CYBERSECURITY MATHEMATICS

CHAPTER 5

PERMUTATION :

EX: SUPPOSE WE HAVE 5 PERSONS (A , B , C , D , E) & WE HAVE 5 CHARIS .

THE QUESTION : COUNT THE NUMBER OF SCENARIOS OF SITTING THESE 5 PERSONS ON 5 CHARIS ??

$$5 * 4 * 3 * 2 * 1 = 120$$

EX: SUPPOSE WE HAVE 3 PERSONS (ALI , MOHAMED , HAMMED) & WE HAVE 3 CHARIS .

THE QUESTION : COUNT THE NUMBER OF SCENARIOS OF SITTING THESE 3 PERSONS ON 3 CHARIS ??

$3 * 2 * 1 = 6$

(A,M,H) , (A, H,M) , (H,M,A)(H,A,M) , (M,H, A) , (M ,A, H)

EX: SUPPOSE WE HAVE 5 PERSONS (ALI , MOHAMED , HAMMED , SHAWQI , OMAR) & WE HAVE 3 CHARIS .

THE QUESTION : COUNT THE NUMBER OF SCENARIOS OF SITTING THESE 5 PERSONS ON 3 CHARIS ??

$$5 * 4 * 3 = 60$$

$60 = (5*4*3*2*1) / (2*1) \rightarrow 60 = 5! / (5-3)!$

5 PERSONS - 3 CHAIRS = 2

$_NP_R$ OR $P(N,R) = N! / (N-R)!$ → The formula of (P)

N: # of persons

R: # of chairs

EX : HOW MANY PERMUTATION OF WORD (LENGTH 5 CHARS) FROM ENGLISH CAPITAL LETTERS ONLY ??

a) REPTATION IS NOT ALLOWED

#OF SCENARIOS = 26 * 25 * 24 * 23 * 22

B) REPTATION IS ALLOWED

OF SCENARIOS = 26 * 26 * 26 * 26 * 26 = 26^5

COMBINATION :

PERSONS : A , B , C , D , E , F

WHAT ABOUT IF WE CHOOSE 3 PERSONS TO BE SIT IN THE THREE CHARIS , IF WE CARE ABOUT WHO SIT ON THE CHAIR SO THE NUMBER OF SENIORS EQUAL 120 BUT , IF WE DO NOT CARE ABOUT WHO SIT ON THE CHAIR THE NUMBER OF COMBINATION : 6 C 3 = 6! / 3 ! * (6 - 3) ! = 20

THE FORMULA OF COMBINATION : C (N , R) = N ! / R ! * (N-R) !

SUMMARY :

THE PERMUTATION IS THE MATHEMATIC WAY TO EXPRESS THE # OF SCENARIOS

IN PERMUTATION THE ORDER IS MUTTER BUT , IN COMBINATION THE ORDER DOES NOT MUTTER , SO IF ORDER IS MUTTER , WE MUST FIND THE PERMUTATION BUT , IF ORDER DOES NOT MUTTER , WE MUST FIND COMBINATION .

THE FORMULA OF COMBINATION : C (N , R) = N ! / R ! * (N-R) !

THE FORMULA OF PERMUTATION : P (N , R) = N ! / (N-R) !

THE BINOMIAL THEOREM :

$$(X+Y)^N = \sum_{j=0}^{n} \binom{n}{j} * x^j * y^{n-j}$$

EX : $(X+Y)^3 \longrightarrow \binom{3}{0} \cdot x^0 \; y^{3-0} + \binom{3}{1} \cdot x^1 \cdot y^{3-1} + \ldots\ldots\ldots$

USE N! / (N-J)!*J!

THE FINAL ANSWER : $X^3 + 3X^2Y + 3XY^2 + Y^3$

EX : $(2T+3)^4$

$2T \to x$, $3 \to y$

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} \cdot x^j \cdot y^{4-j} \longrightarrow (2t+3)^4 = \sum_{j=0}^{4} \binom{4}{j} \cdot (2t)^j \cdot (3)^{4-j}$$

$$\binom{4}{0} \cdot (2t)^0 (3)^{4-0} + \ldots\ldots$$

THE FINAL ANSWER : $16T^4 + 96T^3 + 216T^2 + 216T + 81$

PROBABILITY THEOREM :

PR : $\Omega \longrightarrow R$ P (W) = PROP OF EVERY W OCCURRED

EX : IF WE HAVE A COIN WITH TWO FACES { H , T } WHAT IS PROP OF H AND WHAT IS THE PROP OF T

PR (H) = 1/2

PR (T) = 1/2

EX: IF WE HAVE TWO DICES WITH

$\Omega = \{ (M, N) ;\ M, N \text{ BELONG TO } Z \ ;\ 1 \leq M, N \leq 6 \}$

$PR(M, N) = 1/36$

$\{ (1,1), (1,2), (3,4) \ldots \ldots \}$

$(1,1) \downarrow \frac{1}{36}$, $(1,2) \downarrow \frac{1}{36}$, $(3,4) \downarrow \frac{1}{36}$, $\ldots$

EX: IF WE TWO COINS EACH COIN HAVE TWO FACES

$$\Omega = \{ (H,H),(H,T),(T,H),(T,T) \}$$

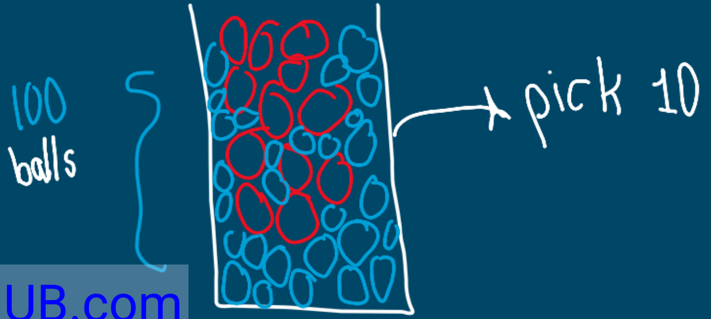
$$\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}$$

EX : URN CONTAIN 100 BALL

21 : RED

74 : BLUE

IF WE PICK 10 BALLS RANDOMLY (WITHOUT REPLACE) WHAT IS THE PROBABILITY THAT EXACTLY 3 OF THEM IS RED



$$Pr\left(\text{EXACTLY 3 OF THEM IS RED}\right) = \frac{\binom{21}{3}\binom{79}{7}}{\binom{100}{10}} = \frac{\frac{21!}{3!\,18!} \times \frac{79!}{7!\,72!}}{\frac{100}{10!\,90!}} = 0.223$$

DEF : A SAMPLE SPACE (OR SET OUTCOMES) IS A FINITE SET EACH OUTCOME W ∈ N IS ASSIGNED A PROBABILITY P(W), WHERE WE REQUIRE THAT PROBABILITY FUNCTION PR: $\Omega \to R$ SATISFY THE FOLLOWING TWO PROPERTIES :

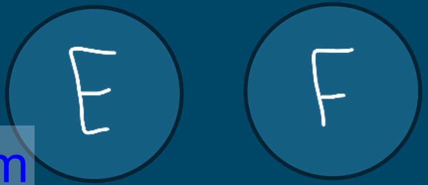
A) $0 \leq PR(W) \leq 1$ FOR ALL $W \in \Omega$

B) $\sum_{W \in \Omega} PR(W) = 1$

DEF : AN EVENT IS ANY SUBSET OF Ω WE ASSIGN A PROBABILITY TO EVENT E BY SETTING $PR(E) = \sum_{W \in \Omega} PR(W)$

IN PARTICULAR $PR(\emptyset) = \emptyset$ BE CONVENTION AND $PR(\Omega) = 1$

DEF : WE SAY TWO EVENTS E AND F ARE DISJOINT IF E OR F = $\emptyset$, ITS CLEAR THAT $PR(E,F) = PR(E) + PR(F)$ IF E , F IS DISJOINT



$PR(E \text{ OR } F) = PR(E) + PR(F) - PR(E \text{ AND } F)$ IF ITS NOT DISJOINT

DEF : THE COMPLEMENT $PR(E^C) = 1 - PR(E)$

EXAMPLE : E = { AT LEAST ONE SIDE 6 OF ROLLING DICE TWICE }

Ω = { (1,1),(1,2) ...... (1,6),(2,1),(2,2) ... } 6 * 6 = 36

PR (E) = 11 / 36 → { (1,6) , (2,6), (3,6) ... (6,6) (6,1) , (6,2) ... (6,5) }

EXAMPLE : E { PROBABILITY OF (NO SIX ARE ROLLED)

$PR(E) = 1 - PR(E^C)$ 1 - 11/36 = 25 / 36

EXAMPLE : E { NO NUMBER HIGHER THAN TWO IS ROLLED)

F = {(1,1) ,(1,2) , (2,1),(2,2) }   PR (F) = 4 / 36 = 1 / 9

EXAMPLE : PR (E OR F) = PR (E) + PR (F) - PR (E AND F)

= 11 / 36 + 4 / 36 = 15 / 36 = 5 / 12

EXAMPLE : G = { DOUBLE } = {(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)}

$P(E \cup G) = PR(E) + PR(G) - P(E \cap G) =$ 11 / 36 + 6 / 36 - 1 / 36 = 4 / 9

EXAMPLE : H { THE SUM OF TWO DICE IS AT LEAST 4 }

$H' = \{(1,1),(1,2),(2,1)\} = 3/36$

$PR(H) = 1 - 3/36 = 33/36 = 11/12$

EXAMPLE : E = { THE FIRST CARD DRAWN IS KING }

F = { THE SECOND CARD DRAWN IS KING }

INDEPENDENT

$PR(E) = 4/52$, $PR(F) = 4/52$

PR (F IF E HAS OCCURRED) = 3 / 51 THIS THE PROP WHEN THE FIRST CARD IS KING

PR (F IF HAS NOT OCCURRED) = 4 / 51 THIS THE PROP WHEN THE FIRST CARD IS NOT KING

$PR(E \cap F) = 4/52 * 3/51 = 0.0045$

DEF : TWO EVENTS E AND F ARE SAID TO BE INDEPENDENT IF $PR(E \cap F) = PR(E) * PR(F)$

COMBINE EVENT

OR

$P(A \cup B)$

NOT DISJOINT *) $PR(A \cup B) = PR(A) + PR(B) - P(A \cap B)$

DISJOINT *) $PR(A \cup B) = PR(A) + PR(B)$

AND

$P(A \cap B)$

$P(A) * P(B|A)$

DEPENDENT

$P(A) * P(B)$

INDEPENDENT

BAYES THEOREM :

$$PR(F|E) = (PR(E|F) * PR(F)) \; / \; PR(E)$$

DEF : PROBABILITY OF F ON E IS DENOTED BY $P(F|E) = PR(F)$ GIVEN THAT E IS OCCURRED

$$PR(F|E) = PR(F \cap E) \; / \; PR(E)$$

$PR(E|F) = PR(E \cap F) \; / \; PR(F)$ ⟶ PROBABILITY OF E ON F

$$PR(F \cap E) = PR(F|E) * PR(E)$$

$$PR(E \cap F) = PR(E|F) * PR(F)$$

$$PR(E \cap F) = PR(E|F) * PR(E)$$

$$PR(E \cap F) = PR(E|F) * PR(E) \;/\; PR(F)$$

PROPOSITION :

a) $PR(E) = PR(E|F) * PR(F) + PR(E|F^C) * PR(F^C)$

b) $PR(E|F) = PR(F|E) * PR(E) / (PR(F|E) * PR(E) + PR(F|E^C) * PR(E^C))$

EXAMPLE : 2 URNS :

URN # 1 : { 10 GOLD COINS , 5 SILVER COINS }

URN # 2 : { 2 GOLD COINS , 8 SILVER COINS }

A URN IS RANDOMLY SELECTED , AND COIN IS PICKED RANDOMLY

WHAT IS THE PROBABILITY OF CHOOSING A GOLD COIN ?

E : { SELECTED THE GOLD COIN }

F : { URN #1 THE URN SELECTED }

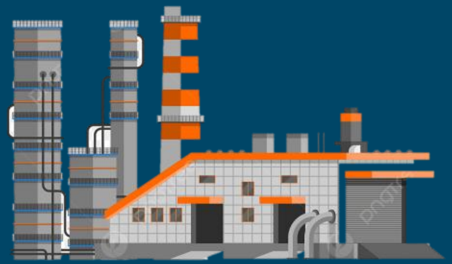
$$PR(E) = PR(E \mid F) * PR(F) + PR(E \mid F^C) * PR(F^C)$$

$$= 10/15 * 1/2 + 2/10 * 1/2 = 13/10$$

EXAMPLE : MANUFACTURING FACTOR . MACHINE A , B AND C PRODUCE 25% , 35 % , 40% BLUBS

MACHINES (DEFECTED OUT OF TOTAL) : 5% , 4% , 2%

WHAT IS THE PROBABILITY OF PICK A RANDOM BLUB TO BE DEFECTED AND FROM MACHINE B ?



PR (A) = 25/100 , PR (B) = 35 / 100 , PR (C) = 40 / 100

PR (D | A) = 5 / 100 , PR (D | B) = 4 / 100 , PR (D | C) = 2 / 100

PR (B | D) = (PR (D | B) * PR (B)) / PR (D | A) * PR (A) + PR (D | B) * PR (B) + PR (D | C) * PR (C)

= (4/ 100 * 35 / 100) / 5 / 100 * 25 / 100 + 4 / 100 * 35/100 + 2/100 * 40 /100 = 28 / 69

EXAMPLE : AN INSURANCE COMPANY INSURED 2000 SCOOTER DRIVERS , 4000 CAR DRIVERS , 6000 TRACK DRIVERS PROBABILITY OF AN ACCIDENT INVOLVING SCOOTER , CAR AND TRACK ARE 0.01 , 0.03 , 0.15 RESPECTIVELY ONE OF INSURED PERSONS MEETS WITHOUT AN ACCIDENT

WHAT IS THE PROBABILITY HE IS SCOOTER DRIVER ?

TOTAL = 2000 + 4000 + 6000 = 12000

PR (S) = 2/12 , PR (C) = 4 / 12 , PR (T) = 6/ 12

PR (A | S) = 1 / 100 , PR (A | C) = 3 / 100 , PR (A | T) = 15 / 100

PR (S | A) = (PR (A | S) * PR (S)) / PR (A | S) * PR (S) + PR (A | C) * PR (C) + PR (A | T) * PR (T)

= (1/ 100 * 2 / 12) / 1 / 100 * 2 / 12 + 3 / 100 * 4 / 12 + 15/100 * 6 / 12 = 1 / 52

RANDOM VARIABLES :

DEF: LET $X : \Omega \longrightarrow \mathbb{R}$ BE A RANDOM VARIABLE , THE PROBABILITY DESNITY FUNCTION X DENOTED BY F(X) , IS DEFINED TO BE :

$$F_X(x) = PR(X = x)$$

OR

$$F_X(x) = PR(X \leq x)$$

THERE ARE A NUMBER OF STANDARD DENSITY FUNCTION THAT OCCUR FREQUENTLY IN DISCRETE PROBABILITY CALCULATIONS

1 - UNIFORM DISTRIBUTION :

$$F_X(J) = PR(X = J) = \begin{cases} \frac{1}{N} & \text{if } j \in S \\ \emptyset & \text{if } j \notin S \end{cases}$$

2- BINOMIAL DISTRIBUTION

$P(X) = \binom{n}{x} P^{X} Q^{N-X}$ P = SUCCESS , Q = FAIL

EXAMPLE : 6 - SIDE DICE ROLLED 12 - TIMES , WHAT IS THE PROBABILITY OF GETTING SIDE (4) 5 - TIMES ?

$P = 1/6$ $PR(X = 5) = \binom{12}{5} (1/6)^{5} (5/6)^{12-5} = 0.028425$

$Q = 5/6$

$N = 12$ TIMES

EXAMPLE : MULTIPLE CHOICE TEST BANK CONTAINS 20 QUESTIONS . WITH ANSWERS A , B , C AND D , ONLY ONE ANSWER CHOICE TO EACH QUESTION REPRESENT THE CORRECT ANSWER

- FIND THE PROBABILITY THAT A STUDENT WILL ANSWER EXACTLY 6 QUESTIONS CORRECT IF HE MAKES RANDOM GUESS ON 20 QUESTIONS.

$$PR(X = 6) = \binom{20}{6} (1/4)^{6} (3/4)^{20-6} = 16.864$$

3- GEOMETRIC DISTRIBUTION

$P(X = X) = P * Q^{X-1}$ P = SUCCESS , Q = FAIL

EXAMPLE : WHAT IS THE PROBABILITY OF GETTING THE 6 ONE TIME ON THE 4 TH ROLLING OF SIX - DICE ?

$(5/6)^{4-1}(1/6) = 0.0945 = 9.45\%$

DEF: LET Y , X BE TWO RANDOM VARIABLES . THE JOINT DENSITY FUNCTION OF X AND Y DENOTED BY $F_{x,y}(X, Y)$ IS PROBABILITY THAT X TAKES THE VALUE X AND Y TAKES THE VALUE Y

A - $F_{x,y}(X, Y) = PR(X = X \text{ AND } Y=Y)$ SIMILARLY , THE CONDITIONAL DENSITY FUNCTION

B - $F_{x,y}(X | Y) = PR(X = X \mid Y=Y)$

EXAMPLE : URN HAS : 3 SILVER COINS AND 4 GOLD COINS

1^{ST} COIN DRAWN RANDOMLY , EXAMINED AND RETURNED

2^{ND} COIN DRAWN RANDOMLY , EXAMINED AND RETURNED

LET X = GOLD COIN DRAWN

Y = SLIVER