Ch.2 :- Limits and Continuity

Q.1 Find the following limits :-

ⓐ $\lim_{t \to -1} \frac{t^2+3t+2}{t^2-t-2} = \frac{0}{0}$

$\lim_{t \to -1} \frac{(t+1)(t+2)}{(t+1)(t-2)} = \lim_{t \to -1} \frac{t+2}{t-2} = \frac{1}{-3}$

ⓑ $\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x} = \frac{0}{0}$

$\lim_{x \to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \lim_{x \to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$

ⓒ $\lim_{x \to 1} \frac{x^4-1}{x^3-1} = \frac{0}{0}$

$\lim_{x \to 1} \frac{(x^2-1)(x^2+1)}{(x-1)(x^2+x+1)} = \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}(x^2+1)}{\cancel{(x-1)}(x^2+x+1)} = \frac{4}{3}$

ⓓ $\lim_{\theta \to 0} \frac{\sin 2\theta}{3\theta} = \frac{1}{3} \cdot 2 \lim_{\theta \to 0} \frac{\sin 2\theta}{2\theta} = \frac{2}{3} \cdot 1$

$= \frac{2}{3}$

OR $\lim_{\theta \to 0} \frac{\frac{\sin 2\theta}{\theta}}{\frac{3\theta}{\theta}} = \frac{\lim_{\theta \to 0} \frac{\sin 2\theta}{\theta}}{\lim_{\theta \to 0} 3} = \frac{2}{3}$

(e) $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\sin(2\theta)} = \frac{0}{0}$

$\lim_{\theta \to 0} \frac{1-\cos\theta}{\sin(2\theta)} \cdot \frac{1+\cos\theta}{1+\cos\theta} = \lim_{\theta \to 0} \frac{\sin^2\theta}{2\sin\theta\cos\theta\,(1+\cos\theta)}$

$= \lim_{\theta \to 0} \frac{\sin\theta}{2\cos\theta\,(1+\cos\theta)}$

$= \frac{\cancel{2}\,0}{2(1+1)} = \frac{0}{4} = 0$

OR

$\lim_{\theta \to 0} \frac{1-\cos\theta}{\sin(2\theta)} = \frac{\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta}}{\lim_{\theta \to 0} \frac{\sin(2\theta)}{\theta}} = \frac{0}{2} = 0$

(f) $\lim_{x \to 0} \frac{1+\sqrt{x}}{1-\sqrt{x}} = \frac{\sqrt{1}}{-\sqrt{1}} = \boxed{-1}$

(g) $\lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{x+1}$

$\lim_{x \to -\infty} \frac{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x\left(1+\frac{1}{x}\right)}$

$\lim \frac{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}{x\left(1+\frac{1}{x}\right)}$

Note:-

$\sqrt{x^2} = |x|$

$\left(\sqrt{x}\right)^2 = x$

$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

$$\lim_{x\to-\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{x\left(1+\frac{1}{x}\right)}$$

$$\lim_{x\to-\infty} \frac{-\cancel{x}\sqrt{1+\cancel{\frac{1}{x^2}}}}{\cancel{x}\left(1+\cancel{\frac{1}{x}}\right)} = \boxed{-1}$$

(1) $\lim_{x\to+\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-x}\right)$ Multipling by the Conjugate

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-x}\right)\left(\frac{\sqrt{x^2+1}+\sqrt{x^2-x}}{\sqrt{x^2+1}+\sqrt{x^2-x}}\right)$$

$$\lim_{x\to\infty}\frac{x+1}{\sqrt{x^2+1}+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{x\left(1+\frac{1}{x}\right)}{|x|\sqrt{1+\frac{1}{x^2}}+|x|\sqrt{1-\frac{1}{x}}}$$

$$= \lim_{x\to\infty}\frac{x\left(1+\cancel{\frac{1}{x}}\right)}{|x|\left(\sqrt{1+\cancel{\frac{1}{x^2}}}+\sqrt{1-\cancel{\frac{1}{x}}}\right)}$$

$$= \lim_{x\to\infty}\frac{x\left(1+\frac{1}{x}\right)}{x\left(\sqrt{1+\frac{1}{x^2}}\right)+\sqrt{1-\frac{1}{x}}}$$

$$= \frac{1}{\sqrt{1}+\sqrt{1}} = \frac{1}{2}$$

(h) $$\lim_{x \to -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}} = \frac{\sqrt[3]{1}}{\sqrt[3]{1}} = 1$$

(j) $$\lim_{t \to 3^+} \frac{\lfloor t \rfloor}{t}$$

$\lfloor t \rfloor$: Greatest Integar function

$$= \frac{\lfloor 3.1 \rfloor}{3} = \boxed{1}$$

(k) $$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$$

By Sandwich Th

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-|x| \leq |x| \sin\left(\frac{1}{x}\right) \leq |x|$$

$$0 = \lim_{x \to 0} -|x| \leq \lim_{x \to 0} |x| \sin\left(\frac{1}{x}\right) \leq \lim_{x \to 0} |x| = 0$$

Then By Sandwich Th. $$\lim_{x \to 0} |x| \sin\left(\frac{1}{x}\right) = 0$$

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

OR $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

$$= \lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

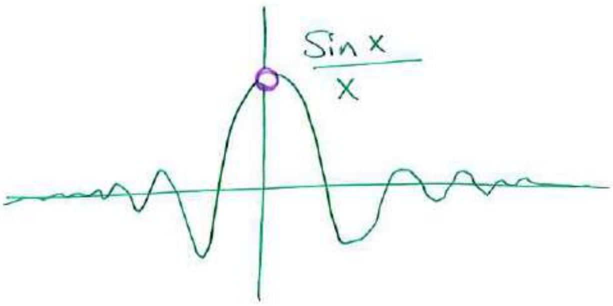
Assume $y = \frac{1}{x}$

$x \to 0$

$y \to \infty$ or $-\infty$

$$= \lim_{y \to \pm\infty} \frac{\sin y}{y}$$

$$= 0$$



$\boxed{Q_2}$ Find the asymptotes of the following functions

Summary :-

$\cancel{(a)\ f(x) = \frac{x+1}{x-1}}$

$\boxed{f(x) = \frac{g(x)}{h(x)}}$

Asymptots :-

Horizantal Asy

$\boxed{y = b}$

deg $g(x) \leq$ deg $h(x)$

$\lim_{x \to \infty} f(x) = b$

OR $\lim_{x \to -\infty} f(x) = b$

Vertical Asy.

$\boxed{x = a}$

Check Zero of Denominator

$\lim_{x \to a^+} f(x) = \pm\infty$

OR $\lim_{x \to a^+} f(x) = \pm\infty$

Oblique Asy

$\boxed{y = ax + b}$

deg $g(x) >$ deg $h(x)$

Long Division

ⓐ $f(x) = \frac{x+1}{x-1}$, D. $\boxed{x \neq 1}$

O. Asy: None

H. Asy: Check $\lim_{x \to \pm\infty} f(x)$

$$\left.\begin{array}{l} \lim_{x \to \infty} \frac{x+1}{x-1} = 1 \\ \text{OR } \lim_{x \to -\infty} \frac{x+1}{x-1} = 1 \end{array}\right\} \longrightarrow \underset{\text{H Asy.}}{\boxed{y = 1}}$$

V. Asy: Check $\lim_{x \to 1} f(x)$

$$\left.\begin{array}{l} \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x+1}{x-1} = \frac{2}{\text{small}+} = \infty \\ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x+1}{x-1} = \frac{2}{\text{small}-} = -\infty \end{array}\right\} \longrightarrow \underset{\text{V. Asy.}}{\boxed{x = 1}}$$

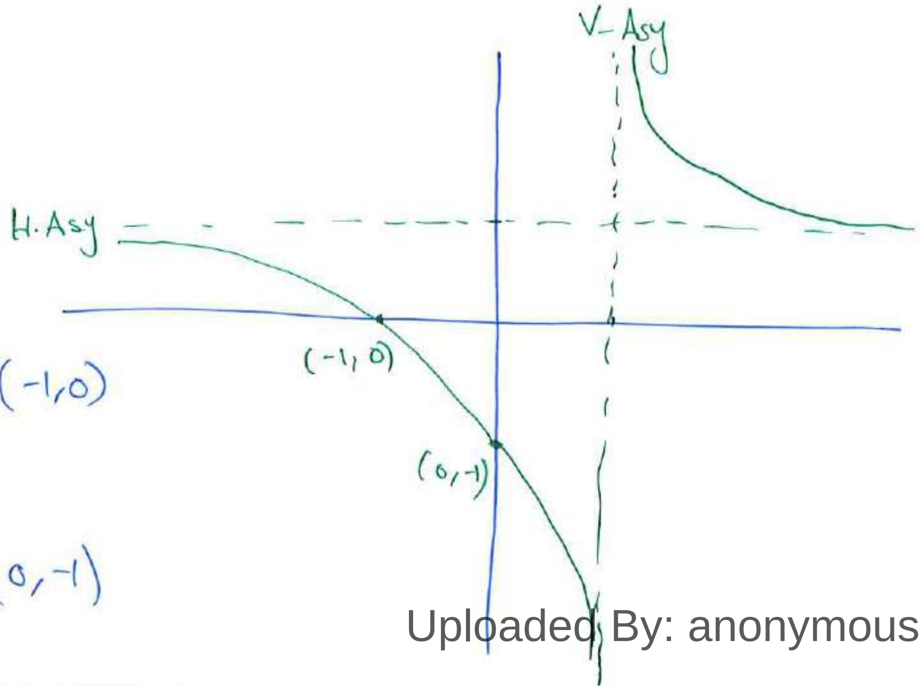
for the graph.

- x intercept $0 = \frac{x+1}{x-1}$

  $0 = x+1 \rightarrow x = -1$ $(-1, 0)$

- y-intercept $y = \frac{0+1}{0-1}$

  $y = -1$ $(0, -1)$



(b) $y = \dfrac{x^3+1}{x^2}$, D. $\boxed{x \neq 0}$

H. Asy : None

O. Asy : $\boxed{y = x}$

$$\begin{array}{r|l} & x \\ x^2 & x^3 + 1 \\ & -x^3 \\ \hline & 1 \end{array}$$

$$\boxed{\frac{x^3+1}{x^2} = x + \frac{1}{x^2}}$$

. V. Asy : check $\lim\limits_{x \to 0} \dfrac{x^3+1}{x^2}$

$$\left.\begin{array}{l} \lim\limits_{x \to 0^+} \dfrac{x^3+1}{x^2} = \dfrac{1}{\text{Small}+} = \infty \\ \lim\limits_{x \to 0^-} \dfrac{x^3+1}{x^2} = \dfrac{1}{\text{Small}-} = -\infty \end{array}\right\} \longrightarrow \text{V. Asy. } \boxed{x = 0}$$

for the graph.

- x-intercept $0 = \dfrac{x^3+1}{x^2}$

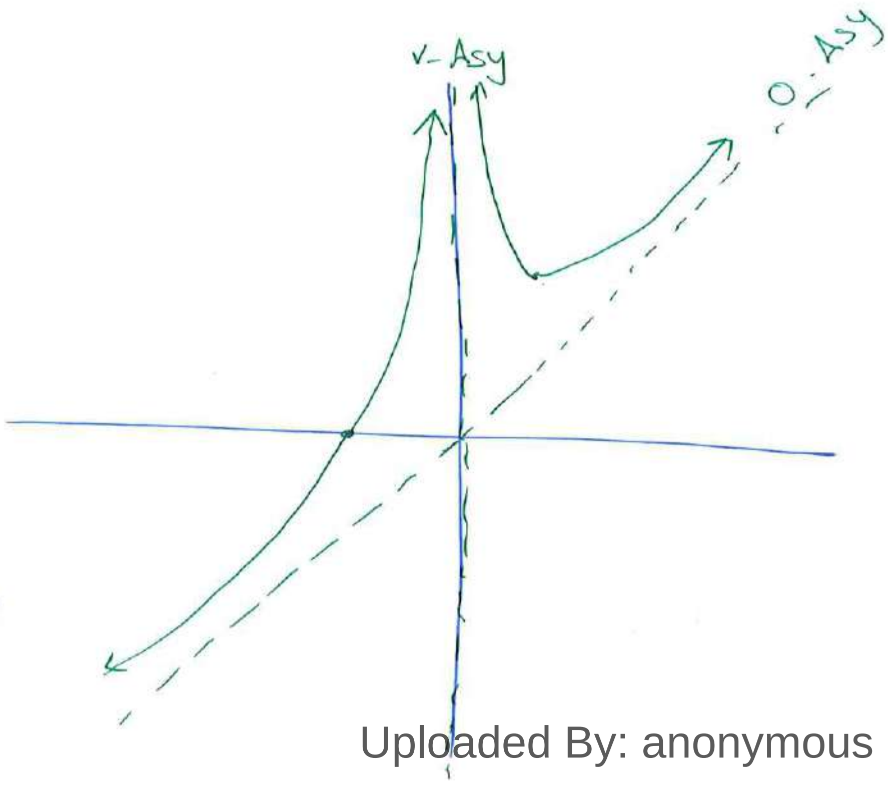
$x^3 + 1 = 0$

$x^3 = -1 \rightarrow x = -1$

$(-1, 0)$

y-intercept $y = \frac{1}{0}$ DNE

No y-intercept



⑥ $f(x) = \dfrac{x^2+1}{x-1}$ $\xrightarrow{}$ D. $\boxed{x \neq 1}$

H. Asy : None

O. Asy $\rightarrow$ $\boxed{y = x+1}$

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2+1 \\ & -x^2 \pm x \\ \hline & x+1 \\ & -x \pm 1 \\ \hline & 2 \end{array}$$

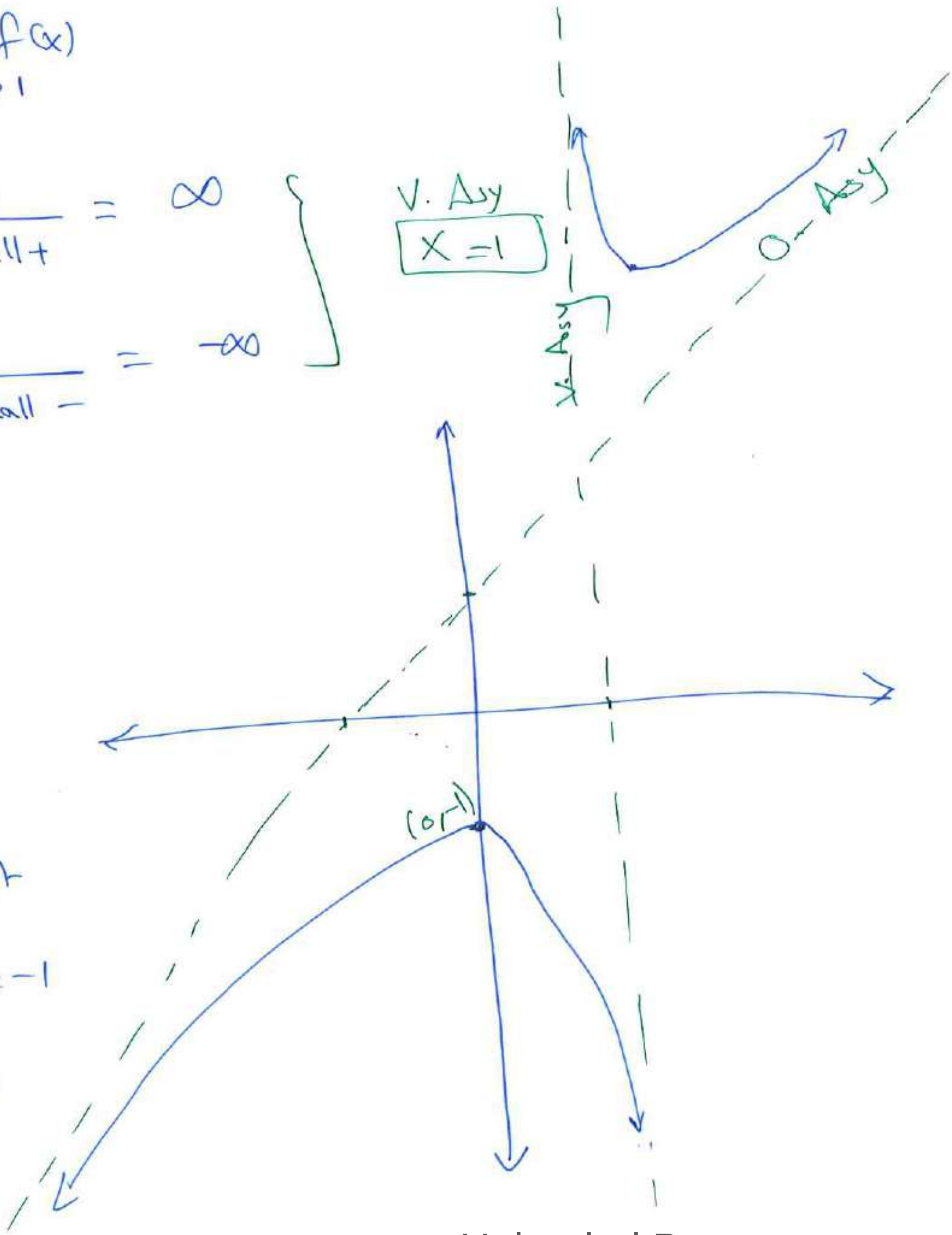
$$\frac{x^2+1}{x-1} = \underbrace{\overset{\text{linear}}{(x+1)}}_{\text{O. Asy}} + \frac{2}{x-1}$$

V. Asy : Check $\lim\limits_{x \to 1} f(x)$

$$\lim_{x \to 1^+} \frac{x^2+1}{x-1} = \frac{2}{\text{Small}+} = \infty$$

$$\lim_{x \to 1^-} \frac{x^2+1}{x-1} = \frac{2}{\text{Small}-} = -\infty$$

V. Asy $\boxed{x = 1}$



for the graph

x-intercept $\quad 0 = \dfrac{x^2+1}{x-1}$

$x^2+1 = 0$

No x-intercept

y-intercept $\quad y = \dfrac{1}{-1} = -1$

$(0,-1)$

(d) $f(x) = \dfrac{x^3+1}{x^2-1}$, $\boxed{x \neq \pm 1}$

$$\begin{array}{r|l} & x \\ \hline x^2-1 & x^3+1 \\ & -x^3 \pm x \\ \hline & x+1 \end{array}$$

- H. Asy : None

$$\frac{x^3+1}{x^2-1} = \underbrace{x}_{\text{O.Asy}} + \frac{x+1}{x^2-1}$$

- O. Asy : $\boxed{y = x}$

- V. Asy : Check $\lim\limits_{x\to 1} f(x)$ <u>and</u> $\lim\limits_{x\to -1} f(x)$

$$\lim_{x\to 1^+} \frac{x^3+1}{x^2-1} = \frac{2}{\text{Small}+} = \infty$$

$$\lim_{x\to 1^-} \frac{x^3+1}{x^2-1} = \frac{2}{\text{Small}-} = -\infty$$

V. Asy $\boxed{x = 1}$

$$\lim_{x\to -1} \frac{x^3+1}{x^2-1} = \frac{0}{0} \longrightarrow \boxed{x=-1} \text{ Removable Discontinuity}$$

$$\lim_{x\to -1} \frac{\cancel{(x+1)}(x^2+x+1)}{\cancel{(x+1)}(x-1)} = \frac{3}{-2}$$

for the graph

x-intercept: $0 = \dfrac{x^3+1}{x^2-1}$

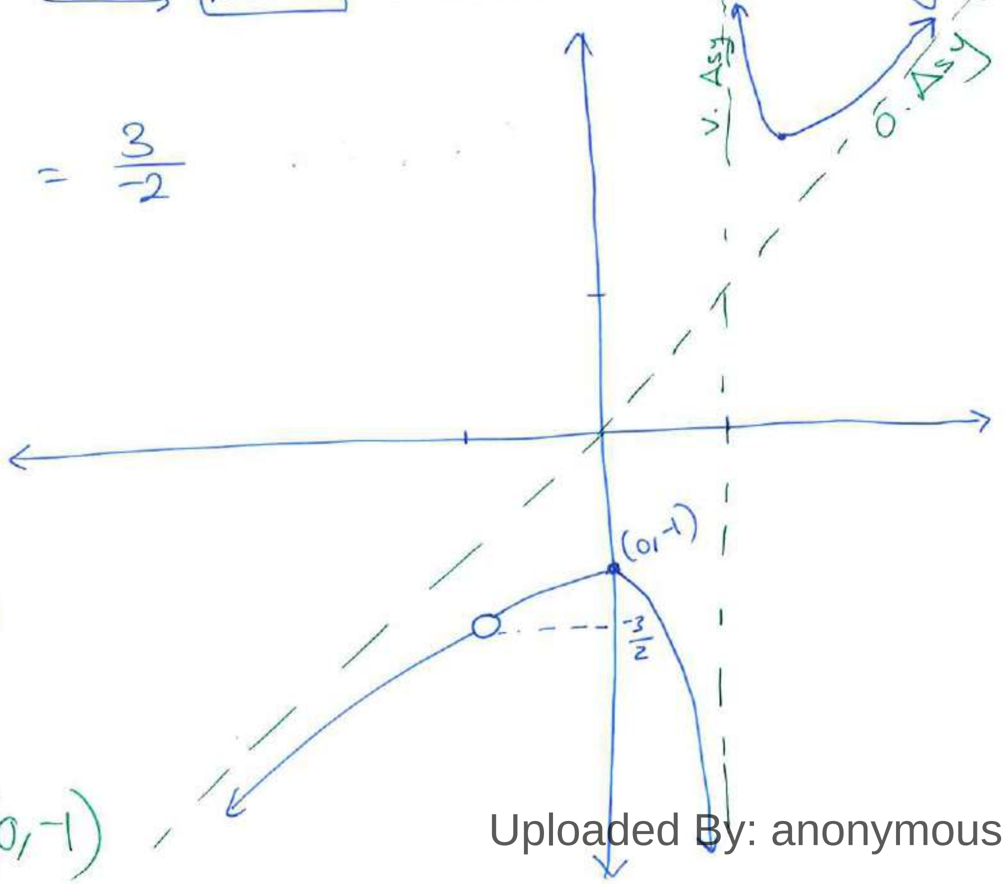
$x^3+1=0$

$x=-1 \notin D$

Reject

y-intercept $y = \dfrac{1}{-1}$

$y=-1$ $(0,-1)$



$\boxed{Q_3}$ Find the value of a, b ??

$$g(x) = \begin{cases} ax + 2b & x \le 0 \\ x^2 + 3a - b & 0 < x \le 2 \\ 3x - 5 & x > 2 \end{cases}$$

g(x) is cont. everywhere

. Then sketch the graph of the function

** Since g(x) is cont at $x = 0$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^-} g(x)$$

$$0 + 3a - b = 2b \longrightarrow \boxed{a = b}$$

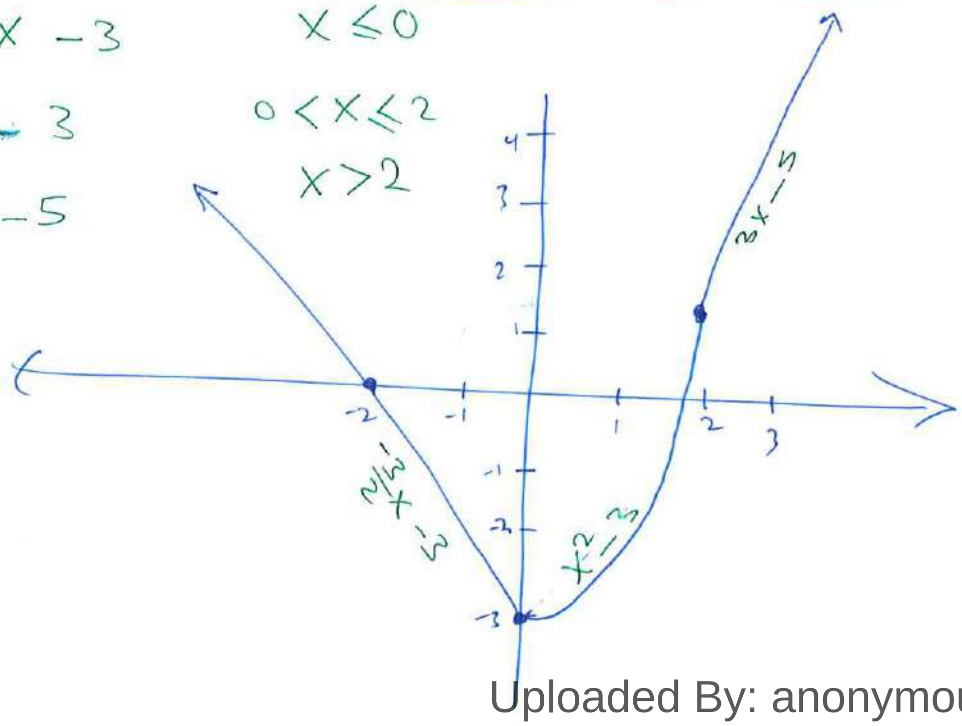
** Since g(x) is cont at $x = 2$

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^-} g(x)$$

$$1 = 4 + 3a - b \longrightarrow 3a - a = -3$$

$$\boxed{b = \frac{-3}{2}}, \quad \boxed{a = \frac{-3}{2}}$$

$$g(x) = \begin{cases} \frac{-3}{2} x - 3 & x \le 0 \\ x^2 - 3 & 0 < x \le 2 \\ 3x - 5 & x > 2 \end{cases}$$



$\boxed{Q4}$ Continuous Extension

$$h(x) = \frac{x^2 + 3x - 10}{x - 2}, \quad \boxed{x \neq 2}$$

$h(2) = \frac{0}{0} \longrightarrow \boxed{x = 2}$ Removable Discontinuity

$$\lim_{x \to 2} \frac{x^2 + 3x - 10}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+5)}{x-2} = 7$$

The the continuous Extension of $h(x)$ at $x = 2$

$$H(x) = \begin{cases} \frac{x^2 + 3x - 10}{x - 2} & x \neq 2 \\ 7 & x = 2 \end{cases}$$

Intermediate Value Theorem :- I. V. T

$\boxed{Q5}$ Use the Intermediate Value Theorem :- I. V. T to show that the $f(x) = x^3 + 2x^2 + 2$ has a root (solution)

Conditions :-

① $f(x)$ is cont. for all x because it's a polynomial

$\longrightarrow f(x)$ is cont on $[-2, 2]$

② $y_0 = 0$ lie between $f(-2)$ and $f(2)$

$f(-2) = -14$ $\qquad f(-2) \leqslant y_0 \leqslant f(2)$

$f(2) = 2$

$\longrightarrow$ Then by I. V.T, $\exists\, c \in [-2, 2]$ such that $f(c) = y_0 = 0$

$\longrightarrow$ This means that c is a root (solution) of the function