

Ch 7 Numerical Integration:

7.1 Quadrature formulas:

Def: Given $a = x_0 < x_1 < \dots < x_n = b$

A formula of the form:

$$Q[f] = \sum_{k=0}^n w_k f(x_k) = w_0 f(x_0) + w_1 f(x_1) + \dots + w_n f(x_n)$$

with the property

$$\int_a^b f(x) dx = Q[f] + E[f]$$

is called a Numerical integration or Quadrature formula.

$E[f]$ is the truncation error for integration.

$\{x_k\}_{k=0}^n$ are called quadrature nodes.

$\{w_k\}_{k=0}^n$ are called the weights

Newton-Cotes formulas: at equally spaced nodes:

1) Trapezoidal Rule:

$$\int_{x_0}^{x_1} f(x) dx \approx \frac{h}{2} (f_0 + f_1), \text{ with Error} = \frac{-h^3}{12} f^{(2)}(\xi)$$

(1)

2) Simpson's $\frac{1}{3}$ Rule:



$$\int_{x_0}^{x_2} f(x) dx \approx \frac{h}{3} (f_0 + 4f_1 + f_2) \quad , \text{ with Error} = \frac{-h^5 f^{(4)}(c)}{90}$$

3) Simpson's $\frac{3}{8}$ Rule:

$$\int_{x_0}^{x_3} f(x) dx = \frac{3h}{8} (f_0 + 3f_1 + 3f_2 + f_3) \quad , \text{ with Error} = \frac{-3h^5 f^{(4)}(c)}{80}$$

Ex: Estimate $\int_0^1 e^x dx = 1.7182818$ by:

1) $T(f) = \frac{1}{2} [e^0 + e^1] = 1.859141$

2) $S_{\frac{1}{3}}(f) = \frac{1}{2} \cdot \frac{1}{3} [e^0 + 4e^{\frac{1}{2}} + e^1] = 1.718861$

3) $S_{\frac{3}{8}}(f) = \frac{3(\frac{1}{3})}{8} [e^0 + 3e^{\frac{1}{3}} + 3e^{\frac{2}{3}} + e^1] = 1.718540$

Note: More partitions $\Rightarrow$ More precise.

• To Derive the formulas we use Newton or Lagrange Interpolations.

Example: Derive Trapezoidal Rule and its truncation error?

$$P_1(x) = \left(\frac{x - x_1}{x_0 - x_1} \right) f_0 + \left(\frac{x - x_0}{x_1 - x_0} \right) f_1$$

$$\begin{aligned} \int_{x_0}^{x_1} P_1(x) dx &= \frac{f_0}{-h} \left(\frac{(x - x_1)^2}{2} \right) \Big|_{x_0}^{x_1} + \frac{f_1}{h} \left(\frac{(x - x_0)^2}{2} \right) \Big|_{x_0}^{x_1} \\ &= \frac{h^2 f_1}{2h} - \frac{h^2 f_0}{-2h} = \frac{h}{2} (f_0 + f_1) \end{aligned}$$

$$E_1(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{2!} f^{(2)}(c)$$

$$\int_{x_0}^{x_1} E_1(x) dx = \int_{x_0}^{x_1} \frac{(x - x_0)(x - x_1)(x - x_2)}{2} f^{(2)}(c) dx$$

Use change of variable:

$$x - x_0 = ht$$

$$x - x_1 = x - x_0 + x_0 - x_1$$

$$= ht - h = h(t - 1)$$

:

$$x - x_k = h(t - k)$$

$$dx = h dt$$

$$\begin{aligned} \Rightarrow \int_{x_0}^{x_1} E_1(x) dx &= \int_0^1 \frac{ht \cdot h(t-1) \cdot h(t-2)}{2} f^{(2)}(c) dt = \frac{h^3}{2} f^{(2)}(c) \int_0^1 (t^2 - t) dt \\ &= -\frac{h^3}{12} f^{(2)}(c) \end{aligned} \quad (3)$$

Derive S_3 Rule:

$$\int_{x_0}^{x_2} f_2(x) dx = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2 \right] dx$$

Use change of variables:

$$\begin{aligned} x - x_0 &= ht \Rightarrow dx = h dt \\ x - x_1 &= h(t-1) \\ x - x_2 &= h(t-2) \end{aligned}$$

$$\Rightarrow \int_0^2 \left[\frac{h(t-1)h(t-2)}{-h(-2h)} f_0 + \frac{ht(t-2)h}{h(-h)} f_1 + \frac{ht h(t-1)}{(2h)(h)} f_2 \right] h dt$$

$$= h \int_0^2 \left[\frac{f_0}{2} (t^2 - 3t + 2) + f_1 (2t - t^2) + \frac{f_2}{2} (t^2 - t) \right] dt$$

$$= h \left[f_0 \left(\frac{1}{3} \right) + \frac{4}{3} f_1 + \frac{f_2}{3} \right] = \frac{h}{3} [f_0 + 4f_1 + f_2]$$

Example:

x	1	2	3	4
$f(x)$	20.1	22.5	25.6	28.9

$$\int_1^4 f(x) dx \approx S_3(f) = \frac{3(1)}{8} (f(1) + 3f(2) + 3f(3) + f(4))$$

or by Trapezoidal:

$$\int_1^4 f(x) dx \approx \frac{3}{2} (f(1) + f(4))$$

(4)

Def: The degree of precision (or accuracy)

of a quadrature formula is the largest positive integer K such that the formula is exact

for x^k , $k = 0, 1, 2, \dots$

(i.e) $\int_{-1}^1 Q_i(x) dx = 0$, $\forall i \leq K$ & $\int_{-1}^1 Q_{K+1}(x) dx \neq 0$

Exempl: Find D.O.P for $T(f)$.

k	$f(x)$	Exact	Formula
$k=0$	1	$\int_0^1 1 dx = 1$	$\frac{1}{2} [1+1] = 1$
$k=1$	x	$\int_0^1 x dx = \frac{1}{2}$	$\frac{1}{2} [0+1] = \frac{1}{2}$
$k=2$	x^2	$\int_0^1 x^2 dx = \frac{1}{3}$	$\frac{1}{2} [0+1] = \frac{1}{2} \quad \times$

$\Rightarrow$ D.O.P = 1

Exempl: Find D.O.P for $S_{\frac{1}{3}}(f)$.

$f(x)$	Exact	Formula
1	$\int_0^2 1 dx = 2$	$\frac{1}{3} [1+4+1] = 2$
x	$\int_0^2 x dx = 2$	$\frac{1}{3} [0+4+2] = 2$
x^2	$\int_0^2 x^2 dx = \frac{8}{3}$	$\frac{1}{3} [0+4+4] = \frac{8}{3}$
x^3	$\int_0^2 x^3 dx = 4$	$\frac{1}{3} [0+4+\cancel{4}^8] = 4$

(5)

$$x^4 \quad \int_0^2 x^4 dx = \frac{2^5}{5} \quad \frac{1}{3} [0 + 4 + 16] = \frac{20}{3} \quad \times$$

$$\Rightarrow \text{D.O.P } (S_3(f)) = 3.$$

$$\Rightarrow \text{Note: } \text{D.P } (T(f)) = 1$$

$$\text{D.P } (S_3(f)) = 3$$

$$\text{D.O.P } (S_3(f)) = 3$$

$$\text{Thm: } \text{Error}_{\text{trun.}} = n f^{(K+1)}(c), \text{ where}$$

$$K = \text{D.O.P} \quad \& \quad n \text{ is a constant.}$$

$$\text{Exempl: } \text{Derive } E_{\text{trun}}(T(f)) \quad (2)$$

$$\text{Since } \text{D.O.P } (T(f)) = 1 \Rightarrow E_{\text{trun}}(T(f)) = n f^{(2)}(c)$$

Need to find n :

Since the error starts from 2nd degree poly. then

$$\text{Let } f(x) = (x - x_0)^2 \Rightarrow f'(x) = 2(x - x_0), \quad f''(x) = 2.$$

$$\text{Exact Error} = \int_{x_0}^{x_1} (x - x_0)^2 dx - \frac{h}{2} [f_0 + f_1]$$

$$= \frac{h^3}{3} - \frac{h^3}{2} = -\frac{h^3}{6}.$$

$$\text{Therefore: } -\frac{h^3}{6} = n \cdot 2 \Rightarrow n = -\frac{h^3}{12}.$$

$$\Rightarrow E_{\text{trun.}} = -\frac{h^3}{12} f^{(2)}(c). \quad (6)$$

Example: Derive $S_{\frac{1}{3}}$ Rule's error.

Since D.O.P ($S_{\frac{1}{3}}(f)$) = 3 $\Rightarrow E_{\text{trunc.}} = n f^{(4)}(c)$.

Let $f(x) = (x-x_0)^4 \Rightarrow f^{(4)}(c) = 24$.

$$\text{Exact error} = \int_{x_0}^{x_2} (x-x_0)^4 dx - \frac{h}{3} [f_0 + 4f_1 + f_2]$$

$$= \frac{32h^5}{5} - \frac{h}{3} [0 + 4h^4 + 16h^4]$$

$$= \frac{32h^5}{5} - \frac{20}{3}h^5 = \frac{-4h^5}{15}$$

$$\text{Now: } \frac{-4h^5}{15} = n \cdot 24 \Rightarrow n = \frac{-h^5}{90}$$

$$\Rightarrow E_{\text{truncation}} = \frac{-h^5}{90} f^{(4)}(c)$$

Example Find D.O.P of $\int_0^{3h} f(x) dx \approx \frac{3h}{4} (3f(h) + f(3h))$
 Exact formula.

$$1 \quad \int_0^{3h} f(x) dx = 3h \quad \frac{3h}{4} (3 \cdot 1 + 1) = 3h$$

$$x \quad \int_0^{3h} x dx = \frac{9h^2}{2} \quad \frac{3h}{4} (3 \cdot h + 3h) = \frac{9h^2}{2}$$

$$x^2 \quad \int_0^{3h} x^2 dx = 9h^3 \quad \frac{3h}{4} (3h^2 + 9h^2) = 9h^3$$

$$x^3 \quad \int_0^{3h} x^3 dx = \frac{81h^4}{4} \quad \frac{3h}{4} (3h^3 + 27h^3) = \frac{90h^4}{4}$$

$$\Rightarrow \text{D.O.P} = 2 \quad (7)$$

Example: Find D.O.P for:

$$\int_a^b f(x) dx \approx Q[f] = (b-a) f\left(\frac{a+b}{2}\right).$$

formula.

$f(x)$

Exact

1

$$\int_a^b 1 dx = b-a$$

$$(b-a)$$

x

$$\int_a^b x dx = \frac{b^2}{2} - \frac{a^2}{2}$$

$$(b-a) \left(\frac{a+b}{2} \right) = \frac{b^2}{2} - \frac{a^2}{2}$$

x^2

$$\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3} \neq (b-a) \left(\frac{a+b}{2} \right)^2$$

$$\Rightarrow \text{D.O.P} = \boxed{1}$$

Note: we can use $a=0$ & $b=1$ to make it easier.

7.2 Composite Trapezoidal and Simpson's Rule:

Thm: Composite Trapezoidal Rule:

Suppose that the Interval $[a, b]$ is subdivided into M sub Intervals $[x_k, x_{k+1}]$ of width $h = \frac{b-a}{M}$.

$$(x_k = a + kh, \quad k = 0, 1, \dots, M).$$

The Composite trapezoidal rule for M subintervals can be expressed in any of three equivalent ways:

$$CT(f) = \frac{h}{2} \sum_{k=1}^M (f(x_{k-1}) + f(x_k))$$

$$\text{or } CT(f) = \frac{h}{2} (f_0 + 2f_1 + 2f_2 + \dots + 2f_{M-1} + f_M).$$

$$\text{or } CT(f) = \frac{h}{2} (f(a) + f(b)) + h \sum_{k=1}^{M-1} f(x_k)$$

Note: In General, h not necessarily be constant (Equal) for all partitions.

Example: Use Composite Trapezoidal of 3 Compositions

to estimate $\int_1^2 \frac{e^{x^2}}{\sin x} dx$.

$$CT(f) = \frac{\frac{1}{3}}{2} \left(f(1) + 2f\left(1\frac{1}{3}\right) + 2f\left(1\frac{2}{3}\right) + f(2) \right)$$

$$= \frac{1}{6} \left(3.2303 + 2 \frac{e^{(\frac{1}{3})^2}}{\sin(\frac{1}{3})} + 2 \frac{e^{(\frac{2}{3})^2}}{\sin(\frac{2}{3})} + \frac{e^{2^2}}{\sin 2} \right) =$$

Proof: Apply the trapezoidal rule over each sub Interval

$[x_{k-1}, x_k]$: then

$$\begin{aligned} \int_a^b f(x) dx &\approx \sum_{k=1}^M \frac{h}{2} (f(x_{k-1}) + f(x_k)) \\ &= \frac{h}{2} (f(x_0) + f(x_1) + f(x_1) + f(x_2) + \dots + f(x_{M-1}) + f(x_M)) \\ &= \frac{h}{2} (f(x_0) + 2f(x_1) + \dots + 2f(x_{M-1}) + f(x_M)) \end{aligned}$$

Error of Composite Trapezoidal:

$$= \frac{-h^3 f^{(2)}(c)}{12} \cdot M = \frac{-h^3 f^{(2)}(c)}{12} \left(\frac{b-a}{h} \right) = \frac{-(b-a)h^2 f^{(2)}(c)}{12}$$

Example: Find # of Compositions needed to estimate

$$\int_1^2 \ln(x+1) dx \quad \text{with accuracy } 10^{-4}.$$

$$f(x) = \ln(x+1) \Rightarrow f'(x) = \frac{1}{x+1} \Rightarrow f''(x) = \frac{-1}{(x+1)^2}$$

$$M_2 = \max_{1 \leq x \leq 2} |f''(x)| = \frac{1}{4}$$



$$\Rightarrow E = -\frac{1}{12} \left(\frac{2-1}{M} \right)^2 (2-1) \left(\frac{1}{4} \right) < 10^{-4}$$

$$\Rightarrow M^2 > 208.33$$

$$\Rightarrow M > 14.4 \Rightarrow \boxed{M = 15}$$

$$\# \text{ of points} = 16.$$

Note: Although the error in $CT(f)$ contains $O(h^2)$ but while in $T(f)$ contains $O(h^3)$, but $CT(f)$ is more accurate since h is smaller because of division.

Example:

t	0	1	2	3
$f(t)$	10	12	13	15

$$\int_0^3 f(t) dt = \frac{1}{2} (f(0) + 2f(1) + 2f(2) + f(3)) = 37.5$$

Example:

t	1	3	4	5
$f(t)$	10	15	20	21

$$\begin{aligned} \int_1^5 f(t) dt &= \frac{2}{2} (f(1) + f(3)) + \frac{1}{2} (f(3) + f(4)) + \frac{1}{2} (f(4) + f(5)) \\ &= \frac{2}{2} (10 + 15) + \frac{1}{2} (15 + 2(20) + 21) = 73. \end{aligned}$$

Example: Estimate: $\int_{-1}^3 x e^{-x} dx$ Using two compositions of $T(f)$

$$CT(f) = \frac{h}{2} (f(-1) + 2f(1) + f(3))$$

Thm: Composite Simpson's $\frac{1}{3}$ Rule:

Suppose that $[a, b]$ is subdivided into $2M$ subIntervals

$[x_k, x_{k+1}]$ of equal width $h = \frac{b-a}{2M}$, $x_k = a + hk$.

for $k = 0, 1, 2, \dots, 2M$. Then the Composite Simpson's rule for $2M$ subIntervals can be expressed as

$$CS(f) = \frac{h}{3} \sum_{k=1}^M (f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k}))$$

$$\text{or } CS(f) = \frac{h}{3} (f_0 + 4f_1 + 2f_2 + 4f_3 + \dots + 4f_{2M-1} + f_{2M})$$

$$\text{or } CS(f) = \frac{h}{3} (f(a) + f(b)) + \frac{2h}{3} \sum_{k=1}^{M-1} f(x_{2k}) + \frac{4h}{3} \sum_{k=1}^M f(x_{2k-1})$$

Proof: $\int_0^{x_{2m}} f(x) dx = \int_0^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \dots + \int_{x_{2m-2}}^{x_{2m}} f(x) dx$

$$= \frac{h_1}{3} (f_0 + 4f_1 + f_2) + \frac{h_2}{3} (f_2 + 4f_3 + f_4) + \dots + \frac{h_m}{3} (f_{2m-2} + 4f_{2m-1} + f_{2m})$$

If h_i are equal:

$$= \frac{h}{3} (f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \dots + 4f_{2m-1} + f_{2m})$$

$$= \frac{h}{3} \sum_{k=1}^m (f_{2k-2} + 4f_{2k-1} + f_{2k})$$

Error for CS(f) (we apply $S_{\frac{1}{3}}$: M times)

$$= -\frac{h^5}{90} f^{(4)}(c) \cdot M = -\frac{h^5}{90} f^{(4)}(c) \cdot \left(\frac{b-a}{2h}\right)$$

$$= \frac{-h^4}{180} f^{(4)}(c) \cdot (b-a).$$

Example Find the number M and the step size h needed to estimate

$$2 \int_1^2 \ln(x+1) dx \quad \text{with } E < 10^{-6}$$

$$f^{(4)}(x) = \frac{-6}{(1+x)^4} \Rightarrow M_4 = \max_{1 \leq x \leq 2} |f^{(4)}(x)| = 0.375$$

then:

$$\left(\frac{2-1}{2M}\right)^4 \cdot \frac{0.375 \cdot (2-1)}{180} < 10^{-6}$$

$$(2M)^4 > 2083.333 \Rightarrow M^4 > 130.2083$$

$$\Rightarrow M > 3.378 \Rightarrow \boxed{M = 4}$$

$$\leftarrow h = \frac{2-1}{2(4)} = \frac{1}{8} = 0.125.$$

7.5 Gauss - Legendre Integration.

Degree of precision = $2n - 1$, where $n = \#$ of points
for $G_n(f)$

Case 1 : Assume we have 2 - points:

$$\Rightarrow \text{D.O.P } (G_2(f)) = 2(2) - 1 = 3.$$

$$\int_{-1}^1 f(x) dx \approx w_1 f(x_1) + w_2 f(x_2) = G_2(f)$$

We need 4 equations to find x_1, x_2, w_1, w_2 :

$$1) E(1) = 0 \Rightarrow \int_{-1}^1 1 dx = 2 = w_1(1) + w_2(1)$$

$$2) E(x) = 0 \Rightarrow \int_{-1}^1 x dx = 0 = w_1 x_1 + w_2 x_2$$

$$3) E(x^2) = 0 \Rightarrow \int_{-1}^1 x^2 dx = \frac{2}{3} = w_1 x_1^2 + w_2 x_2^2$$

$$4) E(x^3) = 0 \Rightarrow \int_{-1}^1 x^3 dx = 0 = w_1 x_1^3 + w_2 x_2^3$$

Solve the four equations:

$$\Rightarrow w_1 = w_2 = 1 \quad \& \quad x_1 = \frac{1}{\sqrt{3}}, \quad x_2 = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow G_2(f) = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right).$$

Example: Estimate $\int_{-1}^1 \frac{1}{x+2} dx$ using:

$$1) T(f) = \frac{2}{2} (f(-1) + f(1)) = 1.3333$$

$$2) S_{\frac{1}{3}}(f) = \frac{1}{3} (f(-1) + 4f(0) + f(1)) = 1.1111$$

$$3) G_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 1.0909$$

$$\text{Exact: } 1.09861$$

Example: Find the Error for $G_2(f)$.

Since D.O.P = 3, then Let $f(x) = x^4$

$$\text{Exact error} = \int_{-1}^1 x^4 dx - G_2(f)$$

$$= \frac{2}{5} - \left[\frac{1}{9} + \frac{1}{9} \right] = \frac{2}{5} - \frac{2}{9} = \frac{8}{45}$$

$$\text{then } \frac{8}{45} = n f^{(4)}(c) = n \cdot 24 \Rightarrow \boxed{n = \frac{1}{135}}$$

$$\Rightarrow E_{\text{true}}(G_2(f)) = \frac{f^{(4)}(c)}{135}$$

Note: If we assume $f(x) = (x+1)^4$, then

$$\int (x+1)^4 dx = \frac{32}{5}, \quad f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right) = \frac{56}{9}$$

$$\Rightarrow \text{Error} = \frac{32}{5} - \frac{56}{9} = \frac{8}{45}$$

Case 2: Gauss-Legendre of 3 points.

$$D.O.P = 2 \cdot 3 - 1 = 5.$$

Then $\int_{-1}^1 f(x) dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$
we need 6 equations. (As we did in $G_2(x)$)

$$\Rightarrow G_3(f) = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

with Truncation error : $E = \frac{f^{(6)}(c)}{15.750}$.

Case 3: Gauss-Legendre of 1 point

$$D.O.P = 2 \cdot 1 - 1 = 1$$

$$\int_{-1}^1 f(x) dx \approx w_1 f(x_1)$$

$$\Rightarrow E(1) = 0 \Rightarrow \int_{-1}^1 1 dx = 2 = w_1$$

$$\Rightarrow E(x) = 0 \Rightarrow \int_{-1}^1 x dx = 0 = w_1 x_1$$

$$\Rightarrow G_1(f) = 2 f(0).$$

Thm: Gauss - Legendre translation:

Suppose that the points $\{x_k\}$ & $\{w_k\}$ are given for N points Gauss-Legendre rule over $[-1, 1]$.
To apply the rule over the Interval $[a, b]$, use

The change of variable:

$$t = \frac{a+b}{2} + \frac{b-a}{2} x$$

$$\begin{aligned} x=1 &\Rightarrow t=b \\ x=-1 &\Rightarrow t=a \end{aligned}$$

$$dt = \frac{b-a}{2} dx$$

$$\Rightarrow \int_a^b f(t) dt = \int_{-1}^1 f\left(\frac{a+b}{2} + \frac{b-a}{2} x\right) \frac{b-a}{2} dx$$

$$\Rightarrow \int_a^b f(t) dt \approx \frac{b-a}{2} \sum_{k=1}^N w_k f\left(\frac{a+b}{2} + \frac{b-a}{2} x_k\right)$$

Example: $\int_1^5 \frac{dt}{t} = \ln(5) - \ln(1) = 1.609438$ (Exact)

Using $G_3(f)$: with $a=1$, $b=5$, $f(t) = \frac{1}{t}$

$$\Rightarrow G_3(f) = \frac{5-1}{2} \left(\frac{5}{9} f\left(\frac{5+1}{2} + \frac{5-1}{2} \left(-\sqrt{\frac{3}{5}}\right)\right) + \frac{8}{9} f\left(\frac{5+1}{2} + \frac{5-1}{2} (0)\right) + \frac{5}{9} f\left(\frac{5+1}{2} + \frac{5-1}{2} \left(\sqrt{\frac{3}{5}}\right)\right) \right)$$

$$\begin{aligned} &= 2 \left(\frac{5}{9} f(3 - 2(0.6)^{\frac{1}{2}}) + \frac{8}{9} f(3) + \frac{5}{9} f(3 + 2(0.6)^{\frac{1}{2}}) \right) \\ &= 1.602694. \end{aligned}$$

Example: Estimate $\int_0^1 e^t dt$ using $G_2(f)$

$$t = \frac{a+b}{2} + \frac{b-a}{2} x = \frac{1}{2} + \frac{1}{2} x$$

$$dt = \frac{1}{2} dx$$

$$\begin{aligned}\Rightarrow G_2(f) &= \frac{1}{2} f\left(\frac{1}{2} + \frac{1}{2}\left(\frac{-1}{\sqrt{3}}\right)\right) + f\left(\frac{1}{2} + \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)\right) \\ &= \frac{1}{2} \left[e^{\frac{1}{2} - \frac{1}{2\sqrt{3}}} + e^{\frac{1}{2} + \frac{1}{2\sqrt{3}}} \right]\end{aligned}$$