi\left(\frac{1-\mu}{\sigma}\right)$$

(μ, σ^2)