Exercises :

Q114: If X is a Random variable s.t $E(X)=3$ and $E(X^2)=13$, use chebyshev's inequality to determine a lowerbound for the $Pr(-2<X<8)$.

By Thm, $Pr(|X-\mu| < K\sigma) \geq 1-\frac{1}{K^2}$

By given : $\mu = 3$, $\sigma^2 = 13-(3)^2 = 4 \rightarrow \sigma = 2$.

$$\Rightarrow Pr(-2<X<8) = Pr(-5<X-3<5)$$

$$= Pr(|X-3|<5) \qquad 5 = 2K \rightarrow K = \frac{5}{2}$$

$$= Pr(|X-3|<\tfrac{5}{2}\,2)$$

$$\geq 1-\frac{1}{(\frac{5}{2})^2} = 1-\left(\frac{2}{5}\right)^2 = 1-\frac{4}{25}$$

$$\geq \frac{21}{25}$$

Q115: let X be a r.v with m.g.f. $M(t)$, $-h<t<h$. Prove that

1. $Pr(X \geq a) \leq e^{-at} M(t)$, $0<t<h$.

and 2. $Pr(X \leq a) \leq e^{-at} M(t)$, $-h<t<0$.

① $a>0$, $X>0$, $0<t<h$:

$$Pr(|X|\geq a) = Pr(X \geq a) = Pr(e^{tX} \geq e^{ta}) \leq \frac{E(e^{tX})}{e^{ta}} = e^{-at} M(t) \text{ , } 0<t<h$$

② $a<0$, $X<0$, $-h<t<0$:

$$Pr(|X| \geq a) = Pr(X \leq -a) = Pr(X \leq a) = Pr(e^{tX} \leq e^{ta}) \leq \frac{E(e^{tX})}{e^{ta}} = e^{-ta} M(t)$$

$-h<t<0$

Q116 : The m.g.f of X exists for all real variance of t and is given by

$$M(t) = \frac{e^{t} - e^{-t}}{2t} \quad , \quad t \neq 0 \quad , \quad M(0) = 1$$

Show that $Pr(X \geq 1) = 0$ and $Pr(X \leq -1) = 0$, by Q115.

$\rightsquigarrow$ $a = 1$, $t > 0$

$$Pr(X \geq 1) \leq e^{-at} M(t)$$

$$\leq e^{-t} \left(\frac{e^{t} - e^{-t}}{2t} \right) = \frac{1 - e^{-2t}}{2t}$$

$$\rightsquigarrow Pr(X \geq 1) \leq \lim_{t \to 0} \frac{1 - e^{-2t}}{2t} = 0$$

$\rightsquigarrow$ $a = -1$, $t < 0$

$$Pr(X \leq -1) \leq e^{-at} M(t)$$

$$\leq e^{t} \left(\frac{e^{t} - e^{-t}}{2t} \right) = \frac{e^{2t} - 1}{2t}$$

$$\rightsquigarrow Pr(X \leq -1) \leq \lim_{t \to 0} \frac{e^{2t} - 1}{2t} = 0$$