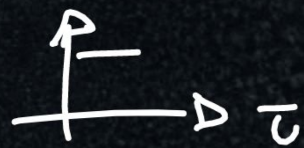
FT Theorems

(10) Integration

$$F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j2\pi f}X(f) + \frac{1}{2}\delta(f)X(0)$$

proof:-

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau$$



$$F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau\right]e^{-j2\pi f t}\,dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} u(t-\tau)\,e^{-j2\pi f t}\,dt\right]d\tau$$

$\rightarrow F[u(t-\tau)] = e^{-j2\pi f\tau}\,F[u(t)]$

$= e^{-j2\pi f t}\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right]$

$$F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \int_{-\infty}^{\infty}\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] e^{-j2\pi f\tau}\, x(\tau)\,d\tau$$

$$= \left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f\tau}\,d\tau$$

$$= \left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] X(f)$$

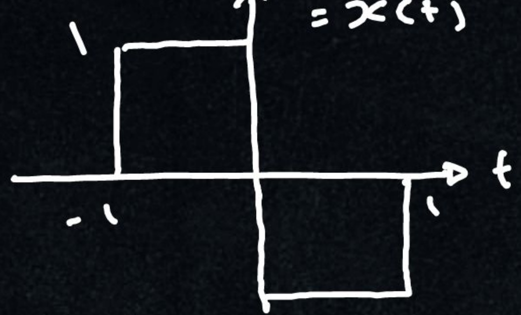
$$\boxed{F\left[\int_{0}^{t} x(\tau)\,d\tau\right] = \frac{X(f)}{j2\pi f} + \frac{1}{2}\delta(f)\,X(0)}$$

Ex:-

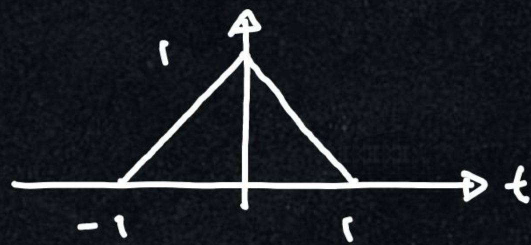
$$F\left[\Pi(t+1/2) - \Pi(t-1/2)\right] = j2\pi f \, \mathrm{sinc}^2(f)$$

Determine the FT of $\Delta(t)$ using integration theorem

$\Pi(t+\frac{1}{2}) - \Pi(t-\frac{1}{2})$
$= x(t)$



$\Delta(t)$



$$\Delta(t) = \int_{0}^{t} x(\tau)\, d\tau$$

$$F[\Delta(t)] = \frac{1}{j2\pi f} X(f) + \frac{1}{2}\delta(f) X(0)$$

$$= \mathrm{sinc}^2(f) + \underbrace{\frac{1}{2}\delta(f) X(0)}_{0} = \boxed{\mathrm{sinc}^2(f)}$$

(10) Modulation theorem

"Amplitude Modulation (AM)

$c(t) = A_c \cos(2\pi f_c t)$ → carrier signal

$m(t) = x(t)$ → message signal

$f_c >> f_m$

$y(t) = x(t)\, c(t)$ [redacted]

$$Y(f) = X(f) * C(f) = X(f) * \frac{A_c}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right]$$

$$Y(f) = \frac{A_c}{2} X(f-f_0) + \frac{A_c}{2} X(f+f_0)$$

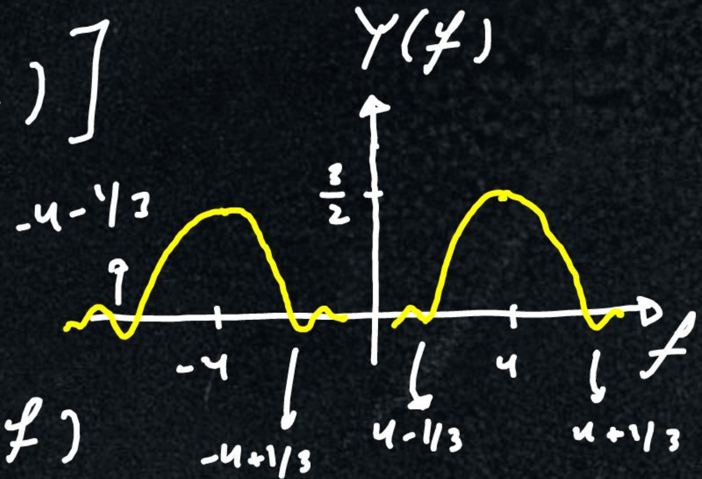
EX:- Determine the FT of $y(t) = \Pi(t/3)\cos(8\pi t)$

let $x(t) = \Pi(t/3)$

$c(t) = \cos(2\pi(4)t)$

$\underbrace{4}_{f_0}$

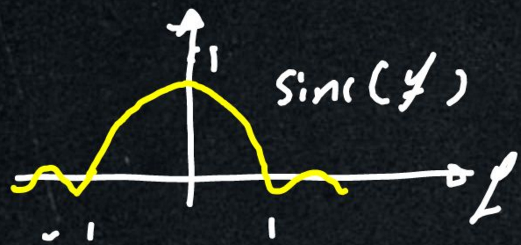
$$Y(f) = \frac{1}{2}\left[X(f-f_0) + X(f+f_0)\right]$$



$$F[\Pi(t)] = \text{sinc}(f)$$

$$F[x(t)] = F[\Pi(t/3)] = 3\,\text{sinc}(3f)$$

$$\Rightarrow Y(f) = \frac{3}{2}\,\text{sinc}(3(f-4)) + \frac{3}{2}\,\text{sinc}(3(f+4))$$



EX :- Determine the FT of $y(t) = \Delta(t/2)\cos(10\pi t)$

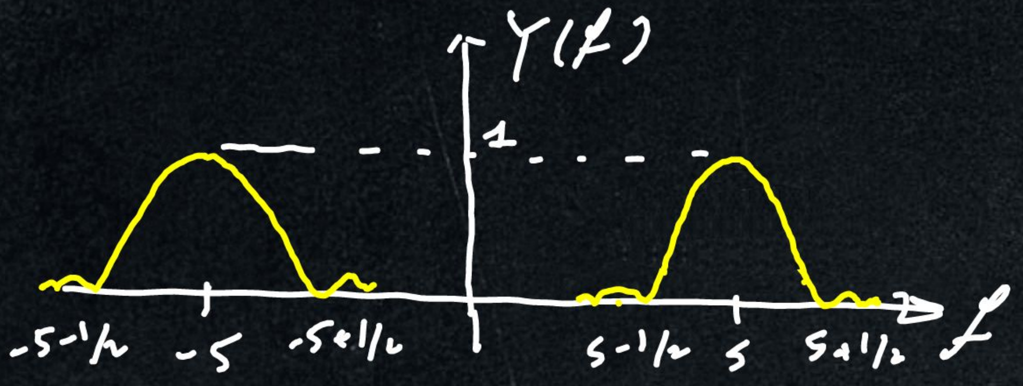
let $x(t) = \Delta(t/2)$

$c(t) = \cos(2\pi \underbrace{(5)}_{f_0} t)$

$$Y(f) = \frac{1}{2}X(f-5) + \frac{1}{2}X(f+5)$$

$$F[\Delta(t)] = \mathrm{sinc}^2(f) \Rightarrow X(f) = 2\,\mathrm{sinc}^2(2f)$$

$$Y(f) = \mathrm{sinc}^2(2(f-5)) + \mathrm{sinc}^2(2(f+5))$$



EX:- Find the spectral representation of the modulated signal $x(t) = A_m \cos(2\pi f_m t)$ by the carrier signal $c(t) = A_c \cos(2\pi f_c t)$.

$$y(t) = x(t)\,c(t) \xrightarrow{FT} Y(f) = X(f) * C(f)$$

$$Y(f) = \tfrac{1}{2} A_c X(f - f_c) + \tfrac{1}{2} A_c X(f + f_c)$$

but $X(f) = \tfrac{1}{2} A_m \delta(f - f_m) + \tfrac{1}{2} A_m \delta(f + f_m)$

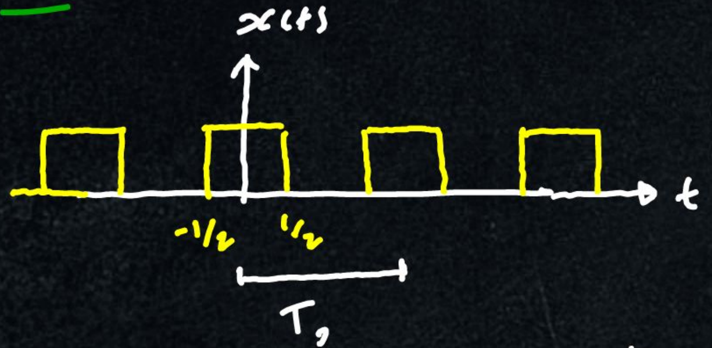
$$Y(f) = \underbrace{k}_{\frac{1}{4}A_c A_m}\,[\,\underbrace{\delta(f - f_c - f_m)}_{f_c + f_m} + \underbrace{\delta(f - f_c + f_m)}_{f_c - f_m} + \underbrace{\delta(f + f_c - f_m)}_{-f_c + f_m} + \underbrace{\delta(f + f_c + f_m)}_{-f_c - f_m}\,]$$

4.4] Fourier Transform of periodic signals

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \, e^{j2\pi n f_0 t}$$

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \, \delta(f - n f_0)$$

EX :. Find the FT of the signal $x(t)$

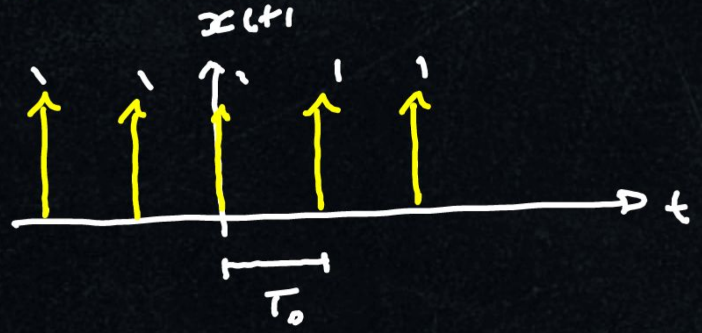


by applying the FS, we get

$$x(t) = \sum_{n=-\infty}^{\infty} \text{sinc}(n f_0) \, e^{j2\pi n f_0 t}$$

$$X(f) = \sum_{n=-\infty}^{\infty} \text{sinc}(n f_0) \, \delta(f - n f_0)$$

EX :- Find the FT of $x(t)$

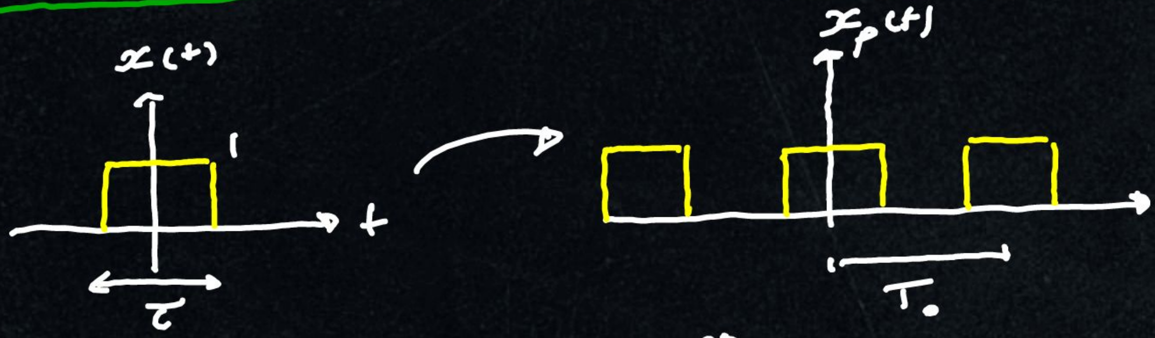


By applying the FS, we get

$$x(t) = \sum_{n=-\infty}^{\infty} f_0 \, e^{j2\pi f_0 t}$$

$$X(f) = \sum_{n=-\infty}^{\infty} f_0 \, \delta(f - n f_0)$$

4.5] FT of periodic signals via convolution theorem



$$x_p(t) = x(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} x(t - nT_0)$$

$x_p(t)$ ↓ periodic

$x(t)$ ↓ non-periodic

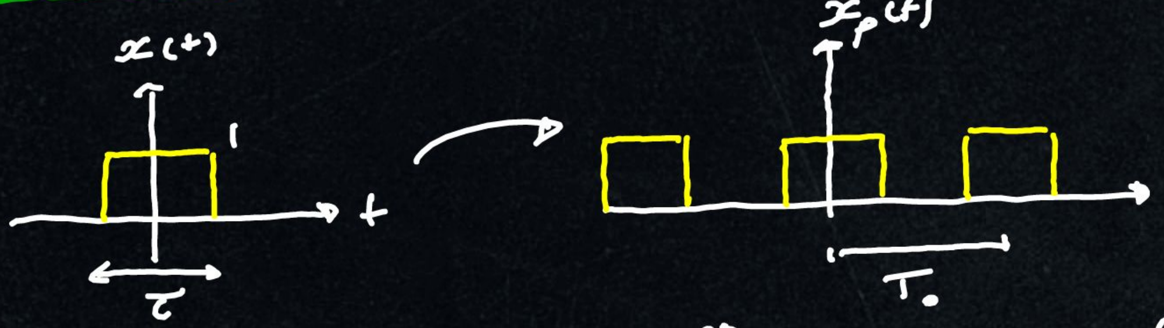
$$X_p(f) = X(f)\left[f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)\right]$$

$$X_p(f) = \sum_{n=-\infty}^{\infty} f_0\, X(nf_0)\, \delta(f - nf_0)$$

$$\Rightarrow \boxed{X_n = f_0\, X(nf_0)}$$

This can be used to find the FS coefficients

4.5] FT of periodic signals via convolution theorem



$$x_p(t) = x(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} x(t - nT_0)$$

$x_p(t)$ ↓ periodic

$x(t)$ ↓ non-periodic

$$X_p(f) = X(f) \left[f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0) \right]$$

$$X_p(f) = \sum_{n=-\infty}^{\infty} f_0 \, X(nf_0) \, \delta(f - nf_0)$$

$$\Rightarrow \boxed{X_n = f_0 \, X(nf_0)}$$

This can be used to find the FS coefficients