

(1)

Bessel's and Legendre Functions

the d.e $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$ (1)

is called Bessel's eq. of order ν . ($\nu \geq 0$).

the solution of eq (1) is given by

$$y = \begin{cases} c_1 J_\nu(x) + c_2 J_{-\nu}(x), & \nu \neq \text{integer} \\ a_1 J_\nu(x) + a_2 Y_\nu(x), & \nu \text{ integer,} \end{cases}$$

where $J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$

(Bessel functions of the first kind)

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

(Bessel's function of the second kind)

ex. Solve $x^2 y'' + xy' + (x^2 - 9)y = 0$ on $(0, \infty)$.

sol. $\nu^2 = 9 \Rightarrow \boxed{\nu = 3}$

$$\therefore y = c_1 J_3(x) + c_2 Y_3(x)$$

(2)

ex. Solve $x^2 y'' + \alpha y' + (x^2 - \frac{1}{9})y = 0$.

sol. $v^2 = \frac{1}{9} \Rightarrow \boxed{v = \frac{1}{3}}$

$\therefore y = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)$.

• The d.e $\boxed{x^2 y'' + x y' + (x^2 - v^2)y = 0}$ (2)

is called the parametric Bessel eq. of order v . (where v is integer)

The general eq. for eq (2) is given by

$\boxed{y = c_1 J_v(\alpha x) + c_2 Y_v(\alpha x)}$, $v = \text{integer}$

[by letting $t = \alpha x$, $\alpha > 0$ and using chain rule see the pdf file].

ex. Solve $x^2 y'' + x y' + (9x^2 - 4)y = 0$

sol. $\alpha^2 = 9 \Rightarrow \boxed{\alpha = 3}$, $v^2 = 4 \Rightarrow \boxed{v = 2}$

$\therefore y = c_1 J_2(3x) + c_2 Y_{-2}(3x)$.

ex. Solve $x^2 y'' + x y' + (2x^2 - 64)y = 0$.

sol. $\alpha^2 = 2 \Rightarrow \boxed{\alpha = \sqrt{2}}$, $v^2 = 64 \Rightarrow \boxed{v = 8}$

$\therefore y = c_1 J_8(\sqrt{2}x) + c_2 Y_{-8}(\sqrt{2}x)$.

(3)

Properties of Bessel functions

$$(1) J_{-m}(x) = (-1)^m J_m(x), \quad m = 0, 1, 2, \dots$$

$$(2) J_m(-x) = (-1)^m J_m(x), \quad m = 0, 1, 2, \dots$$

$$(3) J_m(0) = \begin{cases} 0, & m > 0 \\ 1, & m = 0 \end{cases}, \quad m = 0, 1, 2, \dots$$

$$(4) \lim_{x \rightarrow 0^+} Y_m(x) = -\infty, \quad m = 0, 1, 2, \dots$$

$$(5) x J_\nu'(x) = \nu J_\nu(x) - x J_{\nu+1}(x).$$

$$(6) \frac{d}{dx} [x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x).$$

$$\text{or } \int x^\nu J_{\nu-1}(x) dx = x^\nu J_\nu(x) + C.$$

$$\text{ex. } \int x^3 J_2(x) dx = x^3 J_3(x) + C.$$

$$\text{ex. } \frac{d}{dx} [x J_1(x)] = x J_0(x).$$

$$(7) \frac{d}{dx} [x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x)$$

$$\text{or } \int x^{-\nu} J_{\nu+1}(x) dx = -x^{-\nu} J_\nu(x) + C.$$

(4)

(8) Bessel function of half-integral order $J_{1/2}(x)$.

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\frac{3}{2} + n)} \left(\frac{x}{2}\right)^{2n + \frac{1}{2}}$$

$$= \sqrt{\frac{2}{\pi x}} \sin x.$$

(9) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$

Legendre Functions

The d.e $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ (3) is called Legendre's eq. of order n $-1 \leq x \leq 1$

The solution for eq (3) is given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2-1)^n], n=0,1,2,\dots$$
 (4)

(4) is called Rodrigues' formula or Legendre polynomials.

for ex., $P_0(x) = \frac{1}{2^0 0!} \frac{d^0}{dx^0} [(x^2-1)^0] = 1$

(5)

$$P_1(x) = \frac{1}{2 \cdot 1!} \frac{d}{dx} [(x^2-1)^1] = \frac{1}{2} \cdot 2x = x$$

$$\Rightarrow \boxed{P_1(x) = x}$$

$$P_2(x) = \frac{1}{2^2 \cdot 2!} \frac{d^2}{dx^2} [(x^2-1)^2]$$

$$= \frac{1}{8} \frac{d}{dx} [2(x^2-1)(2x)]$$

$$= \frac{1}{2} \frac{d}{dx} [x^3 - x] = \frac{1}{2} (3x^2 - 1)$$

$$\Rightarrow \boxed{P_2(x) = \frac{1}{2} (3x^2 - 1)}$$

$$\boxed{P_3(x) = \frac{1}{2} (5x^3 - 3x)}$$

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$$\boxed{P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3)}$$

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ملاحظة: مطلوب حفظ P_0, P_1, P_2, P_3 فقط
لأننا نستخدم لاحقاً.

⑥

Particular solutions of the d.e. (3):

$$n=0 : (1-x^2)y'' - 2xy' = 0 \Rightarrow P_0(x) = 1$$

$$n=1 : (1-x^2)y'' - 2xy' + 2y = 0 \Rightarrow P_1(x) = x$$

$$n=2 : (1-x^2)y'' - 2xy' + 6y = 0$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$n=3 : (1-x^2)y'' - 2xy' + 12y = 0$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

⋮

Properties:

$$(1) P_n(-x) = (-1)^n P_n(x)$$

$$(2) P_n(1) = 1$$

$$(3) P_n(-1) = (-1)^n$$

$$(4) P_n(0) = 0, n \text{ odd}$$

$$(5) P_n'(0) = 0, n \text{ even}$$

$$(6) \int_{-1}^1 P_n(x) P_m(x) dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases} \quad (\text{Orthogonality})$$