10.10 The Binomial Series and Applications of Taylor series.

* The Binomial Series:

for $-1 < x < 1$, $(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$

* $\binom{m}{1} = m$

* $\binom{m}{2} = \dfrac{m(m-1)}{2}$

* $\binom{m}{k} = \dfrac{m!}{k!(m-k)!} = \dfrac{m(m-1)\cdots(m-k+1)}{k!}$

* $\binom{-1}{k} = (-1)^k$

Frequently used Taylor series:

* $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$

* $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| <$

* $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$

* $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

* $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

* $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$

for $-1 < x \leq 1$

* $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad , \quad |x| \leq 1$$

Questions: 10, 16, 26, 30, 33

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$\boxed{10}$ Find the first four terms of the binomial series for the function $\dfrac{x}{\sqrt[3]{1+x}}$

$$\frac{x}{\sqrt[3]{1+x}} = x(1+x)^{-\frac{1}{3}}$$

$$= x\left(1+\sum_{k=1}^{\infty}\binom{-\frac{1}{3}}{k}x^k\right)$$

$$= x\left(1+\left(-\tfrac{1}{3}\right)x+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}x^2+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!}x^3+\cdots\right)$$

$$= x - \frac{1}{3}x^2 + \frac{2}{9}x^3 - \frac{14}{81}x^4 + \cdots$$

$\boxed{16}$ Use the series to estimate the integral's values with an error of magnitude less than 10^{-3}.

$$\int_0^{0.2}\frac{e^{-x}-1}{x}\,dx$$

$$e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

$$\frac{e^{-x}-1}{x} = \frac{1}{x}\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots-1\right)$$

$$\int_0^{0.2} \left(-1 + \frac{x}{2} - \frac{x^2}{6} + \frac{x^3}{24} - \cdots\right) dx$$

$$= -x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{96} + \cdots \Big|_0^{0.2}$$

$$\frac{(0.2)^4}{96} \simeq 0.00002 < 0.0001$$

$$\text{so} \simeq -0.2 + \frac{(0.2)^2}{4} - \frac{(0.2)^3}{18} = -0.19044$$

$$|E| < 0.00002$$

$\boxed{26}$ Find a polynomial that will approximate $F(x)$ throughout the given interval with an error of magnitute less than 10^{-3}

$$F(x) = \int_0^x t^2 e^{-t^2}\, dt \quad , \quad [0, 1]$$

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \frac{t^{10}}{5!} + \cdots$$

$$t^2 e^{-t^2} = t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + \frac{t^{10}}{4!} - \frac{t^{12}}{5!} + \cdots$$

$$\int_0^x t^2 e^{-t^2}\,dx = \int_0 \left(t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + \cdots\right)dt$$

$$= \left[\frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7 \cdot 2!} - \frac{t^9}{9 \cdot 3!} + \cdots\right]\Big|_0^x$$

$$= \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7 \cdot 2!} - \frac{x^9}{9 \cdot 3!} + \frac{x^{11}}{11 \cdot 4!} \Big| - \frac{x^{13}}{13 \cdot 5!} + \cdots$$

$$|E| < \frac{(1)^{13}}{13 \cdot 5!} \approx 0.00064$$

$\boxed{30}$ Use series to evaluate the limit.

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\lim_{x \to 0} \frac{1}{x}\left(e^x - e^{-x}\right) = \lim_{x \to 0} \frac{1}{x}\left[\left(\cancel{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(\cancel{1} - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\right]$$

$$= \lim_{x \to 0} \frac{1}{x}\left[2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots\right]$$

$$= \lim_{x \to 0} \left[2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \cdots\right] = 2$$

$\boxed{33}$ use series to evaluate

$$\lim_{y\to 0} \frac{y - \tan^{-1} y}{y^3}$$

$$\tan^{-1}(y) = y - \frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} + \cdots$$

$$\frac{1}{y^3}\left(y - \tan^{-1} y\right) = \frac{1}{y^3}\left[\cancel{y} - \left(\cancel{y} - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \cdots\right)\right]$$

$$= \frac{1}{y^3}\left[\frac{y^3}{3} - \frac{y^5}{5} + \frac{y^7}{7} - \cdots\right]$$

$$= \frac{1}{3} - \frac{y^2}{5} + \frac{y^4}{7} - \cdots$$

$$\lim_{y\to 0} \frac{1}{y^3}\left(y - \tan^{-1} y\right) = \lim_{y \to 0}\left(\frac{1}{3} - \frac{y^2}{5} + \frac{y^4}{7} + \cdots\right)$$

$$= \frac{1}{3}$$