Exercises : True or False :

a. If f and g are increasing on $[a,b]$, then $f+g$ is increasing on $[a,b]$?

True, If $x<y$ belongs to $[a,b]$ Then $f(x) \le f(y)$ and $g(x) \le g(y)$

Adding these inequality, we obtain $f(x)+g(x) \le f(y)+g(y)$.

b. If f and g are increasing on $[a,b]$ then fg is increasing on $[a,b]$? False

$f(x) = g(x) = x$ are increasing on $[-1,0]$

But $fg = x^2$ is decreasing on $[-1,0]$.

c. If f is diffble on (a,b) and $\lim\limits_{x \to a^+} f(x)$ exists and is finite then for each $x \in (a,b)$ there is c between a and x s.t $f(x) - f(a^+) = f'(c)(x-a)$. True.

The function $g(x) = f(x)$ for $x \in (a,b)$ and $g(a) = f(a^+)$ is conti. on $[a,x]$ $\forall x \in (a,b)$. Thus, By Mean Value Theorem, there is a $c \in (a,b)$ s.t

$f(x) - f(a^+) = g(x) - g(a) = g'(c)(x-a) = f'(c)(x-a)$.

d. If f and g are diffble on $[a,b]$ and $|f'(x)| \le 1 \le |g'(x)|$ for all $x \in (a,b)$ then $|f(x) - f(a)| \le |g(x) - g(a)|$ for all $x \in [a,b]$. True.

For any $x \in (a,b)$ By MVT, there are points c, d between a and x such that $|f(x) - f(a)| = (x-a)|f'(c)| \le (x-a)|g'(d)|$

$= |g(x) - g(a)|$.

2.3.1: prove that each of the following inequalities holds:

a. $2x + 0.7 < e^x$ for all $x \geq 1$.

let $f(x) = e^x - 2x - 0.7$: since $x \geq 1$

$f'(x) = e^x - 2 > 0$ By Thm 7.i, f increasing on $[1, \infty)$

$\Rightarrow$ In particular, $e^x - 2x - 0.7 \geq f(1)$

$\geq e - 2.7 \geq 0$.

b. $\log x < \sqrt{x} - 0.6$ for all $x \geq 4$. $(4, \infty)$.

let $f(x) = \sqrt{x} - \log x - 0.6$

since $x \geq 4 \rightarrow f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x} = \frac{x - 2\sqrt{x}}{2x\sqrt{x}} \geq 0$

Hence, By Thm 7i : f increasing on $[4, \infty)$

In particular, $\sqrt{x} - \log x - 0.6 \geq f(4) = 2 - \log 4 > 0$.

c. $\sin^2 x \leq 2|x|$ for all $x \in R$.

let $f(x) = 2|x| - \sin^2 x$ and suppose first that $x \geq 0$. Then

$f'(x) = 2 - 2\sin x \cos x \geq 0$,

By Thm 7i, f increases on $[0, \infty)$ In particular $2x - \sin^2 x \geq f(0) = 0$.

i.e $\sin^2 x \leq 2|x|$ when $x \geq 0$.

$\rightarrow$ If $x < 0$ then by what we just showed $\sin^2 x = \sin^2(-x) \leq 2(-x) = 2|x|$.

$\Rightarrow \sin^2 x \leq 2|x|$ ✓

→ Fallhs →

→ $\bar{f}(c) = f(b) - f(a)$

→ Mean $\bar{f}(c) \neq 0$

d. $1 - \sin x \leq e^x$ for all $x \geq 0$.

let $f(x) = e^x + \sin x - 1$, since $x \geq 0$ then

$\bar{f}(x) = e^x + \cos x \geq 1 + \cos x \geq 0$

Hence, By Theorem 7.i , f increasing on $[0, \infty)$

Inparticular, $e^x - 1 + \sin x \geq f(0) = 0$.

4.3.2 : suppose that $I = (0,2)$, that f is continuous at $x = 0$ and $x = 2$ and that f is diffble on I . If $f(0) = 1$ and $f(2) = 3$, prove that $1 \in \bar{f}(I)$.

→ cont. on I .

$\exists c \in I$ s.t $\bar{f}(c) = 1$.

proof : Apply the MVT to f on $[a,b]$.

By MVT , $\exists c \in (0,2)$ s.t $\bar{f}(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{3-1}{2} = 1$

⊠

4.3.3 : let f be an a real function and recall that an $r \in R$ is called a root of a function f iff $f(r) = 0$. show that if f is diffble on R then its dereveties $\bar{f}$ has at least one root between any two roots of f.

If a and b are roots of f , then By Mean Value Thm :

$\bar{f}(c) = \dfrac{f(b) - f(a)}{b - a} = 0$ For some $c \in (a,b)$.

Hence, c is a root of $\bar{f}$.

⊠

4.3.4: suppose that $a<b$ are extended real numbers and that f is diff'ble on (a,b). If f' is bounded on (a,b), prove that f is uniformly continuous on (a,b).

let $M \in \mathbb{R}$ s.t $|f'(x)| \leq M$

By the MVT : $x,y \in (a,b) \Rightarrow \exists\, c \in (a,b)$ s.t $f'(c) = \dfrac{f(y)-f(x)}{y-x}$

let $\varepsilon > 0$ and set $\delta = \frac{\varepsilon}{M}$, If $x,y \in (a,b)$ and $|x-y| < \delta$ then

$$
\begin{aligned}
|f(y)-f(x)| = |f'(c)\,(y-x)| &= |f'(c)|\,|y-x| \\
&\leq M\,|y-x| \\
&< M\,|x-y| \\
&< M\,\delta \\
&< \varepsilon
\end{aligned}
$$

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So we proved f is uniformly cont.

4.3.5: suppose that f is diff'ble on $\mathbb{R}$. If $f(0)=1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$ prove that $|f(x)| \leq |x| + 1$ for all $x \in \mathbb{R}$.

f is diff'ble on $\mathbb{R}$ and $f'(x)$ is bounded on $\mathbb{R} \Rightarrow f$ is uniformly cont. on $\mathbb{R}$

By MVT : $f(x) - f(0) = f'(c)\,(x-0)$ for some $c \in (0,x)$

let $\varepsilon > 0$, take $\delta = \varepsilon$

if $x \in [0,x]$ and $|x| < \delta$ then $|f(x)-f(0)| = |f'(c)|\,|x|$

$$\leq 1 \cdot \delta < \varepsilon .$$

$|f(x) - 1| \leq |x|$

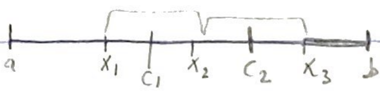
$|f(x)| \leq |f(x)-1| + 1 \leq |x| + 1$

$\Rightarrow \quad |f(x)| \leq |x| + 1 \, , \ \forall x \in \mathbb{R}$.

4.3.6 : suppose that f is diffble on (a,b), conti. on $[a,b]$ and that $f(a) = f(b) = 0$ prove that if $f(c) \neq 0$ for some $c \in (a,b)$ then their exists $x_1, x_2 \in (a,b)$ such that $f'(x_1)$ is positive and $f'(x_2)$ is negative.

①

4.3.8 : suppose that f is twice diffble on (a,b) and that there are points $x_1 < x_2 < x_3$ in (a,b) s.t $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$, prove that their is a point $c \in (a,b)$ s.t $f''(c) > 0$.



$$f'(c_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0$$

$$f'(c_2) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} > 0$$

By MVT to f' on $[c_1, c_2]$

$$(f')'(c) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = \frac{+ - -}{+} = + > 0$$

$c \in (c_1, c_2) \subseteq (a,b)$.

4.3.9: suppose that f is diffble on $(0,\infty)$. If $L = \lim_{x\to\infty} f'(x)$ and $\lim_{n\to\infty} f(n)$ both exists and are finite, prove that $L=0$.

let A represent the limit of $\{f(n)\}$

By the MVT, $f(n+1) - f(n) = f'(c_n)$ for some $c_n \in (n, n+1)$, $n \in \mathbb{N}$

since $c_n \to \infty$ as $n \to \infty$ it follows that

$$0 = A - A = \lim_{n\to\infty} \left(f(n+1) - f(n) \right) = \lim_{n\to\infty} f'(c_n) = L.$$

Done ...