3.3 : The Gamma and the chi squared distribution.

Notation : γ , Γ $\rightarrow$ Gamma .

Def : $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} \, dy$, $\alpha > 0$

Remark: $\Gamma(\alpha)$ exists and positive for all $\alpha > 0$.

Note 1 : $\Gamma(1) = 1$

$$\Gamma(1) = \int_0^\infty y^{1-1} e^{-y} \, dy = \int_0^\infty e^{-y} \, dy = 1 .$$

Note 2: $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$, $\alpha > 1$.

proof : $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} \, dy$ integral by part $u = y^{\alpha-1}$ $dv = e^{-y} dy$

$du = (\alpha-1) y^{\alpha-2} dy$ $v = -e^{-y}$.

$$= (y^{\alpha-1})(-e^{-y}) \Big|_0^\infty - \int_0^\infty -(e^{-y})(\alpha-1) y^{\alpha-2} \, dy$$

$$= \left[\lim_{y \to \infty} (-e^{-y} y^{\alpha-1}) - 0 \right] + (\alpha-1) \int_0^\infty y^{\alpha-2} e^{-y} \, dy$$

$$\stackrel{\alpha>1}{=} 0 + (\alpha-1)\Gamma(\alpha-1)$$

$$= (\alpha-1)\Gamma(\alpha-1) , \alpha > 1 .$$

Note 3: $\Gamma(\alpha) = (\alpha-1)!$ $\alpha = 2, 3, ---$.

$\rightarrow \Gamma(2) = (2-1)\Gamma(1) = (1)(1) = 1!$

$\Gamma(3) = (3-1)\Gamma(2) = 2(1) = 2!$

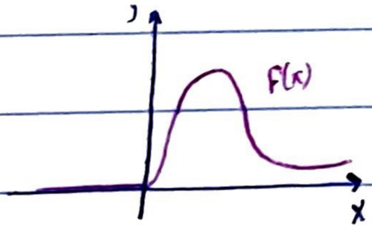
$\Gamma(4) = (4-1)\Gamma(3) = (3)(2)(1) = 3!$.

Def: $0! = 1$.

Def: a Random variable X has a Gamma distribution with parameters α and β if it has the following p.d.f:

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}} & , x > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

Where $\alpha > 0$, $\beta > 0$.



skewed .

Notation: $X \sim Gamma(\alpha, \beta)$.

Question : verify that $f(x)$ is p.d.f .

$\rightarrow f(x) = 0 \quad , x \in (-\infty, 0]$.

$\rightarrow f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{x}{\beta}} x^{\alpha-1} > 0 \quad , x \in (0, \infty) \, , \alpha > 0 \, , \beta > 0$.

$\rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{x}{\beta}} x^{\alpha-1}\,dx$.

$\boxed{* \, \Gamma(\alpha) = \int_{0}^{\infty} y^{\alpha-1} e^{-y}\,dy}$

let $y = \frac{x}{\beta} \Rightarrow dy = \frac{dx}{\beta}$

$\Rightarrow dx = \beta\,dy$.

$$\text{so } \int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} e^{-y} (\beta y)^{\alpha-1} \beta \, dy$$

$$= \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)\cancel{\beta^{\alpha}}} e^{-y} \cancel{\beta^{\alpha}} y^{\alpha-1}\,dy$$

$$= \frac{1}{\Gamma(\alpha)} \underbrace{\int_{0}^{\infty} y^{\alpha-1} e^{-y}\,dy}$$

$$= \frac{1}{\Gamma(\alpha)} \cdot \Gamma(\alpha)$$

$$= 1$$

So $f(x)$ is p.d.f

Question : Find $E(X)$, $var(X)$, γ_X.
Assume $X \sim Gamma(\alpha, \beta)$. } H.W

Question : Find M.g.f of X, $X \sim Gamma(\alpha, \beta)$

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{x}{\beta}} x^{\alpha-1}\, dx$$

$$= \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x\left(\frac{1}{\beta}-t\right)}\, dx$$

$\frac{1}{\beta} - t = \frac{1-\beta t}{\beta}$

$$= \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x\left(\frac{1-\beta t}{\beta}\right)}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_{0}^{\infty} x^{\alpha-1} e^{-x/\frac{\beta}{1-\beta t}}\, dx$$

$$= \frac{1}{\cancel{\Gamma(\alpha)}\beta^{\alpha}} \cancel{\Gamma(\alpha)} \left(\frac{\beta}{1-\beta t}\right)^{\alpha}, \quad \beta < \frac{1}{\beta}$$

$$= \frac{1}{\cancel{\beta^{\alpha}}} \cdot \frac{\cancel{\beta^{\alpha}}}{(1-\beta t)^{\alpha}}, \quad t < \frac{1}{\beta}$$

$$= \frac{1}{(1-\beta t)^{\alpha}} = (1-\beta t)^{-\alpha}, \quad t < \frac{1}{\beta}.$$

$X \sim Gamma(\alpha, \beta)$:

$M(t) = (1-\beta t)^{-\alpha}, \quad t < \frac{1}{\beta}.$

Note : $\alpha > 0$: $\Gamma(\alpha) = \int_{0}^{\infty} y^{\alpha-1} e^{-y}\, dy$

$\alpha > 0, \beta > 0$: $\Gamma(\alpha)\beta^{\alpha} = \int_{0}^{\infty} y^{\alpha-1} e^{-\frac{y}{\beta}}\, dy$.

$\rightarrow E(X) = \mu = \bar{M}(0) = \alpha\beta$.

$\rightarrow Var(X) = \sigma^2 = \bar{\bar{M}}(0) - (\bar{M}(0))^2 = \alpha\beta^2$. check .

$\rightarrow \sigma_X = \sqrt{\sigma^2} = \sqrt{\alpha}\,\beta$.

Def : X is called a chi-squared Random variable with r degree of freedom if its Gamma (α, β) with $\alpha = \frac{r}{2}$, $\beta = 2$.

Notation : $X \sim \chi^2(r)$.

RMK :

$\rightarrow E(X) = \mu_X = r$

$\rightarrow Var(X) = \sigma_X^2 = 2r$

$\rightarrow M(t) = (1-2t)^{-\frac{r}{2}}$, $t < \frac{1}{2}$.

RMK :

$X \sim Gamma(1, \beta)$, $f(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}} & , x > 0 \\ 0 & , \text{elsewhere} \end{cases}$

Def : $\lambda = \frac{1}{\beta}$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

$X \sim exponential(\lambda)$.

exp6: $X \sim Gamma(\alpha, \beta)$, $\alpha > 0, \beta > 0$

$$Y = \frac{2X}{\beta}$$

f: p.d.f of X g: p.d.f of y

F: c.d.f of X G: c.d.f of y

$$\Rightarrow G(y) = Pr(Y \leq y) = Pr\left(\frac{2X}{\beta} \leq y\right)$$
$$= Pr\left(X \leq \frac{\beta y}{2}\right)$$
$$= F\left(\frac{\beta y}{2}\right)$$

$$\Rightarrow g(y) = G'(y) = \frac{dF}{dy}\left(\frac{\beta y}{2}\right) = f\left(\frac{\beta y}{2}\right)\frac{\beta}{2}$$

$$= \frac{\beta}{2}\begin{cases} \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\left(\frac{\beta y}{2}\right)^{\alpha-1} e^{-\frac{\beta y}{2}/\beta} & , \frac{\beta y}{2} > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

$$= \begin{cases} \frac{1}{\Gamma(\alpha)\cancel{\beta^{\alpha}}} \cdot \frac{\cancel{\beta}}{2} \cdot \frac{\cancel{\beta^{\alpha-1}}\, y^{\alpha-1}}{2^{\alpha-1}} e^{-\frac{y}{2}} & , y > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

$$= \begin{cases} \frac{1}{\Gamma(\alpha)2^{\alpha}} y^{\alpha-1} e^{-\frac{y}{2}} & , y > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

$$Y = \frac{2X}{\beta} \sim Gamma(\alpha, 2)$$

$$\Rightarrow Y \sim \chi^2(r) \quad , \quad r = 2\alpha$$

$$\boxed{\begin{matrix} Gamma(\alpha, \beta) \rightarrow \chi^2(2\alpha) \\ X \rightarrow \frac{2X}{\beta} \end{matrix}}$$