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MATH1321

Calculus 2

Chapter 10.2

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10.2 Infinite Series

$a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n$

$S_1 = a_n$ sum of 1^{st} term

$S_2 = a_1 + a_2$ sum of 1^{st} two terms

$S_3 = a_1 + a_2 + a_3$ sum of 1^{st} three terms

$\vdots$

$S_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{k=1}^{n} a_k$ sum of the 1^{st} n^{th} terms

Main Goal is to find if

$\sum_{n=1}^{\infty} a_n$ → conv. by test...

$\sum_{n=1}^{\infty} a_n$ → div. by test...

Test 1: n^{th} Partial Sum Test (for conv. and div.)

If $\lim_{n\to\infty} S_n = L^{\mathbb{R}}$ then $\sum_{n=1}^{\infty} a_n$ conv. to L.

If $\lim_{n\to\infty} S_n$ div. then $\sum_{n=1}^{\infty} a_n$ div.

Ex. Use n^{th} Partial sum test to test the conv./div. of ?

1) $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$ → Telescoping

$S_n = \sqrt{2} - \sqrt{1} + \sqrt{3} - \sqrt{2} + \sqrt{4} - \sqrt{3} + \dots + \sqrt{n+1} - \sqrt{n}$

$S_n = \sqrt{n+1} - 1$

$\lim_{n\to\infty} S_n = \infty \longrightarrow \sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$ div. by n^{th} Partial Sum test.

2) $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ $\quad \frac{A}{n} + \frac{B}{n+1}$

*cover method A=1, B=-1 *

$= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ → Telescoping

$S_n = (1 - \cancel{\frac{1}{2}}) + (\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}) + (\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}) + \dots + (\cancel{\frac{1}{n}} - \frac{1}{n+1})$

$S_n = 1 - \frac{1}{n+1}$ $\quad \lim_{n\to\infty} S_n = 1 - 0 = 1$

$\longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+n}$ conv. to 1 by n^{th} Partial Sum test.

3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$ → Telescoping

$S_n = (1 - \cancel{\frac{1}{\sqrt{2}}}) + (\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}) + (\cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{\sqrt{4}}}) + \dots + (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}})$

$S_n = 1 - \frac{1}{\sqrt{n+1}}$ $\quad \lim_{n\to\infty} S_n = 1$

$\longrightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$ conv. to 1 by n^{th} Partial sum test.

4) $\sum_{n=1}^{\infty} \frac{1}{3^n}$

$S_n = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots + \frac{1}{3^n}$

we can't use the n^{th} Partial sum test.

5) $\sum_{n=1}^{\infty} (\ln\sqrt{n+1} - \ln\sqrt{n})$ → Telescoping

$S_n = \cancel{\ln\sqrt{2}} - \underbrace{\ln\sqrt{1}}_{} ^{0} + \cancel{\ln\sqrt{3}} - \cancel{\ln\sqrt{2}} + \dots + \ln\sqrt{n+1} - \cancel{\ln\sqrt{n}}.$

$S_n = \ln\sqrt{n+1}$ $\quad \ln(\lim_{n\to\infty} \sqrt{n+1}) = \infty$

$\longrightarrow \sum_{n=1}^{\infty} a_n$ div. by n^{th} Partial Sum test.

Test 2: n^{th} term Test (for div. only)

Look for $\sum_{n=1}^{\infty} a_n$

check if $\lim_{n\to\infty} a_n \neq 0$

Then $\sum_{n=1}^{\infty} a_n$ div.

also If $\lim_{n\to\infty} a_n$ fails to exist $(\infty, -\infty, \text{DNE})$ Then $\sum_{n=1}^{\infty} a_n$ div.

Ex. 1) $\sum_{n=1}^{\infty} \frac{2n+5}{1-3n}$

$\lim_{n\to\infty} \frac{2n+5}{1-3n} = \frac{2}{-3} \neq 0$

This infinite series div. by n^{th} term test.

2) $\sum_{n=1}^{\infty} \frac{1}{n}$ Harmonic Series

$\lim_{n\to\infty} \frac{1}{n} = 0$ n^{th} term test does not work

$S_n = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \dots)$

$\geq 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + \dots$

$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty$

$\longrightarrow \sum_{n=1}^{\infty} \frac{1}{n} \geq \infty$ so it is div.

Note

1) If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ may be div or con.

2) If $\sum_{n=1}^{\infty} a_n$ conv. then $\lim_{n\to\infty} a_n = 0$.

converse is not true $(\sum_{n=1}^{\infty} \frac{1}{n})$

$\lim_{n\to\infty} a_n = 0$ does not mean $\sum_{n=1}^{\infty} a_n$ con.

3) $\sum_{n=1}^{\infty} \sqrt{5n}$

$\lim_{n\to\infty} \sqrt{5n} = \infty$

$\longrightarrow \sum_{n=1}^{\infty} \sqrt{5n}$ div by n^{th} term test.

4) $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 \dots$

$\longrightarrow = (1-1) + (1-1) + (1-1) + \dots = 0.$

$\longrightarrow = 1 + (-1+1) + (-1+1) + (-1+1) + \dots = 1.$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n+1}$

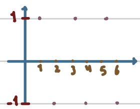
$= \text{DNE}.$

so its div.

$a_1 = 1$ $a_2 = -1$

$a_3 = 1$ $a_4 = -1$

$a_5 = 1$ $a_6 = -1$



Remark

Assume $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$

Then: 1) $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$. (sum of 2 conv. series = conv.)

2) $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$. (subtract of 2 conv. series = conv.)

3) $\sum_{n=1}^{\infty} (k a_n) = kA$.

4) Assume $\sum_{n=1}^{\infty} C_n$ div. Then $\sum_{n=1}^{\infty} (a_n \pm C_n)$ div.

and $\sum_{n=1}^{\infty} k C_n$ div. $(k \neq 0)$

5) $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 4^n}$ $\quad \lim_{n\to\infty} \frac{2^n + 4^n}{3^n + 4^n}$ $(/4^n)$

$= \lim_{n\to\infty} \frac{(\frac{2}{4})^n + 1}{(\frac{3}{4})^n + 1} = \frac{1 + \lim_{n\to\infty} (\frac{1}{2})^n}{1 + \lim_{n\to\infty} (\frac{3}{4})^n}$

$= 1.$

$\longrightarrow \sum_{n=1}^{\infty} a_n$ div by n^{th} term test.

Geometric Series:

has the form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 \dots$

r: ratio الأساس

Geometric Series

conv. — div.

if $|r| < 1$ — if $|r| \geq 1$

$-1 < r < 1$

Test 3: Geometric Series Test (strong)

conv. to sum $= \dfrac{a}{1-r}$

where a = first term

r = ratio

Ex. Find sum of?

1) $1 + \frac{1}{2} + \frac{1}{4} + \dots + (\frac{1}{2})^{n-1} + \dots$ conv. to 2 since

$= \sum_{n=1}^{\infty} (\frac{1}{2})^{n-1} = \dfrac{1}{1-\frac{1}{2}} = 2.$ $\quad r \in (-1, 1).$

2) $1 - \frac{1}{3} + \frac{1}{9} + \dots + (\frac{-1}{3})^{n-1} + \dots$ conv. to $\frac{3}{4}$ since

$= \sum_{n=1}^{\infty} (\frac{-1}{3})^{n-1} = \dfrac{1}{1+\frac{1}{3}} = \frac{3}{4}.$ $\quad r \in (-1, 1)$

3) Check $\sum_{n=1}^{\infty} (\frac{4}{3})^n$?

$= \frac{4}{3} + \frac{16}{9} + \frac{64}{27} + \dots + (\frac{4}{3})^n + \dots$

div. since $r = \frac{4}{3} \notin (-1, 1)$.

4) Check $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n}$?

$= \frac{3}{2} - \frac{3}{4} + \frac{3}{8} + \dots + (-1)^{n+1} \frac{3}{2^n} + \dots$

con. to 1 since $r = \frac{-1}{2} \in (-1, 1)$.

Ex. Consider $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$, find x s.t this series conv. and find it's sum ?

$= 1 - (x+1) + (x+1)^2 + \dots + (-1)^n (x+1)^n + \dots$

$|r| < 1 \longrightarrow |-(x+1)| < 1$

$|x+1| < 1 \longrightarrow -1 < x+1 < 1$

$\therefore -2 < x < 0$.

sum $= \dfrac{a}{1-r} = \dfrac{1}{1+x+1}$

$= \dfrac{1}{2+x}.$

Ex. Express the following repeated decimals as ratio of two integers ?

1) $0.\bar{7} = 0.7777\dots = 0.7 + 0.07 + \dots$

$= \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \dots \quad r = \frac{1}{10}$

since $r \in (-1, 1) \longrightarrow$ series conv.

to $\dfrac{a}{1-r} = \dfrac{7}{9}.$

2) $0.\overline{23} = 0.232323\dots$

$= \frac{23}{100} + \frac{23}{100^2} + \frac{23}{100^3} + \dots \quad r = \frac{1}{100}$

sum $= \dfrac{a}{1-r} = \dfrac{23}{99}.$

3) $0.0\bar{5} = 0.0555\dots = 10^{-1} \times 0.\bar{5}$

$= \frac{1}{10} \times \frac{5}{9} = \frac{5}{90}.$

Ex. Find the sum of $\sum_{n=0}^{\infty} (\frac{5}{2^n} + \frac{1}{3^n})$?

$= \sum_{n=0}^{\infty} \frac{5}{2^n} + \sum_{n=0}^{\infty} \frac{1}{3^n} = (5 + \frac{5}{2} + \frac{5}{4} + \dots) + (1 + \frac{1}{3} + \frac{1}{9} + \dots)$

both geometric series so

sum $= \dfrac{a}{1-r} + \dfrac{a}{1-r} = 10 + \dfrac{3}{2}$

$= \dfrac{23}{2}$ #