# 1 Boas, problem p.564, 12.1-1

Solve the following differential equations by series and by another elementary method and check that the results agree:

$$xy' = xy + y \tag{1}$$

- by series: substituting the power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ in (1) we have

$$xy' - xy - y = 0 \iff x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^n\right) - \sum_{n=0}^{\infty} a_n x^n = 0 \tag{2}$$

calling $n = m+1$ and substituting,
$$\sum_{m=0}^{\infty}(m+1)a_{m+1}x^{m+1} - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0 \tag{3}$$

$$\sum_{m=0}^{\infty}\left[(m+1)a_{m+1} - a_m\right]x^{m+1} - a_0 - \sum_{m=0}^{\infty} a_{m+1}x^{m+1} = 0 \tag{4}$$

The only term with a 0-th power of $x$ is $a_0$, which tells us $a_0 = 0$. Asking for the coefficient of the $(m+1)$-th power of $x$ in (4) to be zero we get

$$a_{m+1} = \frac{1}{m}a_m = \frac{1}{m}\frac{1}{m-1}a_{m-1} = \ldots = \frac{1}{m!}a_1 \tag{5}$$

So the solution for (1) is

$$y(x) = \sum_{m=0}^{\infty} a_{m+1}x^{m+1} = a_1 \sum_{m=0}^{\infty} \frac{1}{m!}x^{m+1} \tag{6}$$

Factoring out one power of $x$, one recognizes the power series of the exponential $e^x$ and writes the solution as

$$y(x) = a_1 x e^x \tag{7}$$

- by separation of variables:

$$\frac{dy}{y} = (1 + \frac{1}{x})dx \implies \ln y = x + \ln x + \ln c \implies y(x) = c\, x e^x \tag{8}$$

which is the same as in (7).

# 2 Boas, problem p.564, 12.1-10

Solve the following differential equations by series and by another elementary method and check that the results agree:

$$y'' - 4xy' + (4x^2 - 2)y = 0 \tag{9}$$

- by series: substituting the power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ in (9) we have

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 4x \sum_{n=1}^{\infty} n a_n x^{n-1} + (4x^2 - 2) \sum_{n=0}^{\infty} a_n x^n = 0 \tag{10}$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n - 4a_1 x - 4\sum_{n=2}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^{n+2} - 2a_0 - 2a_1 x - 2\sum_{n=2}^{\infty} a_n x^n = 0$$

$$2a_2 + 6a_3 x - 4a_1 x - 2a_0 - 2a_1 x + \sum_{n=2}^{\infty}\Big[(n+2)(n+1)a_{n+2} - 2(1+2n)a_n + 4a_{n-2}\Big]x^n = 0$$

the first terms give

$$a_2 = a_0\,, \qquad a_3 = a_1 \tag{11}$$

while the recursion formula is

$$(n+2)(n+1)a_{n+2} - 2(1+2n)a_n + 4a_{n-2} = 0\,, \qquad \text{for } n \geq 2. \tag{12}$$

Consider first the case of $n = 2p$ even. Using (11), we can use the recursion formula to obtain $a_4$. By repeated use of the recursion formula, we can obtain $a_6\,, a_8\,, \ldots$. After computing a few values, it appears that the general form is

$$a_{2p} = \frac{a_0}{p!}\,. \tag{13}$$

Note that (13) is also valid for $p = 0$ and $p = 1$. To test the validity of (13), we insert this equation into (12):

$$\frac{2(p+1)(2p+1)}{(p+1)!} - \frac{2(1+4p)}{p!} + \frac{4}{(p-1)!} \stackrel{?}{=} 0\,. \tag{14}$$

Simple algebra verifies the validity of the equation above. Next, consider the case of $n = 2p+1$ odd. Using (11), we can use the recursion formula to obtain $a_5$. By repeated use of the recursion formula, we can obtain $a_7\,, a_9\,, \ldots$. After computing a few values, it appears that the general form is

$$a_{2p+1} = \frac{a_1}{p!}\,. \tag{15}$$

Note that (13) is also valid for $p = 0$ and $p = 1$. To test the validity of (15), we insert this equation into (12):

$$\frac{2(p+1)(2p+3)}{(p+1)!} - \frac{2(3+4p)}{p!} + \frac{4}{(p-1)!} \stackrel{?}{=} 0\,. \tag{16}$$

Simple algebra verifies the validity of the equation above. Hence, we conclude that

$$\begin{aligned} y(x) = \sum_{n=0}^{\infty} a_n x^n &= \sum_{p=0}^{\infty} a_{2p}x^{2p} + \sum_{p=0}^{\infty} a_{2p+1}x^{2p+1} \\ &= a_0 \sum_{p=0}^{\infty} \frac{x^{2p}}{p!} + a_1 \sum_{p=0}^{\infty} \frac{x^{2p+1}}{p!} \\ &= a_0 \sum_{p=0}^{\infty} \frac{[x^2]^p}{p!} + a_1 x \sum_{p=0}^{\infty} \frac{[x^2]^p}{p!} \\ &= (a_0 + a_1 x)e^{x^2}\,. \end{aligned} \tag{17}$$

- reduction of the order: one checks that the equation is solved by $y_0(x) = e^{x^2}$; then we look for another solution of the form $y(x) = u(x)y_0(x)$:

$$y' = u'y_0 + uy_0', \qquad y'' = u''y_0 + 2u'y_0' + uy_0'' \tag{18}$$

$$y'' - 4xy' + (4x^2 - 2)y = u[y_0'' - 4xy_0' + (4x^2 - 2)y_0] + 2u'y_0' + u''y_0 - 4xu'y_0 = 0 \tag{19}$$

$$e^{x^2}[u'' - 4xu' + 4xu'] = 0 \implies u'' = 0 \implies u = A + Bx \tag{20}$$

$$y = (A + Bx)e^{x^2} \tag{21}$$

## 3 Boas, problem p.586, 12.11-5

Solve the following differential equations by the method of Frobenius:

$$2xy'' + y' + 2y = 0 \tag{22}$$

Substituting the generalized power series $y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$ in (22) we have

$$2x\sum_{n=0}^{\infty}(n+s)(n+s-1)a_n x^{n+s-2} + \sum_{n=0}^{\infty}(n+s)a_n x^{n+s-1} + 2\sum_{n=0}^{\infty} a_n x^{n+s} = 0 \tag{23}$$

$$\text{the } n = 0 \text{ term gives } \; [2s(s-1)a_0 + sa_0]x^{n+s-1} = 0 \implies 2s^2 - s = 0 \implies s = 0, \tfrac{1}{2} \tag{24}$$

$$\text{while the other terms are } \; \sum_{n=1}^{\infty}\Big[2(n+s)(n+s-1)a_n + (n+s)a_n + 2a_{n-1}\Big]x^{n+s-1} = 0 \tag{25}$$

$$\tag{26}$$

For $s = 0$ we have

$$\sum_{n=1}^{\infty}\Big[n(2n-1)a_n + 2a_{n-1}\Big]x^{n-1} = 0 \tag{27}$$

$$a_n = \frac{-2}{n(2n-1)}a_{n-1} = \ldots = \frac{(-2)^n}{n!(2n-1)!!}a_0 \tag{28}$$

Where the double factorial is $m!! = m(m-2)(m-4)\ldots$. Also, we note that $(2n)! = 2n(2n-1)(2n-2)(2n-3)\ldots = 2^n(2n-1)!!n!$. Inserting these coefficients back in the series gives

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\frac{(-2x)^n}{n!(2n-1)!!}a_0 = \sum_{n=0}^{\infty}\frac{(-4x)^n}{(2n)!}a_0 = a_0\sum_{n=0}\frac{(-1)^n(2\sqrt{x})^{2n}}{(2n)!} = a_0\cos(2x^{1/2}). \tag{29}$$

For $s = \frac{1}{2}$ we have instead

$$\sum_{n=1}^{\infty}\Big[n(2n+1)a_n + 2a_{n-1}\Big]x^{n-\frac{1}{2}} = 0 \implies a_n = \frac{-2}{n(2n+1)}a_{n-1} = \ldots = \frac{(-2)^n}{n!(2n+1)!!}a_0 \tag{30}$$

$$\implies y(x) = a_0\sum\frac{(-1)^n(2\sqrt{x})^{2n+1}}{(2n+1)!} = a_0\sin(2x^{1/2}) \tag{31}$$

The general solution is then given by the linear combination of (29), (31):

$$y = A\cos(2x^{1/2}) + B\sin(2x^{1/2}) \tag{32}$$

## 4 Boas, problem p.587, 12.11-14

Solve $y'' = -y$ by the Frobenius method.

We take the generalized power series $y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$ so that

$$y'' + y = \sum_{n=0}^{\infty} \left[ (n+s)(n+s-1)a_n x^{n+s-2} + a_n x^{n+s} \right] = 0 \tag{33}$$

Taking the $n = 0$ term gives $s(s-1)a_0 = 0$, that is, $s = 0, 1$. For $s = 0$ we have

$$n(n-1)a_n + a_{n-2} = 0 \implies a_n = \frac{-a_{n-2}}{n(n-1)} \implies \begin{cases} a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1 \\ a_{2n} = \frac{(-1)^n}{(2n)!} a_0 \end{cases} \tag{34}$$

These two series are those defining the sine and the cosine, so that we have found the well known result $y = a_0 \cos x + a_1 \sin x$.

If we now take $s = 1$ we have

$$(n+1)nb_n + b_{n-2} = 0 \implies b_n = -\frac{b_{n-2}}{n(n+1)} \tag{35}$$

The solution for $b_0 \neq 0$ is then

$$y(x) = \sum_{n=0} b_n x^{n+1} = b_1 \sin x + b_0 \sum_{n=1} \frac{(-1)^n}{(2n)!} x^{2n} \tag{36}$$

In this expression we are missing the $x^0$ term that would give the expansion of the cosine, and one easily check that this is not a solution of the differential equation. That happens because in (33) for $s = 1$ the first term coefficient reads $(n+1)(n+0)b_n$; then, $b_0$ coefficient has a 0 in front, which cancels it out from the rest of the problem, so we are calculating the solution *modulo* the constant $b_0$.

## 5 Boas, problem p.567, 12.2-2

Show that $P_l(-1) = (-1)^l$.

Using eq. (2.6) on p. 565 of Boas, the general solution to the Legendre differential equation is:

$$\begin{aligned} y(x) &= a_0 \left[ 1 - \frac{l(l+1)}{2!} x^2 + \frac{l(l+1)(l-2)(l+3)}{4!} x^4 - \dots \right] \\ &\quad + a_1 \left[ x - \frac{(l-1)(l+2)}{3!} x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{5!} x^5 - \dots \right]. \end{aligned} \tag{37}$$

If $\ell$ is even, then the Legendre polynomial is defined to be the polynomial proportional to $a_0$ (up to an overall normalization determined by convention). If $\ell$ is odd, then the Legendre polynomial is defined to be the polynomial proportional to $a_1$ (up to an overall normalization determined by convention). It immediately follows that if $\ell$ is even, then $P_\ell(x)$ is an even function of $x$, whereas if $\ell$ is odd, then $P_l(x)$ is an odd function of $x$. This means that

$$P_\ell(-x) = (-1)^l P_\ell(x). \tag{38}$$

The normalization convention for the Legendre polynomials defines $P_l(1) = 1$. Hence, inserting $x = 1$ into (38) yields

$$P_l(-1) = (-1)^l \tag{39}$$

Note that eq. (38) is also an immediate consequence of the Rodrigues' formula,

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l\,, \tag{40}$$

and provides another way of deriving (39).

## 6 Boas, problem p.567, 12.2-4

We will solve Legendre equation

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0 \tag{41}$$

using the method of reduction of order: given the known solution $P_l(x)$, we look for an independent solution of the form $y(x) = P_l(x)v(x)$ and then solve for $v(x)$ in (41):

$$(1-x^2)(v''P_l + 2v'P_l' + vP_l'') - 2x(v'P_l + vP_l') + l(l+1)P_l = 0 \tag{42}$$

$$(1-x^2)(v''P_l(x) + 2v'P_l'(x)) - 2xP_l(x)v' = 0 \implies (1-x^2)P_l(x)v'' + 2((1-x^2)P_l'(x) - xP_l(x))v' = 0$$

$$\implies \frac{v''}{v'} = 2\frac{xP_l(x) - (1-x^2)P_l'}{(1-x^2)P_l(x)} = 2\frac{x}{1-x^2} - 2\frac{P_l'}{P_l} = \frac{1}{1-x} - \frac{1}{1+x} - 2\frac{P_l'}{P_l} \tag{43}$$

which is solved by

$$\ln v' = -\ln(1-x) - \ln(1+x) - 2\ln P_l = \ln\frac{1}{(1-x)(1+x)P_l^2},\ \text{that is,} \tag{44}$$

$$v(x) = \int \frac{dx}{(1-x)(1+x)P_l^2} \tag{45}$$

The second solution of the Legendre equation is then

$$Q_l(x) = P_l(x)v(x) \tag{46}$$

We evaluate this expression for the two cases $l = 0, 1$:

- $l = 0$: $P_0(x) = 1$, so the other solution is

$$Q_0(x) = \int dx\frac{1}{(1-x)(1+x)} = \tfrac{1}{2}\int dx\left(\frac{1}{1-x} + \frac{1}{1+x}\right) = \tfrac{1}{2}\ln\frac{1+x}{1-x} \tag{47}$$

- $l = 1$: $P_1(x) = x$, so the other solution is

$$Q_1(x) = x\int dx\frac{1}{(1-x)(1+x)x^2} = x\int dx\left(\tfrac{1}{2}\frac{1}{1-x} + \tfrac{1}{2}\frac{1}{1+x} + \frac{1}{x^2}\right) = \tag{48}$$

$$= \frac{x}{2}\ln\frac{1+x}{1-x} - 1 \tag{49}$$

## 7 Boas, problem p.568, 12.3-1

We will use the hint in problem 12.3-6: if we write

$$\frac{d}{dx}uv = D(uv) = (D_u + D_v)uv \tag{50}$$

where $D_u$, $D_v$ are operators that act only separately on $u, v$, we have

$$\frac{d^n}{dx^n}uv = (D_u + D_v)^n uv = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) D_u^k D_v n - kuv = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) D_u^k u \, D_v^{n-k} v = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{d^k}{dx^k} u \frac{d^{n-k}}{dx^{n-k}} v \tag{51}$$

where we have used the expansion of the $n$-th power of a binomial formed by the two operators $D_u, D_v$ (which commute with each other). $\left( \begin{array}{c} n \\ k \end{array} \right) = \frac{n(n-1)\ldots(n-k+1)}{k!}$ is the binomial coefficient.

## 8 Boas, problem p.569, 12.4-2

By Rodrigues' formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \tag{52}$$

we have, after applying Leibniz' rule (51)

$$P_l(x) = \frac{1}{2^l l!} \sum_{k=0}^{l} \left( \begin{array}{c} l \\ k \end{array} \right) \frac{d^k}{dx^k} (x+1)^l \frac{d^{l-k}}{dx^{l-k}} (x-1)^l \tag{53}$$

Now, every time we differentiate $(x-1)^l$ we lower the exponent by one; in particular, when we differentiate $l$ times, we are left with a constant; when we calculate $P_l(1)$ any factor of $(x-1)$ will become zero, so that the only non zero contribution comes from the 0-th term in the sum: this gives

$$P_l(1) = \frac{2^l}{l!} \cdot 2^l \cdot l! = 1 \tag{54}$$

where the term $2^l$ comes from $(x+1)^l$ for $x = 1$ and $\frac{d^l}{dx^l}(x-1)^l = l \frac{d^{l-1}}{dx^{l-1}}(x-1)^{l-1} = l(l-1)\frac{d^{l-2}}{dx^{l-2}}(x-1)^{l-2} = \ldots = l!$.

## 9 Boas, problem p.569, 12.4-4

We want to prove that

$$\int_{-1}^{1} x^m P_l(x) dx = 0, \quad \text{for } m < l \tag{55}$$

Substituting Rodrigues' formula (52) we have

$$\begin{aligned} \int_{-1}^{1} x^m P_l(x) dx &= \int_{-1}^{1} \frac{x^m}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l dx \propto & (56) \\ &\propto x^m \frac{d^{l-1}}{dx^{l-1}} (x^2-1)^l \bigg|_{-1}^{1} - \int_{-1}^{1} m x^{m-1} \frac{d^{l-1}}{dx^{l-1}} (x^2-1)^l dx = & (57) \\ &= 0 - m x^{m-1} \frac{d^{l-2}}{dx^{l-2}} (x^2-1)^l \bigg|_{-1}^{1} + \int_{-1}^{1} m(m-1) x^{m-2} \frac{d^{l-2}}{dx^{l-2}} (x^2-1)^l dx = \ldots & (58) \end{aligned}$$

in the first passage we have neglected the constant $\frac{1}{2^l l!}$ and integrated by parts; in the second passage, we see that the first term is null because after we have differentiated $(l-1)$ times we will still have at least one factor of $(x-1)$ and one of $(x+1)$ (as you can quickly check by using Leibniz' rule), which are zero when evaluated at $\pm 1$. The same happens for all the other terms evaluated at $\pm 1$, so that, after we have integrated by parts $m$ times (assuming $m < l$), we are left with

$$m! \int_{-1}^{1} \frac{d^{l-m}}{dx^{l-m}} (x^2-1)^l dx = m! \left. \frac{d^{l-m-1}}{dx^{l-m-1}} (x^2-1)^l \right|_{-1}^{1} = 0 \tag{59}$$

for the same argument we used above. Note that this does not hold for $m > l$, because in that case between (58) and (59) we reach a step in which $l-k=0$ and we have $\int x^{m-k}(x^2-1)^l \neq 0$

## 10 Boas, problem p.574, 12.5-10

Express the following polynomial as a linear combination of the Legendre polynomials:

$$f(x) = x^4 \tag{60}$$

The first five Legendre polynomials are:

$$P_0 = 1\,, \qquad P_1 = x\,, \qquad P_2 = \frac{1}{2}(3x^2-1)\,, \qquad P_3 = \frac{1}{2}(5x^3-3x)\,, \qquad P_4 = \frac{1}{8}(35x^4-30x^2+3) \tag{61}$$

We are going to expand $x^4$ as a linear combination of the Legendre polynomials, with unknown coefficients; these will be found imposing that the factors for the different powers of $x$ coincide. Because we have $x^4$, $f(x) = \sum_0^4 c_n P_n$ must contain $P_4$; in particular, $c_4 = \frac{8}{35}$, so that the coefficient of $x^4$ is 1. Then we must put to zero the coefficient of $x^3$: $x^3$ only appears in $P_3$ so we can put $c_3 = 0$. Right now, our function is written as

$$f(x) = \sum_0^2 c_n P_n + \frac{8}{35} P_4(x) \tag{62}$$

Now we fix to zero the coefficient of $x^2$: it appears in $P_4$ and $P_2$ and it is

$$\frac{3}{2}c_2 + \frac{-30}{35} = 0 \implies c_2 = \frac{4}{7}$$

A term linear in $x$ appears only in $P_1$, so we can set $c_1 = 0$. Finally, the constant term is given by

$$c_0 - \frac{1}{2}c_2 + \frac{3}{8}c_4 = 0 \implies c_0 = \frac{1}{5} \tag{63}$$

Then we have found

$$x^4 = \frac{1}{5}P_0(x) + \frac{4}{7}P_2(x) + \frac{8}{35}P_4(x) \tag{64}$$

## 11 Boas, problem p.577, 12.6-6

We want to show that $P_l$ and $P_l'$ are orthogonal on [-1,1] in two ways:

- we can use the fact that the Legendre polynomials are either even or odd functions of $x$ (depending on whether $\ell$ is even or odd, respectively), as shown in problem 5. Then, if $P_l$ is odd, its derivative $P_l'$ is even, and vice versa. In general, if $f(x)$ is an even function of $x$ and $g(x)$ is an odd function of $x$, then

$$\int_{-a}^{a} f(x)g(x) = 0\,. \tag{65}$$

This is easily proven by changing the integration variable to $y = -x$, in which case

$$\int_{-a}^{a} f(x)g(x) = -\int_{+a}^{-a} f(-y)g(-y)dy = \int_{-a}^{a} f(-y)g(-y)dy = -\int_{-a}^{a} f(y)g(y)dy\,, \tag{66}$$

As the integral is equal to minus itself, it must be equal to zero. Hence, we conclude that

$$\int_{-1}^{1} P_l(x)P_l'(x) = 0\,. \tag{67}$$

- we can also use the result of problem 9: remember that $P_l$ is a polynomial of order $l$ and $P_l'$ is a polynomial of order $(l-1)$. Then

$$\int_{-1}^{1} P_l(x)P_l'(x) \tag{68}$$

is given by a sum of terms which have the form $c_n \int_{-1}^{1} x^m P_l(x)dx$, where $m = 0, 1, \ldots, l-1$, that is, $m < l$, so they are all zero and the two functions are orthogonal.

## 12 Boas, problem p.615, 12.23-2

The generating functional of the Legendre polynomials is

$$\Phi(x,h) = \frac{1}{\sqrt{1-2xh+h^2}} = \sum_{l=0}^{\infty} h^l P_l(x)\,; \tag{69}$$

for $x = 0$ this gives

$$\Phi(0,h) = \sum_{l=0}^{\infty} h^l P_l(0) = \frac{1}{\sqrt{1+h^2}} = \sum c_l h^l\,, \tag{70}$$

But the function $\Phi(0,h)$ looks exactly like the power of a binomial:

$$\Phi(0,h) = (1+h^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} h^{2n} \tag{71}$$

Here we can read the Legendre polynomials in zero as

$$P_{2n+1}(0) = 0\,; \tag{72}$$

$$P_{2n}(0) = \binom{-1/2}{n} = \frac{-\frac{1}{2}(-\frac{1}{2}-1)\ldots(-\frac{1}{2}-n+1)}{(n)!} = \frac{(-1^n)(2n-1)!!}{2^n n!}. \tag{73}$$