Ch.6 Eigen Value And Eigen Vectors

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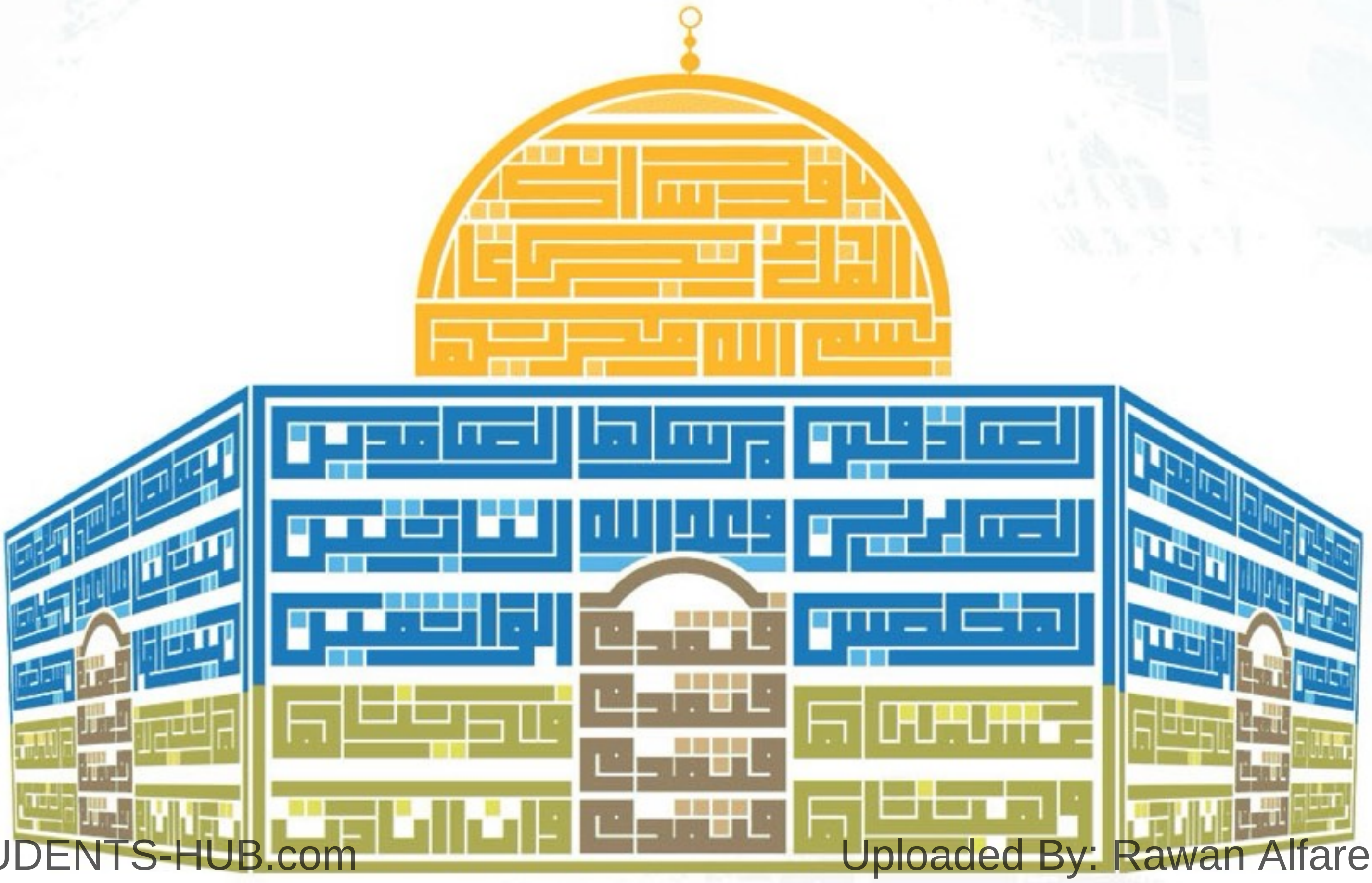

introduction

$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

→ $AX = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3X$ ⟶ قدرنا نكتب الجواب بدلالة λx

→ $y = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $Ay = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ ⟶ $3y$ لكن هون ما بنقدر نكتبها بدلالة

Remarks:-

Given $A_{n\times n}$. Is there $X \in \mathbb{R}^n$ such that, $AX = \lambda X$, $X \neq 0$

λ is scalor , Scalor: Complex number

Def:-

Given $A = (a_{ij})$ be a matrix. A scalor $\lambda \in \mathbb{C}$ is called eigen value for A if $\exists$ $X \neq 0 \in \mathbb{R}^{n\times n}$ such that $\underline{AX = \lambda X}$

EX:- $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda = 3$

$AX = 3X$, $\left.\begin{matrix} \lambda = 3 \text{ is eigen value for } A \\ X = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ is eigen vector for } A. \end{matrix}\right\}$ → * ولكن ممكن يكون موجود eigen value لما فلازم نوجدهم

⟹ find all eigen values and all eigen vectors

Remarks:-

Given $A = (a_{ij})_{n\times n}$

λ is eigen value for A ⟺ their exsists $X \neq 0 \in \mathbb{R}^n$ s.t $AX = \lambda X$

⟺ their exsists $X \neq 0 \in \mathbb{R}^n$ s.t $AX - \lambda X = 0$

⟺ their exists $X \neq 0 \in \mathbb{R}^n$ s.t $(A - \lambda I)X = 0$ → كتبنا I لأنه مستحيل نلاقي matrix – رقم فلذلك ضربنا λ بـ I. عشان تصير matrix – matrix

* $(A - \lambda I)X = 0$ → $X \neq 0$ has the non zero solution.

so, $(A - \lambda I)$ is singular.

so, $\det(A - \lambda I) = 0$

* so first we find eigen value by $|A - \lambda I| = 0$ ⟶ find λ

* then find eigen vector by use $(A - \lambda I)X = 0$ ⟶ find X

Remark:-

let A be $n\times n$ matrix, we define characterstic Polynomial of A as

* $P(A) = \det(A - \lambda I) = |A - \lambda I|$

* set $P(A) = 0$ to find the eigen values ⟹ find all roots of $P(A)$.

* Now, we want to solve the example that we've written it above :-

EX. $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$, find all eigen values of A and corresponding eigen vectors.

1. $|A - \lambda I| = 0$ $\quad \left| \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$

$$\begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) + 2 = 6 + -5\lambda + \lambda^2$$

$P(\lambda) = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 3)(\lambda - 2) = 0$

$\lambda = 3, \quad \lambda = 2$

لكل eigen value
في الها eigen vector

to find eigen vectors :-

$\Rightarrow$ ① for $\lambda = 2$, solve $(A - 2I)x = 0$

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\left.\begin{matrix} x_2 = \alpha \\ x_1 = \alpha \end{matrix}\right\} \Rightarrow x = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha \neq 0$$

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② $\Rightarrow$ for $\lambda = 3$

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$x_2 = \beta$

$x_1 - 2\beta = 0 \rightarrow x_1 = 2\beta$

$$x = \begin{bmatrix} 2\beta \\ \beta \end{bmatrix} = \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \beta \neq 0$$

Remarks:-

1. $P(\lambda)$ is polynomial of degree n.

2. $p(\lambda)$ has at most n roots in $\mathbb{R}$.
but in complex number, it has exactly $\underline{n}$ roots (even if some roots were repeated).

3. $p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots\cdots (\lambda - \lambda_n)$

Example 2:

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

info:
- $\deg(p(\lambda)) = 3$
- Sign for $p(\lambda) = (-1)^3 = -1$

$\det(A - \lambda I)$

$$\begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = \begin{array}{c} + \\ - \\ + \end{array}\begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 0 & \lambda-1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ \lambda-1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} -3 & 1 \\ \lambda-1 & 1-\lambda \end{vmatrix} + 0$$

$$= (2-\lambda)\left[(-2-\lambda)(1-\lambda) - (\lambda-1)\right] + \left[+3(1-\lambda) + (\lambda-1)\right]$$

$$= (2-\lambda)(1-\lambda)\left[(-2-\lambda)+1\right] + \left[3 - 3\lambda + \lambda - 1\right]$$

$$= (2-\lambda)(1-\lambda)\left[(-1-\lambda)\right] + \left[2 - 2\lambda\right]$$

$$= -(2-\lambda)(1-\lambda)(1+\lambda) + 2(1-\lambda)$$

$$= -(2-\lambda)(1-\lambda^2) + 2(1-\lambda)$$

$$= (1-\lambda)\left[-(2-\lambda)(1+\lambda) + 2\right]$$

$$-(\cancel{2} + 2\lambda - \lambda - \lambda^2) + \cancel{2}$$

$$-\lambda + \lambda^2$$

$$= (1-\lambda)(\lambda)(\lambda-1)$$
$$= -\lambda(1-\lambda)^2 = 0$$

$$\lambda = 0, \quad \underbrace{\lambda = 1, \ \lambda = 1}_{\text{repeated roots.}}$$

① eigen vectors for $\lambda = 0$

$$A - 0 = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 1 & -3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_3 &= \alpha \\ x_2 &= \alpha \\ x_1 &= \alpha \end{aligned} \longrightarrow X = \underbrace{\begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix}}_{\text{eigen vector.}} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \alpha \neq 0$$

$\Rightarrow$ Basis for eigen spaces of $\lambda_1 = 0$ is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$\Rightarrow$ $\dim(E(\lambda_1 = 0)) = 1$

② eigen vectors for $\lambda = 1$

$$A - I = \begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = \beta_1$

$x_2 = \beta_2$

$x_1 - 3\beta_2 + \beta_1 = 0$

$\boxed{x_1 = 3\beta_2 - \beta_1}$

$$X = \underbrace{\begin{bmatrix} 3\beta_2 - \beta_1 \\ \beta_2 \\ \beta_1 \end{bmatrix}}_{\text{Eigen Vector}} = \beta_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \beta_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\beta_1, \beta_2 \neq 0$ are scalars.

$\Rightarrow$ Basis for Eigen vectors when $\lambda_2 = 1$ are $\left\{ \underbrace{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\text{Basis.}} \right\}$

$\Rightarrow$ $\dim(E(\lambda = 1)) = 2$

Example 3:

$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, find 1. Eigen vectors and values.
2. basis and dim .

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4$
$= 5 - 2\lambda + \lambda^2$
$= \lambda^2 - 2\lambda + 5$

$A = 1$, $B = -2$, $C = 5$

المميز سالب
⇓
So we have Complex roots.

$\dfrac{-B \mp \sqrt{B^2 - 4AC}}{2A}$

$= \dfrac{2 \mp \sqrt{4 - 20}}{2} = \dfrac{2 \mp 4i}{2}$ → $\lambda = 1 - 2i$, $\lambda = 1 + 2i$

① Eigen Vectors for $\lambda_1 = 1 - 2i$

$A - (1-2i)I = \begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 2i \\ 2i & 2 \end{bmatrix} \xrightarrow{iR_1 + R_2} \begin{bmatrix} -2 & 2i \\ 0 & 0 \end{bmatrix}$

$2i^2 + 2$
$-2 + 2 = 0$

note $i * i = -1$

$x_2 = \alpha$
$-2x_1 + 2i\alpha = 0$
$-2x_1 = -2i\alpha$
$\boxed{x_1 = i\alpha}$

Eigen vectors $\left\{ \begin{pmatrix} i\alpha \\ \alpha \end{pmatrix} \right\}$, $\alpha \neq 0$ scalor.

basis $= \alpha \underbrace{\begin{pmatrix} i \\ 1 \end{pmatrix}}_{\text{basis}}$

dim $= 1$

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② Eigen vector when $\lambda = 1 + 2i$

$A - (1+2i)I = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & -2i \\ -2i & 2 \end{bmatrix} \xrightarrow{-iR_1 + R_2} \begin{bmatrix} -2 & -2i \\ 0 & 0 \end{bmatrix}$

ROPs
$-(-2i)i + 2$
$= +2i^2 + 2 = 0$
$= (+2)(-1) + 2 = 0$

$\boxed{x_2 = \beta}$

$-2x_1 - 2i\beta = 0$
$-2x_1 = +2i\beta$
$\boxed{x_1 = -i\beta}$

Eigen Vectors $= \begin{bmatrix} -i\beta \\ \beta \end{bmatrix}$, $\alpha \neq 0$

* Basis $= \begin{pmatrix} -i \\ 1 \end{pmatrix}$, dim $= 1$

* Eigen space $= \begin{bmatrix} -i\beta \\ \beta \end{bmatrix}$, β scalar $= \beta \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Could be Complex or Real number

A is non singular matrix since no $\lambda = 0$

properties about Eigen values :

1. if $\lambda_1 = a + ib$ is a complex eigenvalue, then $\lambda_2 = a - ib$ is also an eigenvalue (conjecunt).

ex. if $\lambda = -i$ is an eigen value for A, then $\lambda = i$ is also eigen value

2. $\lambda = 0$ is eigen value for $A_{n \times n}$, $\Leftrightarrow$ $p(0) = 0$,
$\Leftrightarrow$ 0 is a root of $p(\lambda)$.
$\Leftrightarrow$ $|A - 0I| = 0$
$|A| = 0 \rightarrow$ A is singular.

* $\lambda = 0$ is not an Eigen Value for $A \Leftrightarrow A$ is non singular

* يعني اذا كانه جواب $\lambda = 0$ مش دايمًا بتكونه eigen value
منم نفحص الشروط الكي فوق.

3. the product of All roots (eigen Values) of A , will equal the det(A)

$\{\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n\} \rightarrow (\lambda_1)(\lambda_2)(\lambda_3)(\lambda_4) \ldots (\lambda_n) = \det(A)$.

4. the Sum of all roots (all eigen Values) will give you Trace of A.

$tr(A) = \sum_{i=1}^{n} a_{ii} \rightarrow$ the Sum of all elements in main diagonal.

ex. $\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \rightarrow \lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 1$ $\Big\} \rightarrow$ المثال محلول فوق بالتفاصيل لكن لتوضيح بعض النقاط من الخصائص.

tr(A)

① since $\exists \boxed{\lambda = 0}$ so A is Singular

to make sure from property 3 $\quad \det(A) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$
$= 0 \cdot 1 \cdot 1 = 0$

② $\lambda_1 + \lambda_2 + \lambda_3 = 0 + 1 + 1 = 2$.
$tr(A) = 2 + 2 - 2 = 2$

Same thing on the last example.

$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $\lambda_1 = 1+2i$
$\lambda_2 = 1-2i$

$\det(A) = 1+4 = 5$

① $\lambda \neq 0$ so A is non singular

② $\lambda_1 . \lambda_2 = \det(A) = (1+2i).(1-2i)$
$= 1 - \cancel{2i} + \cancel{2i} + 4$
$= 5$

وحسبناها بطريقة ثانية
$\det(A) = 5$.

③ $\lambda_1 + \lambda_2 = 1+2i + 1-2i = 2$
$tra(A) = 1+1 = 2$

5. if A is a real matrix, $a_{ij} \in \mathbb{R}$ and $\lambda = a+ib$ is a complex eigen value with x complex eigen vector. , then $\bar{\lambda} = a-ib$ is complex eigen value with x complex eigon vector. ($\bar{x}$) of x.

نسبة للمثال السابق

$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $\lambda_1 = 1+2i \longrightarrow x_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\lambda_2 = \bar{\lambda}_1 = 1-2i \longrightarrow x_2 = \bar{x}_1 = \begin{bmatrix} \overline{(-i)} \\ \overline{(1)} \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

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Important Example

if $A_{4\times4}$, eigen values : $\lambda_1 = 1-i$, $\lambda_2 = i$
find $\det(A)$
$\lambda_3 = 1+i$ $\lambda_4 = -i$

① الماتركس السابق تبعها $4 =$
فلذلك لازم يكون عندك 4 أجوبة
لـ λ.

② لما إني في Complex roots فبالتالي
لازم كل root يكون في إله معاكسه.

$\Rightarrow \det(A) = \lambda_1 . \lambda_2 . \lambda_3 . \lambda_4$
$= (i)(-i)(1-i)(1+i)$
$= (1)(1+1 \cancel{-i} \cancel{+i})$
$= 2$
so A is nonsingular.

$\Rightarrow tr(A) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$
$= \cancel{i} \cancel{-i} + 1 \cancel{-i} + 1 \cancel{+i}$
$= 2$

رَبَّنَا تَقبَّلْ مِنَّا إِنَّكَ أنتَ السَّمِيعُ العَلِيمُ

22 | January | 2025

روان الفارس

اللهم استخدمنا ولا تستبدلنا