Chapter 7.1, Problem 22E

Problem

Use the unique factorization for the integers theorem and the definition of logarithm to prove that log3(7) is irrational.

Theorem

Unique Factorization of Integers Theorem (Fundamental Theorem of Arithmetic)

Given any integer *n* > 1, there exist a positive integer *k*, distinct prime numbers *p*1*, p*2*, …, pk*, and positive integers *e*1*, e*2*, …, ek* such that

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \dots p_k^{e_k},$$

and any other expression for *n* as a product of prime numbers is identical to this except, perhaps, for the order in which the factors are written.

Step-by-step solution

Step 1 of 2

Suppose that $\log_3 7$ is rational.

Then $\log_3 7 = \frac{p}{q}$ for some integers p and q with $q \neq 0$.

As logarithms are positive valued, p and q must be positive.

Use definition of logarithm to write,

$$3^{\frac{p}{q}} = 7$$

Apply b^{th} power to both sides,

$$\left(3^{\frac{p}{q}}\right)^q = 7^q$$

$$3^p = 7^q$$

Step 2 of 2

Let $N = 3^p = 7^q$, and take its prime factorization.

As $N = 3^p$, the prime factors are all 3 and also as $N = 7^q$, the prime factors are all 7 .

Since 3 and 7 are co-prime, no integer power of 3 is equal to an integer power of 7 .

This is a contradiction to the unique factorization theorem.

So the supposition is wrong, and therefore, $\log_3 7$ is irrational.