

10.9 Convergence of Taylor series.

10 Use substitution to find the Taylor series at $x=0$ of the function $\frac{1}{2-x}$.

$$\frac{1}{2-x} = \frac{1}{2\left(1 - \frac{x}{2}\right)}$$

$$= \frac{1}{2} \left[\frac{1}{1 - \frac{x}{2}} \right]$$

Taylor series at $\alpha=0$ of $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$

Notice that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

So, by substitution we have :-

geometric series with
 $a=1$ & $r=x$

$$\frac{1}{1 - \frac{x}{2}} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$\text{Now, } \frac{1}{2-x} = \frac{1}{2} \left[\frac{1}{1 - \frac{x}{2}} \right]$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

$$= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \dots$$

12 Use power series operations to find the Taylor series at $x=0$ for the function $x^2 \sin x$

★ Remember:

Taylor series generated by $\sin x$ at $x=0$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$x^2 \sin x = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}$$

$$= x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \dots$$

18 Use power series operations to find the Taylor series at $x=0$ for the function $\sin^2 x$.

we know that :-

$$\Rightarrow \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

* Remember :-

$$\cos(2x) = 1 - 2\sin^2 x.$$

$$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$$

$$\Rightarrow \cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$$

$$\Rightarrow -\frac{1}{2} \cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!} = -\frac{1}{2} + \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \dots$$

$$\Rightarrow \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos(2x)$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \dots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}$$

(28) Use power series operations to find the Taylor series at $x=0$ for the function $\ln(x+1) - \ln(1-x)$.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

Remember $\Rightarrow$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$
$$= 1 - x + x^2 - x^3 + \dots$$

$$\int \frac{1}{1+x} dx = \int (1 - x + x^2 - x^3 + \dots) dx$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\Rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\int \frac{1}{1-x} dx = \int (1 + x + x^2 + x^3 + \dots) dx$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

Now, $\star + \star + \star$

$$\ln(1+x) - \ln(1-x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \dots$$

$$= \sum_{n=0}^{\infty} 2 \frac{x^{2n+1}}{2n+1}$$

35 Estimate the error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate the value of $\sin x$ at $x = 0.1$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$P_3(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}$$

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x$$

$$f^{(4)}(x) = \sin x$$

$$\text{Since } n = 3 \Rightarrow f^{(4)}(x) = \sin x$$

$$|f^{(4)}(x)| \leq M \quad \text{on } [0, 0.1]$$

$$|\sin x| \leq 1 \quad \text{on } [0, 0.1]$$

$$\Rightarrow \text{So, } M = 1$$

$$|R_3(0.1)| \leq \frac{M |0.1 - 0|^{3+1}}{(3+1)!}$$

$$|R_3(0.1)| \leq \frac{(1) |0.1|^4}{4!}$$

$$\leq \frac{0.1^4}{4!} \approx 4.167 \times 10^{-6}$$

Note: You can use alternating series estimation theorem to solve this question.

37 For approximately what values of x can you replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitude no greater than 5×10^{-4} ?

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

By alternating series estimation theorem:

$\Rightarrow |error| < | \text{first neglected term} |$ then

$$|error| < \frac{|x^5|}{5!}, \text{ so:}$$

$$\frac{|x^5|}{5!} < 5 \times 10^{-4} \Rightarrow |x|^5 < 5! (5 \times 10^{-4}) = 0.06$$

$$\Rightarrow |x| < \sqrt[5]{0.06} \approx 0.5698$$

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The approximation $e^x = 1 + x + \frac{x^2}{2}$ is used when x is small. Use the Remainder Estimation Theorem to estimate the error when $|x| < 0.1$.

$$e^x = \underbrace{1 + x + \frac{x^2}{2}}_{P_2(x)} \quad \left(\begin{array}{l} \text{Taylor Polynomial of order 2} \\ n=2 \end{array} \right)$$

$$P^{(n+1)}(x) = P^{(3)}(x) = e^x,$$

$$\underline{\text{but}} \quad |x| < 0.1 \Rightarrow |P^{(3)}(x)| < |e^x| = e^x < e^{0.1} < 3^{0.1}$$

Hence, By remainder estimation theorem.

$$\begin{aligned} |R_2(x)| &< \frac{3^{0.1} |x|^3}{3!} \\ &< \frac{3^{0.1} (0.1)^3}{3!} = 1.87 \times 10^{-4} \end{aligned}$$

10.10 : The Binomial series & Applications of Taylor series.

10 Find the first four terms of the binomial series for the

function $\frac{x}{\sqrt[3]{1+x}}$

remember:

The Binomial series for
 $(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$

$$\binom{m}{k} = \frac{m(m-1)(m-2)\dots(m-k+1)}{k!}$$

$$\frac{x}{\sqrt[3]{1+x}} = x (1+x)^{-\frac{1}{3}}, \quad m = -\frac{1}{3}$$

$$= x \left[1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{3}}{k} x^k \right]$$

$$= x \left[1 + \binom{-\frac{1}{3}}{1} x^1 + \binom{-\frac{1}{3}}{2} x^2 + \binom{-\frac{1}{3}}{3} x^3 + \binom{-\frac{1}{3}}{4} x^4 + \dots \right]$$

$$= x \left[1 + -\frac{1}{3} x + \frac{(-\frac{1}{3})(-\frac{1}{3}-1)}{2} x^2 + \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{1}{3}-3+1)}{3!} x^3 + \dots \right]$$

$$= x - \frac{x^2}{3} + \frac{4}{9(2)} x^3 + \frac{(\frac{4}{9})(-\frac{7}{3})}{6} x^4 + \dots$$

$$\frac{x}{\sqrt[3]{1+x}} = x - \frac{x^2}{3} + \frac{2}{9} x^3 - \frac{14}{81} x^4 + \dots$$

16 Use the series to estimate the integral's values with an error of magnitude less than 10^{-3} .

$$\int_0^{0.2} \frac{e^{-x} - 1}{x} dx.$$

we know, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\Rightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$\Rightarrow \frac{e^{-x} - 1}{x} = \frac{1}{x} (e^{-x} - 1) = \frac{1}{x} \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} - 1 \right).$$

$$= \frac{1}{x} \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \right) - 1$$

$$= \frac{1}{x} \left(-x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \right)$$

$$= -1 + \frac{x}{2!} - \frac{x^2}{3!} + \frac{x^3}{4!} - \dots$$

$$\Rightarrow \int_0^{0.2} \frac{e^{-x} - 1}{x} dx = \int_0^{0.2} \left(-1 + \frac{x}{2!} - \frac{x^2}{3!} + \frac{x^3}{4!} - \dots \right) dx.$$

$$= -x + \frac{x^2}{2 \cdot 2!} - \frac{x^3}{3 \cdot 3!} + \frac{x^4}{4 \cdot 4!} - \dots \Big|_0^{0.2}$$

$$= -0.2 + \frac{(0.2)^2}{2 \cdot 2!} - \frac{(0.2)^3}{3 \cdot 3!} + \frac{(0.2)^4}{4 \cdot 4!} - \dots$$

now, with an error of magnitude less than 10^{-3} ($|\text{error}| < 10^{-3}$),

we find the first term to be numerically less than 10^{-3} :

$$\frac{(0.2)^3}{3 \cdot 3!} = \frac{0.008}{18} = 4.4 \times 10^{-4} < 10^{-3}$$

So, the first neglected term is $\frac{(0.2)^3}{3 \cdot 3!}$.

$$\int_0^{0.2} \left(\frac{e^{-x} - 1}{x} \right) dx \approx -(0.2) + \frac{(0.2)^2}{4} \quad \left[\text{The estimated value of the integral} \right]$$

26 Find a polynomial that will approximate $F(x)$ throughout the given interval with an error of magnitude less than 10^{-3} .

$$F(x) = \int_0^x t^2 e^{-t^2} dt, \quad [0, 1].$$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Rightarrow e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

$$t^2 e^{-t^2} = t^2 \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+2}}{n!}$$

$$t^2 e^{-t^2} = t^2 - \frac{t^4}{1!} + \frac{t^6}{2!} - \frac{t^8}{3!} + \frac{t^{10}}{4!} - \frac{t^{12}}{5!} + \dots$$

$$\begin{aligned} \Rightarrow F(x) &= \int_0^x t^2 e^{-t^2} dt = \int_0^x \left(t^2 - \frac{t^4}{1!} + \frac{t^6}{2!} - \frac{t^8}{3!} + \frac{t^{10}}{4!} - \frac{t^{12}}{5!} + \dots \right) dt \\ &= \left[\frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7 \cdot 2!} - \frac{t^9}{9 \cdot 3!} + \frac{t^{11}}{11 \cdot 4!} - \frac{t^{13}}{13 \cdot 5!} + \dots \right]_0^x \\ &= \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7 \cdot 2!} - \frac{x^9}{9 \cdot 3!} + \frac{x^{11}}{11 \cdot 4!} - \frac{x^{13}}{13 \cdot 5!} + \dots \end{aligned}$$

$\Rightarrow$ Now, with an error of magnitude less than 10^{-3} , we find that the first term to be numerically less than 10^{-3} (0.001) is $\frac{1}{13 \cdot 5!}$.

$$|\text{error}| < \left| \frac{x^{13}}{13 \cdot 5!} \right| = \frac{|x|^{13}}{13 \cdot 5!} < \frac{1}{13 \cdot 5!} \approx 6.4 \times 10^{-4} < 10^{-3}.$$

$$\Rightarrow \text{So, } F(x) \approx \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7 \cdot 2!} - \frac{x^9}{9 \cdot 3!} + \frac{x^{11}}{11 \cdot 4!}.$$

Use Series to evaluate the Limits:

30 $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \dots$$

$$\frac{e^x - e^{-x}}{x} = 2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \dots$$

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x} = \lim_{x \rightarrow 0} \left(2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \dots \right) = \boxed{2}$$

33 $\lim_{y \rightarrow 0} \frac{y - \tan^{-1} y}{y^3}$

$$\tan^{-1} y = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{2n+1}, \quad |y| \leq 1$$

$$= y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \dots$$

$$\lim_{y \rightarrow 0} \frac{y - \tan^{-1} y}{y^3} = \lim_{y \rightarrow 0} \frac{y - y + \frac{y^3}{3} - \frac{y^5}{5} + \frac{y^7}{7} - \dots}{y^3}$$

$$= \lim_{y \rightarrow 0} \left(\frac{1}{3} - \frac{y^2}{5} + \frac{y^4}{7} - \dots \right)$$

$$= \boxed{\frac{1}{3}}$$