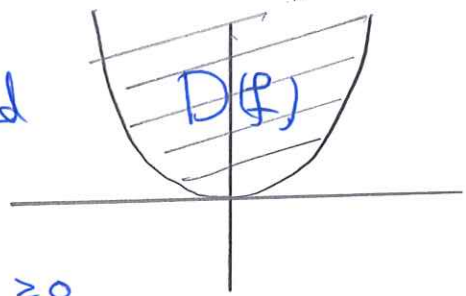


14.1 Domains and Ranges

$$(1) \quad z = f(x, y) = \sqrt{y - x^2}$$

$$D_f = \{(x, y) : y - x^2 \geq 0\} = \{(x, y) : y \geq x^2\}$$

Domain is closed and unbounded



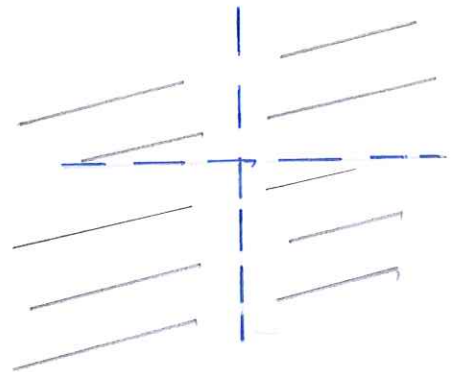
$$R_f = [0, \infty) \quad , \quad \left[\begin{array}{l} y - x^2 \geq 0 \\ z = \sqrt{y - x^2} \geq 0 \Rightarrow z \geq 0 \end{array} \right]$$

$$(2) \quad f(x, y) = \frac{1}{xy} = z$$

$$D_f = \{(x, y) : x \neq 0 \text{ and } y \neq 0\}$$

Domain is unbounded and open

$$R_f = (-\infty, 0) \cup (0, \infty)$$



$$(3) \quad f(x, y) = e^{-x^2 - y^2} = \frac{1}{e^{x^2 + y^2}} > 0$$

D_f = Entire plane which is clopen (closed & open)

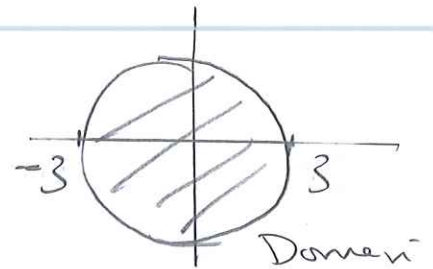
$$R_f = (0, 1] \quad \left\{ \begin{array}{l} x^2 + y^2 \geq 0 \Rightarrow 0 < e^{-x^2 - y^2} \leq e^0 = 1 \\ \Rightarrow 0 < z \leq 1 \end{array} \right.$$

(2) (49)

$$(4) \quad f(x, y) = \sqrt{9 - x^2 - y^2}$$

$$D_f = \{(x, y) : 9 - x^2 - y^2 \geq 0\} = \{(x, y) : x^2 + y^2 \leq 9\}$$

Domain is closed and bounded.



$$\begin{aligned} \text{Range :} \quad & 0 \leq x^2 + y^2 \leq 9 \\ & -9 \leq -x^2 - y^2 \leq 0 \\ & 0 \leq 9 - x^2 - y^2 \leq 9 \\ \therefore & 0 \leq \underbrace{\sqrt{9 - x^2 - y^2}}_z \leq 3 \end{aligned}$$

$$\Rightarrow \text{Range : } [0, 3]$$

Recall

$$\begin{cases} D(\sin^{-1}x) = [-1, 1] \\ \text{Range}(\sin^{-1}x) = \\ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \end{cases}$$

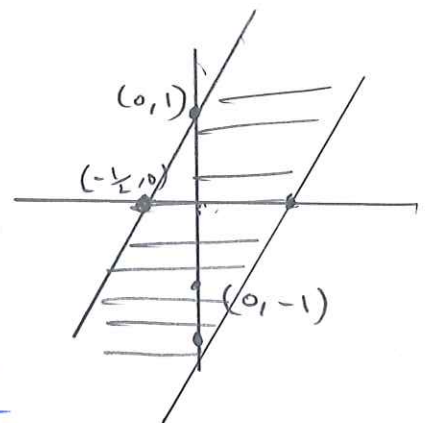
$$(5) \quad f(x, y) = 4 \sin^{-1}(y - 2x)$$

$$\begin{aligned} D_f &= \{(x, y) : -1 \leq y - 2x \leq 1\} \\ &= \{(x, y) : 2x - 1 \leq y \leq 2x + 1\} \end{aligned}$$

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Domain is unbounded and closed.

$$\begin{aligned} \text{Range :} \quad & -\frac{\pi}{2} \leq \sin^{-1}(y - 2x) \leq \frac{\pi}{2} \\ & -4\frac{\pi}{2} \leq 4 \sin^{-1}(y - 2x) \leq 4\frac{\pi}{2} \\ \Rightarrow & -2\pi \leq \underbrace{4 \sin^{-1}(y - 2x)}_z \leq 2\pi \end{aligned}$$



$$\Rightarrow \text{Range } [-2\pi, 2\pi]$$

$$(6) \quad z = f(x, y) = 9 - x^2 - y^2 = 9 - (x^2 + y^2)$$

$D_f :=$ Entire plane.

$$R_f : \quad z = 9 - \underbrace{(x^2 + y^2)}_{\geq 0} \leq 9$$

$$x^2 + y^2 \geq 0 \Rightarrow -x^2 - y^2 \leq 0$$

$$z = 9 - x^2 - y^2 \leq 9$$

$$\text{Range}(f) = (-\infty, 9]$$

$$(7) \quad f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

$$D_f = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq 0\} = \text{Entire space}$$

$$R_f = [0, \infty)$$

$$(8) \quad f(x, y, z) = \frac{1}{x^2 + y^2 + z^2}$$

$$D_f = \{(x, y, z) : (x, y, z) \neq 0\}$$

$$R_f = (0, \infty)$$

$$(9) \quad w = xy \ln z$$

$$D_w = \{(x, y, z) : z > 0\} = \text{Half space}$$

$$R_w = (-\infty, \infty).$$

$$14.2 \text{ ① } \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \left(\frac{0}{0} \right)$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{(\sqrt{x} - \sqrt{y})} \cdot \left(\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right) =$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{x(x-y)(\sqrt{x} + \sqrt{y})}{(x-y)} = \lim_{(x,y) \rightarrow (0,0)} x(\sqrt{x} + \sqrt{y}) = \boxed{0}$$

(Q16)

$$\text{Example ② } \lim_{(x,y) \rightarrow (2,-4)} \frac{y+4}{x^2y - xy + 4x^2 - 4x} = \left(\frac{0}{0} \right)$$

$$= \lim_{(x,y) \rightarrow (2,-4)} \frac{y+4}{xy(x-1) + 4x(x-1)}$$

$$= \lim_{(x,y) \rightarrow (2,-4)} \frac{y+4}{x(x-1)[y+4]} = \lim_{(x,y) \rightarrow (2,-4)} \frac{1}{x(x-1)} = \boxed{\frac{1}{2}}$$

$$\text{Example ③ } \lim_{(x,y) \rightarrow (0,0)} y^2 \sin\left(\frac{1}{x}\right)$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1, \quad x \neq 0$$

$$-y^2 \leq y^2 \sin\left(\frac{1}{x}\right) \leq y^2, \quad x \neq 0$$

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} -y^2 \leq \lim_{(x,y) \rightarrow (0,0)} y^2 \sin\left(\frac{1}{x}\right) \leq \lim_{(x,y) \rightarrow (0,0)} y^2$$

By Sandwich thm

$= 0$

$\left(\frac{2}{4}\right)(42)$

(Q21)
 Example: (4) $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \left(\frac{0}{0}\right)$

Let $u = x^2 + y^2$, $(x,y) \rightarrow (0,0)$ then $u \rightarrow 0$

$$\lim_{u \rightarrow 0} \frac{\sin(u)}{u} \stackrel{\text{L.H}}{=} \lim_{u \rightarrow 0} \frac{\cos u}{1} = 1$$

Example (5): $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - xy^2}{x^2 + y^2} = \left(\frac{0}{0}\right)$

polar coordinates:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$x^2 + y^2 = r^2 \quad \& \quad (x,y) \rightarrow (0,0) \Rightarrow r \rightarrow 0$$

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - xy^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta - r \cos \theta \cdot r^2 \sin^2 \theta}{r^2}$$

$$= \lim_{r \rightarrow 0} \frac{r^3 \cos \theta (\cos^2 \theta - \sin^2 \theta)}{r^2}$$

$$= \lim_{r \rightarrow 0} r \underbrace{\cos \theta (\cos 2\theta)}_{\text{bounded}} = 0$$

$$Q67) \lim_{(x,y) \rightarrow (0,0)} \ln \left(\frac{3x^2 + 3y^2 - x^2 y^2}{x^2 + y^2} \right) \quad \text{[polar coordinates]}$$

$$= \ln \left(\lim_{r \rightarrow 0} \frac{(3r^2 \cos^2 \theta + 3r^2 \sin^2 \theta - r^4 \sin^2 \theta \cos^2 \theta)}{r^2} \right)$$

$$= \ln \left(\lim_{r \rightarrow 0} (3 - r^2 \sin^2 \theta \cos^2 \theta) \right) = \boxed{\ln 3}.$$

$$Q64) \lim_{(x,y) \rightarrow (0,0)} \frac{2x}{x^2 + y^2 + x} = \lim_{r \rightarrow 0} \frac{2r \cos \theta}{r^2 + r \cos \theta}$$

$$= \lim_{r \rightarrow 0} \frac{2 \cos \theta}{r + \cos \theta} = 2, \quad \text{if } \cos \theta \neq 0$$

$$= \text{DNE} \quad \text{if } \cos \theta = 0$$

$$\text{Exempli: } \lim_{(x,y) \rightarrow (0,0)} \frac{4x^2 y^2}{x^4 + y^4} = \lim_{r \rightarrow 0} \frac{4r^4 \sin^2 \theta \cos^2 \theta}{r^4 \cos^4 \theta + r^4 \sin^4 \theta}$$

$$= \lim_{r \rightarrow 0} \frac{4 \sin^2 \theta \cos^2 \theta}{\cos^4 \theta + \sin^4 \theta} = \frac{4 \sin^2 \theta \cos^2 \theta}{\sin^4 \theta + \cos^4 \theta}$$

So, the limit DNE, since the limit varies for different θ .

Two - Path Test for Nonexistence of a limit.

If a function $f(x, y)$ has different limits along two different paths in the domain of f as $(x, y) \rightarrow (x_0, y_0)$, then.

$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$ does not exist.

$$\begin{cases} y = kx \\ y = kx^2 \end{cases}$$

Two paths.

Exempl(1): Show that $f(x, y) = \frac{2x^2y}{x^4 + y^2}$

has no limit as $(x, y) \rightarrow (0, 0)$.

sol: Along $y = x$:

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{2x^2y}{x^4 + y^2} = \lim_{x \rightarrow 0} \frac{2x^3}{x^4 + x^2} = \lim_{x \rightarrow 0} \frac{2x}{x^2 + 1} = 0$$

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Along $y = x^2$:

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{2x^2y}{x^4 + y^2} = \lim_{x \rightarrow 0} \frac{2x^4}{x^4 + x^4} = \boxed{1}$$

$$\Rightarrow \lim_{(x, y) \rightarrow (0, 0)} \frac{2x^2y}{x^4 + y^2} \text{ DNE.}$$

Example: $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^2}{x^3 + 3y^3}$

Along $y = kx$, $x \neq 0$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^2}{x^3 + 3y^3} = \lim_{x \rightarrow 0} \frac{2x(kx)^2}{x^3 + 3(kx)^3}$$

$$= \lim_{x \rightarrow 0} \frac{2k^2 x^3}{x^3(1+3k^3)} = \frac{2k^2}{1+3k^3}$$

If $k = 0$, $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0$

If $k = 1$, $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \frac{2}{4} = \frac{1}{2}$

$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} f(x,y)$ DNE by Two-Path Test.

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Example: $\lim_{(x,y) \rightarrow (1,-1)} \left(\frac{xy+1}{x^2-y^2} \right)$

Along $x = 1 \Rightarrow \lim_{y \rightarrow -1} \frac{y+1}{1-y^2} = \lim_{y \rightarrow -1} \frac{1+y}{(1+y)(1-y)} = \boxed{\frac{1}{2}}$

$\left(\frac{2}{3}\right) (43)$

$$\text{Along } y = -1 \Rightarrow \lim_{x \rightarrow 1} \frac{-x+1}{x^2-1} = \lim_{x \rightarrow 1} \frac{(1-x)}{(x-1)(x+1)}$$

$$= \frac{-1}{x+1} = -\frac{1}{2}$$

$\therefore \lim_{(x,y) \rightarrow (1,-1)} f(x,y)$ DNE by Two-path test.

Example: $\lim_{(x,y) \rightarrow (1,1)} \frac{xy^2-1}{y-1}$

Along $x=1$: $\lim_{(x,y) \rightarrow (1,1)} \frac{xy^2-1}{y-1} = \lim_{y \rightarrow 1} \frac{y^2-1}{y-1} = \boxed{2}$

Along $y=x$: $\lim_{(x,y) \rightarrow (1,1)} \frac{xy^2-1}{y-1} = \lim_{(x,y) \rightarrow (1,1)} \frac{x^3-1}{x-1} = \boxed{3}$

$\therefore \lim_{(x,y) \rightarrow (1,1)} f(x,y)$ DNE.

Continuity:

Def: A function $f(x, y)$ is Continuous at the point (x_0, y_0) if

- 1) f is defined at (x_0, y_0)
- 2) $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$ exists
- 3) $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$

A function is Continuous if it is Continuous at every point of its domain.

Example: Show that

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

is Continuous at every point except the origin.

① $f(0, 0) = 0$ defined.

② $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ Along $y=x$ $\lim_{x \rightarrow 0} \frac{2x^2}{2x^2} = \boxed{1}$

Along $y=x^2$: $\lim_{x \rightarrow 0} \frac{2x^3}{x^2+x^6} = \boxed{0}$

$\therefore$ limit DNE as $(x,y) \rightarrow (0,0)$.

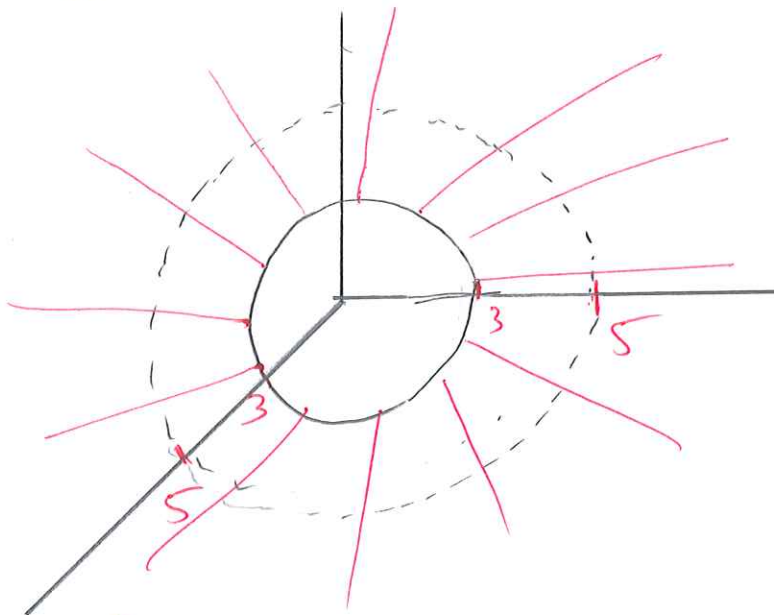
$\Rightarrow f(x,y)$ is cont. everywhere except $(0,0)$

Q40
Example: Where $f(x,y,z) = \frac{1}{4 - \sqrt{x^2+y^2+z^2-9}}$

is continuous.

$$D_f = \left\{ (x,y,z) : x^2+y^2+z^2-9 \geq 0 \text{ \& } 4 - \sqrt{x^2+y^2+z^2-9} \neq 0 \right\}$$

$$= \left\{ (x,y,z) : x^2+y^2+z^2 \geq 9 \text{ \& } x^2+y^2+z^2 \neq 25 \right\}$$



f is continuous on D_f .