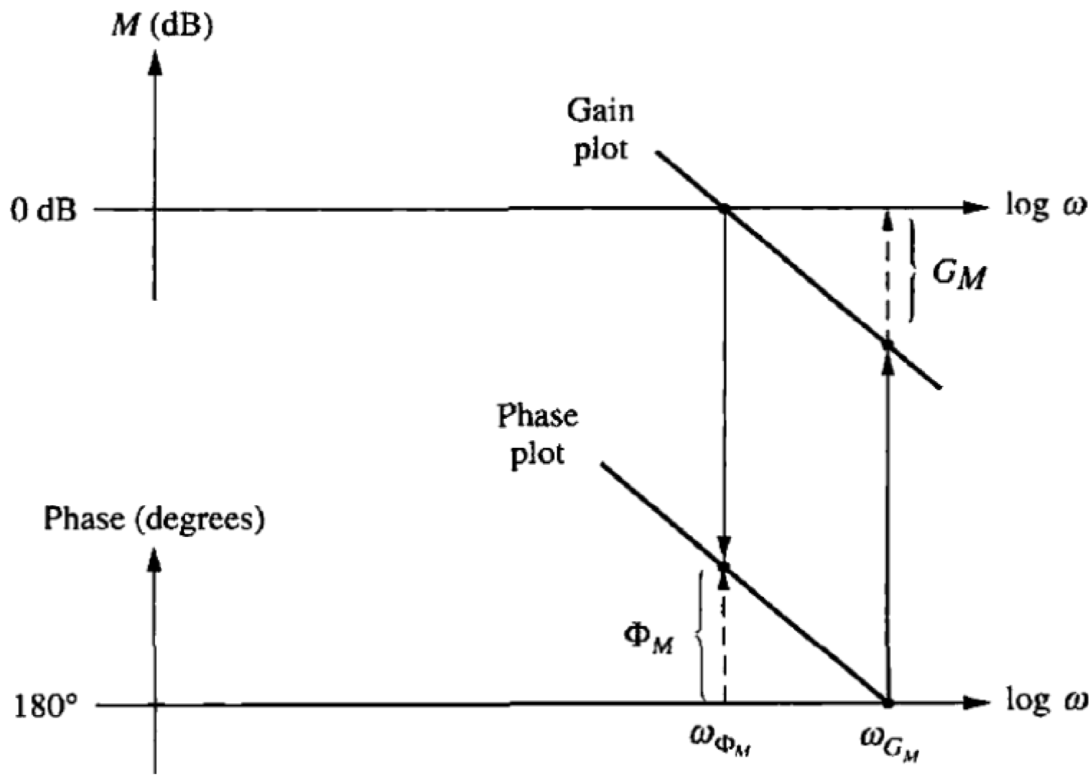
Analysis of Bode Plots

Informal definitions:

- The **gain margin** is the factor by which the gain can be increased before instability results.
- The **phase margin** is the amount of phase by which G(jω) exceeds -180 degrees when |KG(jω)|=1

- These are easily measured on Bode diagrams.



Steady-State Error Characteristics from Frequency Response

TABLE 7.2 Relationships between input, system type, static error constants, and steady-state errors

		Type 0		Type 1		Type 2	
Input	Steady-state error formula	Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

To find K_p, consider the following Type 0 system:

$$G(s) = K\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)} \tag{10.74}$$

A typical unnormalized and unscaled Bode log-magnitude plot is shown in Figure 10.51(a). The initial value is

$$20\log M = 20\log K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i} \tag{10.75}$$

But for this system

$$K_p = K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i} \tag{10.76}$$

which is the same as the value of the low-frequency axis. Thus, for an unnormalized and unscaled Bode log-magnitude plot, the low-frequency magnitude is $20\log K_p$ for a Type 0 system.

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Velocity Constant

To find K_v for a Type 1 system, consider the following open-loop transfer function of a Type 1 system:

$$G(s) = K\frac{\prod_{i=1}^{n}(s+z_i)}{s\prod_{i=1}^{m}(s+p_i)} \tag{10.77}$$

A typical unnormalized and unscaled Bode log-magnitude diagram is shown in Figure 10.51(b) for this Type 1 system. The Bode plot starts at

$$20\log M = 20\log K\frac{\prod_{i=1}^{n} z_i}{\omega_0\prod_{i=1}^{m} p_i} \tag{10.78}$$

The initial −20 dB/decade slope can be thought of as originating from a function,

$$G'(s) = K\frac{\prod_{i=1}^{n} z_i}{s\prod_{i=1}^{m} p_i} \tag{10.79}$$

$G'(s)$ intersects the frequency axis when

$$\omega = K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i} \tag{10.80}$$

But for the original system (Eq. (10.77)),

$$K_v = K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i} \tag{10.81}$$

which is the same as the frequency-axis intercept, Eq. (10.80). Thus, we can find K_v by extending the initial −20 dB/decade slope to the frequency axis on an unnormalized and unscaled Bode diagram. The intersection with the frequency axis is K_v.

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Acceleration Constant

To find K_a for a Type 2 system, consider the following:

$$G(s) = K\frac{\prod_{i=1}^{n}(s+z_i)}{s^2\prod_{i=1}^{m}(s+p_i)} \tag{10.82}$$

A typical unnormalized and unscaled Bode plot for a Type 2 system is shown in Figure 10.51(c). The Bode plot starts at

$$20\log M = 20\log K\frac{\prod_{i=1}^{n}z_i}{\omega_0^2\prod_{i=1}^{m}p_i} \tag{10.83}$$

The initial -40 dB/decade slope can be thought of as coming from a function,

$$G'(s) = K\frac{\prod_{i=1}^{n}z_i}{s^2\prod_{i=1}^{m}p_i} \tag{10.84}$$

$G'(s)$ intersects the frequency axis when

$$\omega = \sqrt{K\frac{\prod_{i=1}^{n}z_i}{\prod_{i=1}^{m}p_i}} \tag{10.85}$$

But for the original system (Eq. (10.82)),

$$K_a = K\frac{\prod_{i=1}^{n}z_i}{\prod_{i=1}^{m}p_i} \tag{10.86}$$

Thus, the initial -40 dB/decade slope intersects the frequency axis at $\sqrt{K_a}$.

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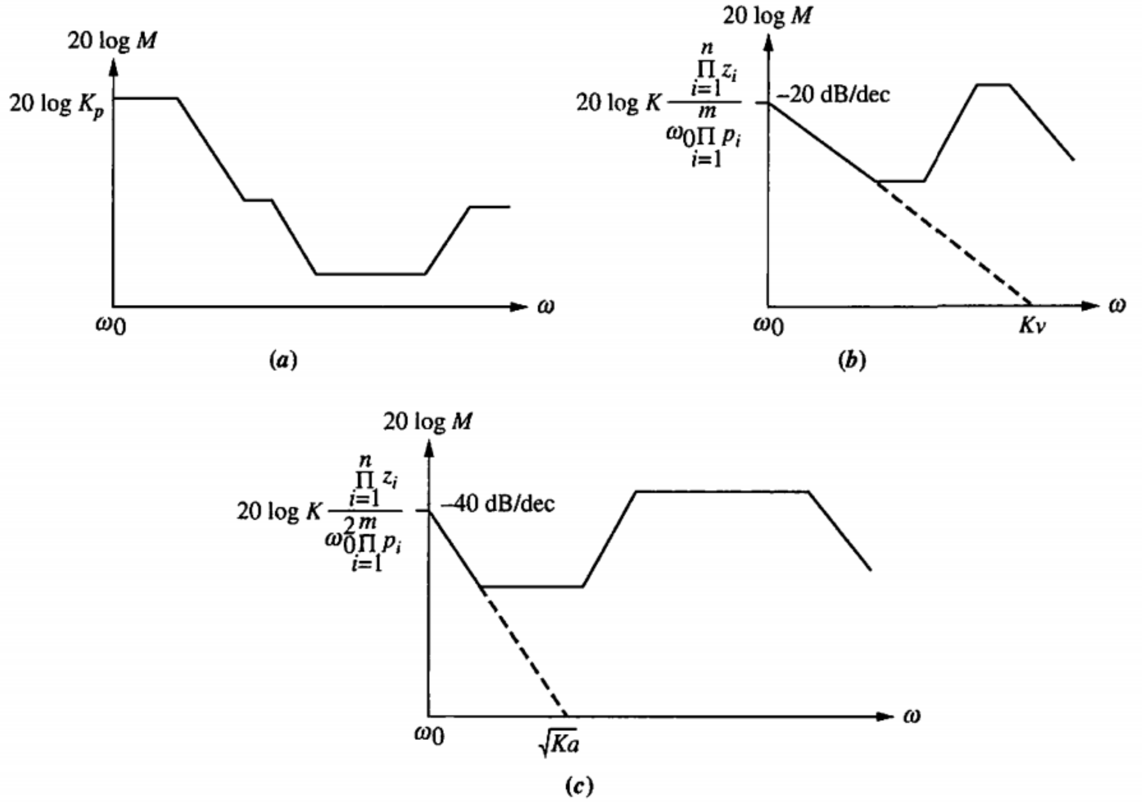

FIGURE 10.51 Typical unnormalized and unscaled Bode log-magnitude plots showing the value of static error constants: **a.** Type 0; **b.** Type 1; **c.** Type 2

Relation Between Closed-Loop Transient and Closed-Loop Frequency Responses

Consider the following second order system

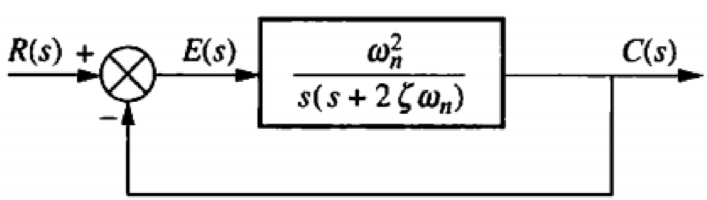

FIGURE 10.38 Second-order closed-loop system

$$\frac{C(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{10.49}$$

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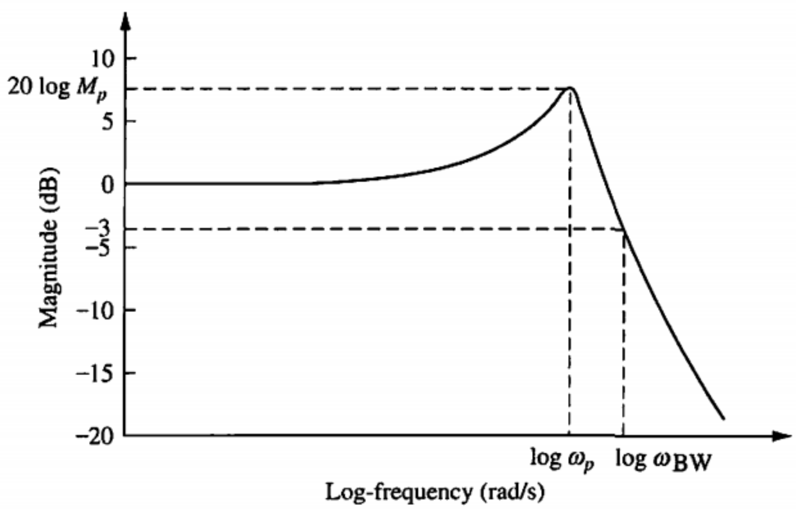
Let us now find the frequency response of Eq. (10.49), define characteristics of this response, and relate these characteristics to the transient response. Substituting $s = j\omega$ into Eq. (10.49), we evaluate the magnitude of the closed-loop frequency response as

$$M = |T(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}} \tag{10.51}$$

A representative sketch of the log plot of Eq. (10.51) is shown in Figure 10.39.

We now show that a relationship exists between the peak value of the closed-loop magnitude response and the damping ratio. Squaring Eq. (10.51), differentiating with respect to ω^2, and setting the derivative equal to zero yields the maximum value of M, M_p, where

$$\boxed{M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}} \tag{10.52}$$



at a frequency, ω_p, of

$$\boxed{\omega_p = \omega_n\sqrt{1 - 2\zeta^2}} \tag{10.53}$$

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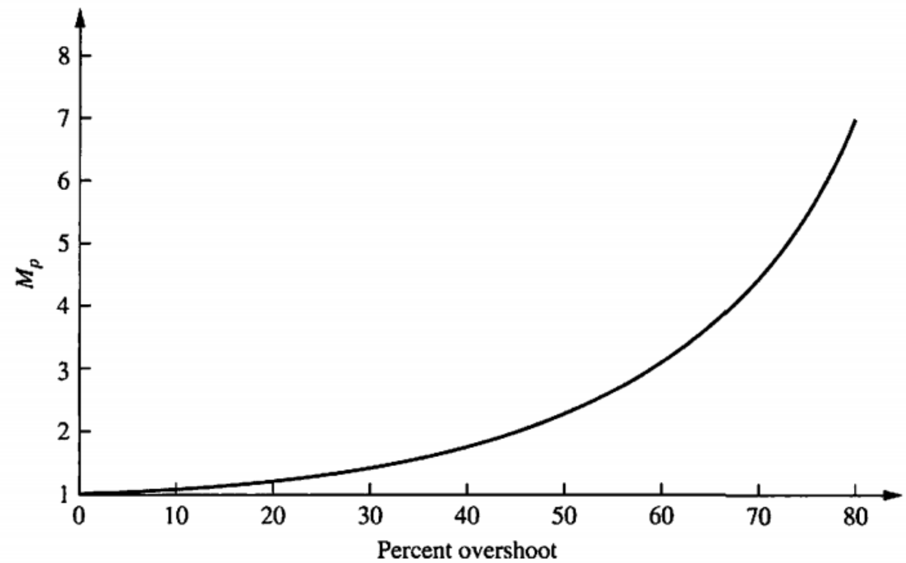
Response Speed and Closed-Loop Frequency Response

Another relationship between the frequency response and time response is between the speed of the time response (as measured by settling time, peak time, and rise time) and the *bandwidth* of the closed-loop frequency response, which is defined here as the frequency, ω_{BW}, at which the magnitude response curve is 3 dB down from its value at zero frequency (see Figure 10.39).



The bandwidth of a two-pole system can be found by finding that frequency for which $M = 1/\sqrt{2}$ (that is, −3 dB) in Eq.(10.51). The derivation is left as an exercise for the student. The result is

$$\boxed{\omega_{BW} = \omega_n\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}} \tag{10.54}$$

To relate ω_{BW} to settling time, we substitute $\omega_n = 4/T_s\zeta$ into Eq. (10.54) and obtain

$$\boxed{\omega_{BW} = \frac{4}{T_s\zeta}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}} \tag{10.55}$$

Similarly, since, $\omega_n = \pi/(T_p\sqrt{1-\zeta^2})$,

$$\boxed{\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}} \tag{10.56}$$

To relate the bandwidth to rise time, T_r, we use Figure 4.16, knowing the desired ζ and T_r. For example, assume $\zeta = 0.4$ and $T_r = 0.2$ second. Using Figure 4.16, the ordinate $T_r\omega_n = 1.463$, from which $\omega_n = 1.463/0.2 = 7.315$ rad/s. Using Eq. (10.54), $\omega_{BW} =$ 10.05 rad/s. Normalized plots of Eqs. (10.55) and (10.56) and the relationship between bandwidth normalized by rise time and damping ratio are shown in Figure 10.41.

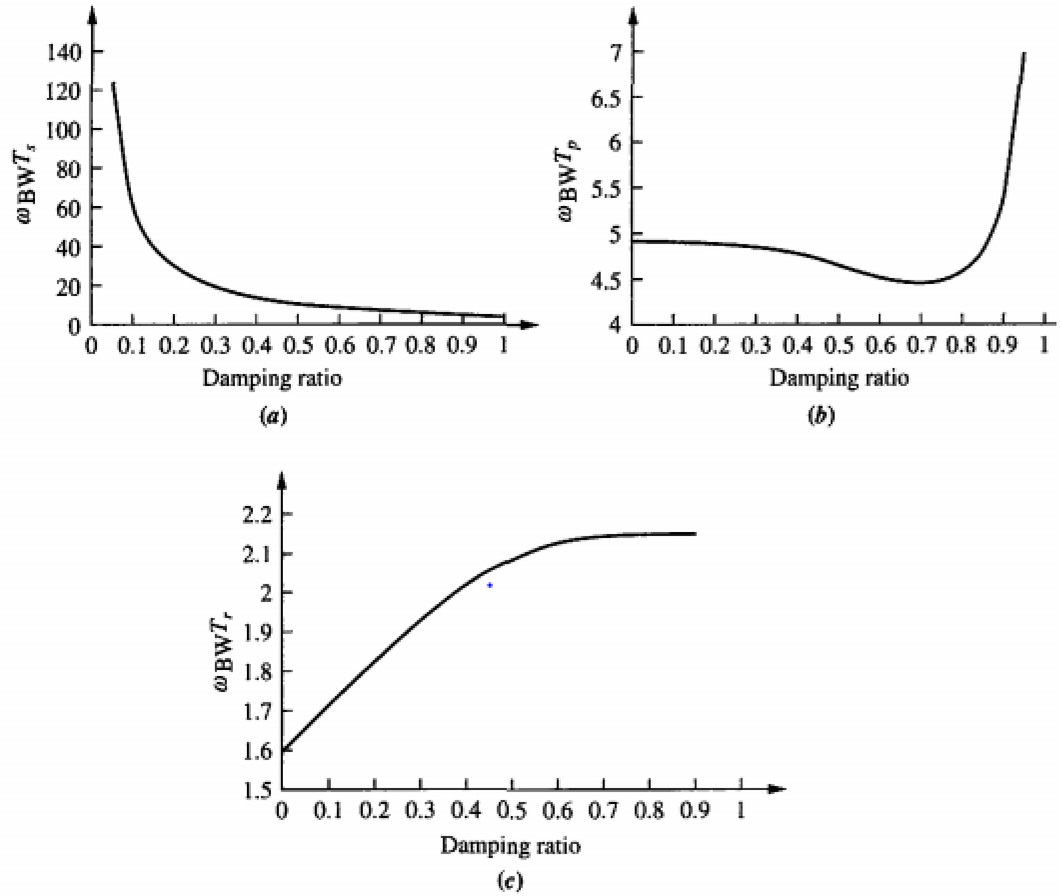


FIGURE 10.41 Normalized bandwidth vs. damping ratio for **a.** settling time; **b.** peak time; **c.** rise time

Relation Between Closed-Loop Transient and Open-Loop Frequency Responses

Damping Ratio from Phase Margin

Let us now derive the relationship between the phase margin and the damping ratio. This relationship will enable us to evaluate the percent overshoot from the phase margin found from the open-loop frequency response.

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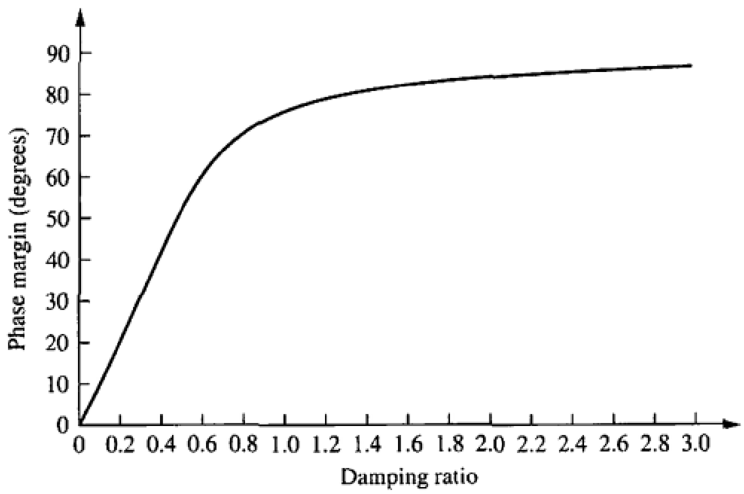
The difference between the angle of Eq. (10.72) and -180° is the phase margin, ϕ_M. Thus,

$$\boxed{\begin{aligned}\Phi_M &= 90 - \tan^{-1}\frac{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}{2\zeta}\\ &= \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}\end{aligned}} \tag{10.73}$$

Equation (10.73), plotted in Figure 10.48, shows the relationship between phase margin and damping ratio.



Transient Response Design via Gain Adjustment

PROBLEM: For the position control system shown in Figure 11.2, find the value of preamplifier gain, K, to yield a 9.5% overshoot in the transient response for a step input. Use only frequency response methods.

SOLUTION: We will now follow the previously described gain adjustment design procedure.

1. Choose $K = 3.6$ to start the magnitude plot at 0 dB at $\omega = 0.1$ in Figure 11.3.
2. Using Eq. (4.39), a 9.5% overshoot implies $\zeta = 0.6$ for the closed-loop dominant poles. Equation (10.73) yields a 59.2° phase margin for a damping ratio of 0.6.

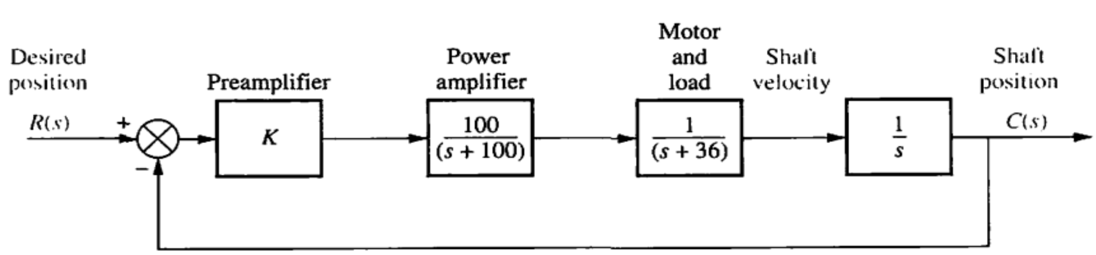


FIGURE 11.2 System for Example 11.1

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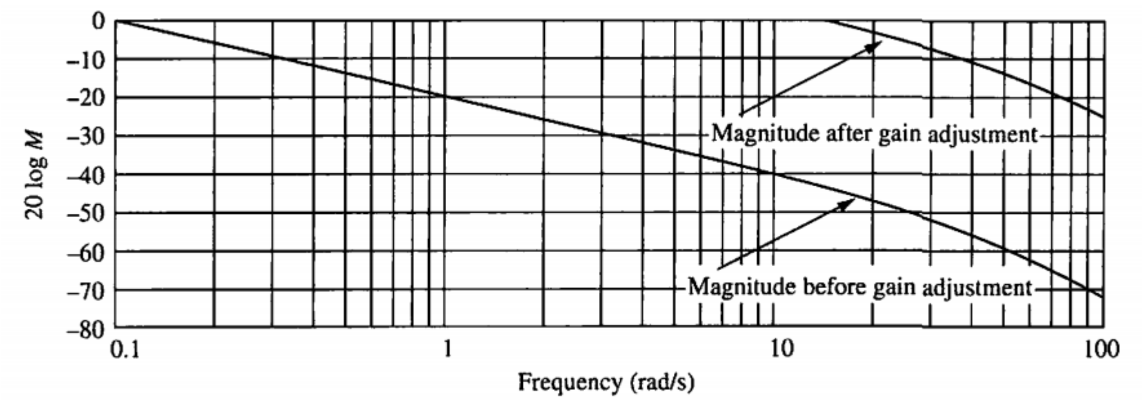

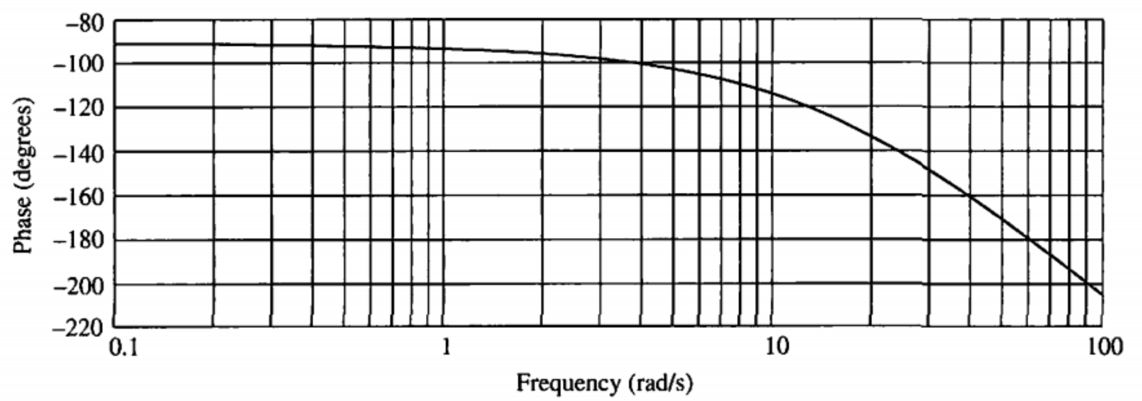

3. Locate on the phase plot the frequency that yields a 59.2° phase margin. This frequency is found where the phase angle is the difference between −180° and 59.2°, or −120.8°. The value of the phase-margin frequency is 14.8 rad/s.
4. At a frequency of 14.8 rad/s on the magnitude plot, the gain is found to be −44.2 dB. This magnitude has to be raised to 0 dB to yield the required phase margin. Since the log-magnitude plot was drawn for $K = 3.6$, a 44.2 dB increase, or $K = 3.6 \times 162.2 = 583.9$, would yield the required phase margin for 9.48% overshoot.

   The gain-adjusted open-loop transfer function is

$$G(s) = \frac{58{,}390}{s(s+36)(s+100)} \tag{11.1}$$

Table 11.1 summarizes a computer simulation of the gain-compensated system.

TABLE 11.1 Characteristic of gain-compensated system of Example 11.1

Parameter	Proposed specification	Actual value
K_v	—	16.22
Phase margin	59.2°	59.2°
Phase-margin frequency	—	14.8 rad/s
Percent overshoot	9.5	10
Peak time	—	0.18 second

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