

Ch 11 Parametric Equations and Polar Coordinates

Note Title

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11.1 Parametrizations of Plane Curves

DEFINITION If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

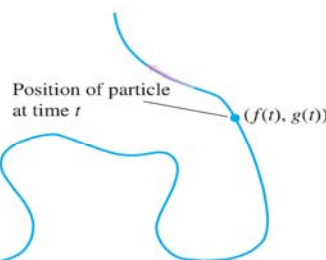
over an interval I of t -values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

In this case, t is a **parameter** for the curve.

If $t \in I = [a, b]$ closed interval, then the point $(f(a), g(a))$ is the **initial point** and $(f(b), g(b))$ is the **terminal point** of the curve.

The equations and interval together constitute a **parametrization** of the curve.

Examples: The following parametric eqs describe the position $P(x, y)$ of a particle moving in the xy -plane.



- Identify the path traced by the particle.
- Graph its cartesian eq on the given interval.
- Describe the direction of the motion.

1) $x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq 2\pi$

sol: a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$,

which is the circle centered at the origin with radius 1.

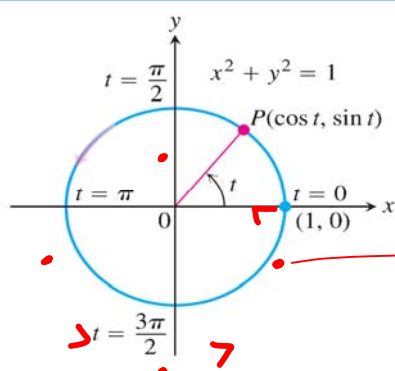
b) At $t=0$, $(x, y) = (\cos 0, \sin 0) = (1, 0)$. $\longrightarrow$ initial point

at $t = \frac{\pi}{2}$, $(x, y) = (\cos \frac{\pi}{2}, \sin \frac{\pi}{2}) = (0, 1)$,

at $t = \pi$, $(x, y) = (-1, 0)$,

at $t = \frac{3\pi}{2}$, $(x, y) = (0, -1)$, and

at $t = 2\pi$, $(x, y) = (1, 0) \longrightarrow$ terminal point.



initial point = terminal point

- c) The particle moves once on the circle $x^2 + y^2 = 1$ counterclockwise starting at the point $P(1, 0)$ and the arrow shows the direction of increasing t .

2) $x = a \sin 2t$, $y = a \cos 2t$, $0 \leq t \leq \pi$

sol: (a) $x^2 + y^2 = a^2 \sin^2 2t + a^2 \cos^2 2t = a^2$

which is the circle centered at the origin with radius a .

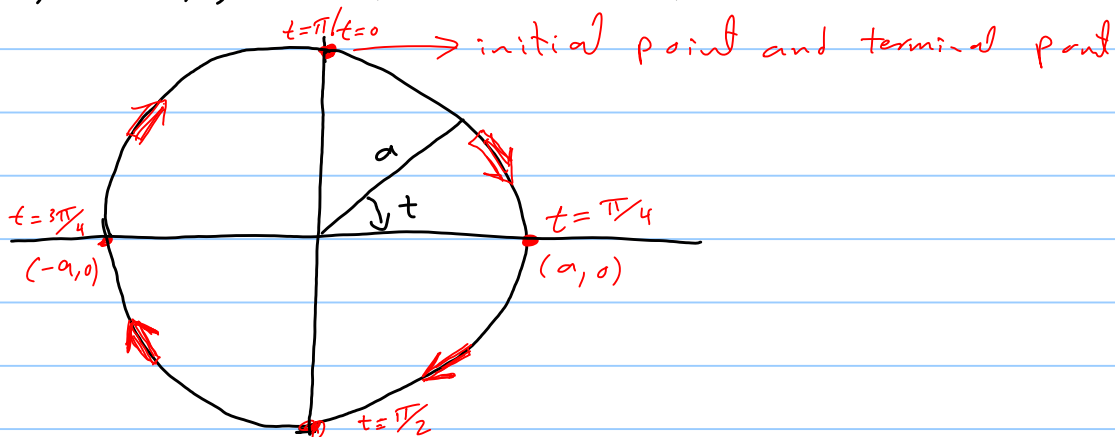
(b) At $t=0$, $(x, y) = (a \sin 0, a \cos 0) = (0, a)$. $\rightarrow$ initial point

at $t = \frac{\pi}{4}$, $(x, y) = (a \sin \frac{\pi}{2}, a \cos \frac{\pi}{2}) = (a, 0)$,

at $t = \frac{\pi}{2}$, $(x, y) = (0, -a)$,

at $t = \frac{3\pi}{4}$, $(x, y) = (-a, 0)$, and

at $t = \pi$, $(x, y) = (0, a)$ $\rightarrow$ terminal point.



(b)

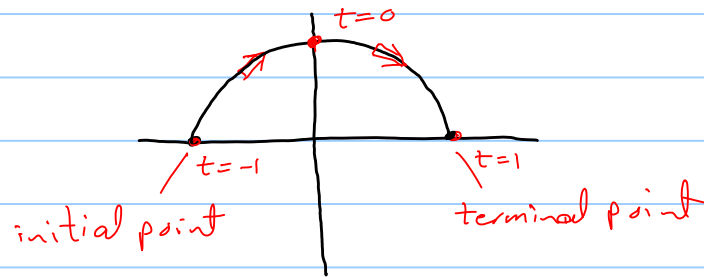
- c) The particle moves once on the circle $x^2 + y^2 = a^2$ clockwise starting at the point $P(0, 1)$, and the arrow shows the direction of increasing t .

Remark: In Example 1 above, if the interval of t is $0 \leq t \leq \pi$, then the graph of the path is the upper half of the circle.

3) $x = t, \quad y = \sqrt{1-t^2}, \quad -1 \leq t \leq 1$

sol: a) $y = \sqrt{1-x^2} \Rightarrow x^2 + y^2 = 1$. which is the circle centered at the origin with radius 1.

(b) At $t = -1$: $(x, y) = (-1, 0)$, $\rightarrow$ initial point
at $t = 0$: $(x, y) = (0, 1)$, and
at $t = 1$: $(x, y) = (1, 0)$ $\rightarrow$ terminal point



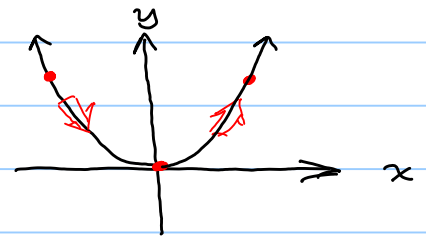
c) The particle moves on the upper half of the circle in clockwise direction.

Remark: Clearly, any curve can be represented by many different set of parametric eqs.

4) $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$

sol: (a) $y = (3t)^2 \Rightarrow y = x^2$, which is parabola open up.

(b) If $t = -1$, then $(x, y) = (-3, 9)$,
 if $t = 0$, then $(x, y) = (0, 0)$,
 and if $t = 1$, then $(x, y) = (3, 9)$,



(c) The particle moves down from the left hand side of the parabola, passes through the origin, then moves up on the right hand side of the graph.

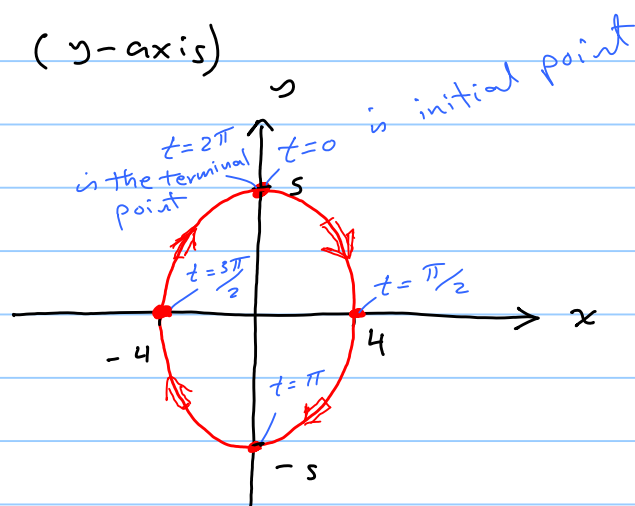
5) $x = 4 \sin t, \quad y = 5 \cos t, \quad 0 \leq t \leq 2\pi$

sol: $\frac{x}{4} = \sin t, \quad \frac{y}{5} = \cos t \Rightarrow$

$$\frac{x^2}{4^2} + \frac{y^2}{5^2} = \sin^2 t + \cos^2 t = 1 \quad \text{which is eqn of}$$

an ellipse with axis $x=0$ (y -axis)

- b) At $t=0$, $(x,y) = (0,5)$,
 at $t = \frac{\pi}{2}$, $(x,y) = (4,0)$
 $\vdots$
 at $t = 2\pi$, $(x,y) = (0,5)$



- c) The particle moves once on the ellipse $\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$ clockwise and the arrow shows the direction of increasing t .

6) The following is Example 1 in book.

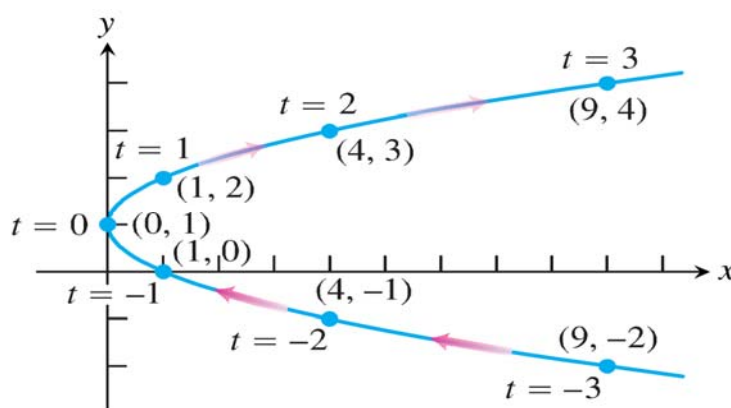


FIGURE 11.2 The curve given by the parametric equations $x = t^2$ and $y = t + 1$ (Example 1).

$$-\infty < t < \infty$$

7) $x = -\sec t$, $y = \tan t$ $-\frac{\pi}{2} < t < \frac{\pi}{2}$

sol: a) $x^2 - y^2 = \sec^2 t - \tan^2 t = 1$ which is the eqn of a hyperbola with axis $y=0$ (x -axis)

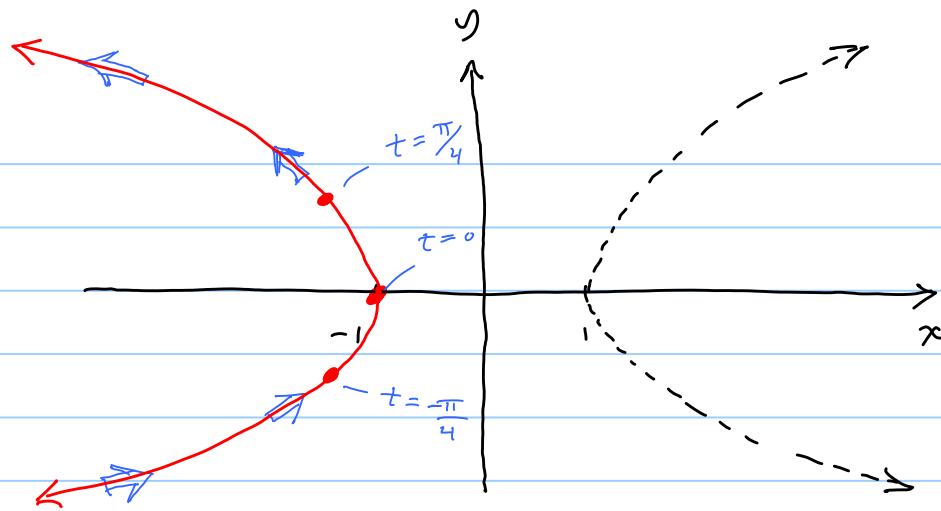
b) As $t \rightarrow -\frac{\pi}{2}$, $x \rightarrow -\infty$, $y \rightarrow -\infty$

at $t = -\frac{\pi}{4}$, $x = -\sqrt{2}$, $y = -1$

at $t = 0$, $x = -1$, $y = 0$

at $t = \frac{\pi}{4}$, $x = -\sqrt{2}$, $y = 1$

As $t \rightarrow \frac{\pi}{2}$, $x \rightarrow -\infty$, $y \rightarrow \infty$



Parametric Egn of a Line

a) If L is a line passes through the two points (x_0, y_0) and (x_1, y_1) , then we can parametrize this line as follows:

$$x = x_0 + (x_1 - x_0)t, \quad y = y_0 + (y_1 - y_0)t, \quad -\infty < t < \infty$$

b) If L is a line passes through the point (x_0, y_0) with slope m then we can parametrize this line as follows:

$$x = x_0 + t, \quad y = y_0 + mt, \quad -\infty < t < \infty$$

Example: Find two different parametrization for the line segment with endpoints $(-1, 3)$ and $(3, -2)$

سأ: لاحظ في البداية أننا بصدد إيجاد معادلة وسيطة لقطعة مستقيمة وليست للخط (مستقيم كله وعليه فإننا **لا نأخذ** $-\infty < t < \infty$ لذا يجب إيجاد قيم t تكون نقطة البداية $(-1, 3)$ ونقطة النهاية $(3, -2)$.
يستخدم (a)

$$x = -1 + (3 - (-1))t, \quad y = 3 + (-2 - 3)t \Rightarrow$$

$$x = -1 + 4t, \quad y = 3 - 5t \quad 0 \leq t \leq 1$$

$$\underline{\text{تستقيم } t} \quad m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{-5}{4}$$

$$\Rightarrow x = -1 + t, \quad y = 3 - \frac{5}{4}t \quad 0 \leq t \leq 4$$

لا حظ أن في $t=4$ هي التي تجعل قيمة (x, y) مساوية لـ $(3, -2)$.