7.3: Exponential Function

* Since $\ln x$ is an increasing function on $\mathbb{R}$

$\longrightarrow$ $\ln(x)$ 1-1 function

$\longrightarrow$ $(\ln(x))^{-1}$ exist

* $\ln(e) = 1$ $\longrightarrow$ $e = \ln^{-1}(1) \cong 2.7$

Def. For any real number x, we define the natural exponential function to be the inverse func. of $\ln x$.

$$\exp(x) = e^x = \ln^{-1}(x)$$

* Properties of exponential fun $y = e^x$

① $\text{Dom}(e^x) = \text{Range}(\ln x) = (-\infty, \infty)$

$\text{Range}(e^x) = \text{Dom}(\ln x) = (0, \infty)$ e^x is positive fun.

② $e^x > 0$ for all x

③ $e^{\ln x} = x$ for all $x \in (0, \infty)$

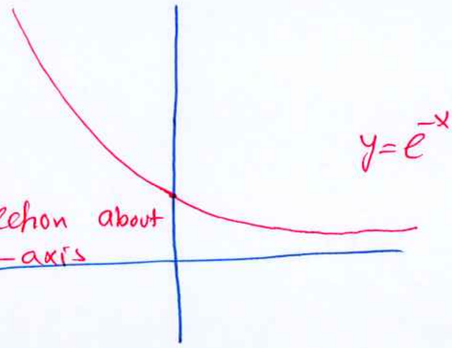
(4) $\ln e^x = x$. for all $x \in \mathbb{R}$

(5) $e^{x_1} e^{x_2} = e^{x_1 + x_2}$

$$\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$$

$$e^{-x} = \frac{1}{e^x}$$

Graph of $y = e^{-x}$



$$\left(e^{x_1}\right)^{x_2} = e^{x_1 \cdot x_2}$$

(6) $\lim\limits_{x \to \infty} e^x = \infty$ $\lim\limits_{x \to \infty} e^{-x} = 0$ $\lim\limits_{x \to 0} e^x = 1$

$\lim\limits_{x \to -\infty} e^x = 0$ $\lim\limits_{x \to -\infty} e^{-x} = \infty$

(7) $\frac{d}{dx}\left(e^x\right) = e^x$

In general :- If u is a diff. function then

$$\frac{d}{dx}\left(e^u\right) = e^u \frac{du}{dx}$$

(8) $\int e^x dx = e^x + C$ In general $\int e^{ax} dx = \frac{e^{ax}}{a} + C$

Theorem:- $e = \lim_{x\to 0}(1+x)^{1/x} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}$

The General exponential function a^x :- $\boxed{y = a^x}$

Def for any real number $a>0$

The exponential function with base a is defined as

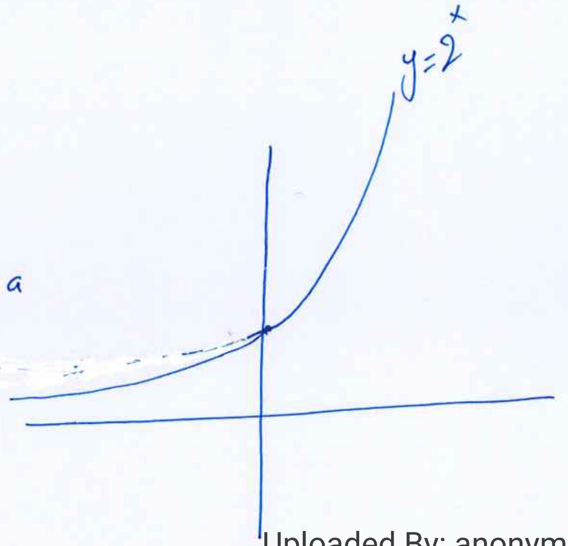
$$a^x = e^{x\ln a}$$

Some properties of $f(x) = a^x$

① $Dom_{a^x} = (-\infty, \infty)$, $Range_{a^x} = (0, \infty)$

② $a^x > 0 \quad \forall x \in \mathbb{R}$

③ If $y = a^x \longrightarrow \frac{dy}{dx} = a^x \ln a$



④ $\int a^x\,dx = \frac{a^x}{\ln a} + C$

In general $y = a^u$

$$\frac{dy}{dx} = a^u \cdot (\ln a)\left(\frac{du}{dx}\right)$$

⑤ $a^{x_1} a^{x_2} = a^{x_1+x_2}$

Exp:- $y = 5^{\sqrt{s}}$ $\quad \frac{dy}{ds} = 5^{\sqrt{s}}(\ln 5)\left(\frac{1}{2\sqrt{s}}\right)$

$\frac{a^{x_1}}{a^{x_2}} = a^{x_1-x_2}$

$(a^{x_1})^{x_2} = a^{x_1 x_2}$ $\quad a^{-x} = \frac{1}{a^x}$

6. $\lim_{x \to \infty} a^x = \infty$ $\qquad \lim_{x \to -\infty} a^x = 0$

7. for any positive real number $a \neq 1$

The function $\log_a x$ is the inverse function of a^x

$$f(x) = a^x \longrightarrow f^{-1}(x) = \log_a x$$

- $f \circ f^{-1}(x) = x$, if $x > 0$
- $f^{-1} \circ f(x) = x$ if $x \in \mathbb{R}$

8. $\log_a x = \dfrac{\ln x}{\ln a}$

$$\frac{d}{dx}\left(\log_a x\right) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a} \cdot \frac{1}{x}$$

$$\frac{d}{dx}\left(\log_a u\right) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{(\ln a) u} \cdot \frac{du}{dx}$$

Q1 Simplify the following expression

(a) $e^{2\ln 4} + \ln(\sqrt{e^6})$

$= 16 + 3 = 19$

(b) $\log_9 27 + \log_2 \sqrt{8} - \ln e^3$

$\log_{3^2} 3^3 + \log_2 2^{\frac{3}{2}} - 3$

$\log_{3^2} (3^2)^{\frac{3}{2}} + \frac{3}{2} - 3 = 0$

Q2 Find the domain of $f(x) = \ln\left(\log_3 (e^{2x} - 3)\right)$

$\text{Dom}(f) = \{x : \log_3 (e^{2x} - 3) > 0$

$\{x : \frac{\ln(e^{2x} - 3)}{\ln 3} > 0\}$

$\{x : e^{2x} - 3 > 1\}$

$\{x : e^{2x} > 4\}$

Dom.

$\{x : 2x > 2\ln 2\} = \{x : x > \ln 2\}$

Find the derivative of the following

① $y = e^{1+x^2}$

$$\frac{dy}{dx} = (2x)\, e^{1+x^2}$$

② $y = 3 \log_8 \left(\log_2 x \right)$

$$= 3 \frac{\ln \left(\log_2 x \right)}{\ln 8}$$

$$= \frac{3}{\ln 8} \left[\ln (\ln x) - \underbrace{\ln (\ln 2)}_{\text{constant}} \right]$$

$$= \frac{3}{\ln 8} \left[\frac{1}{\ln x} \cdot \frac{1}{x} \right] = \frac{1}{\ln 2} \left(\frac{1}{x \ln x} \right)$$

③ $y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \; dt$

$$\frac{dy}{dx} = \ln \left(e^{2x} \right) \left[2e^{2x} \right] - \ln \left(e^{4\sqrt{x}} \right) \left[\frac{4}{2\sqrt{x}} \; e^{4\sqrt{x}} \right]$$

$$= 4x\, e^{2x} \; - \; 8\, e^{4\sqrt{x}}$$

4) $y = 3^{\tan x} + \log_3 \sqrt[3]{x^3+1}$

$$\frac{dy}{dx} = 3^{\tan x}(\ln 3)(\sec^2(x)) + \frac{1}{3(x^3+1)\ln(3)}(3x^2)$$

$$= \ln 3 \left(3^{\tan x}\right)\sec^2 x + \frac{x^2}{(\ln 3)(x^3+1)}$$

5) $y = \left[\log_2 \sin\left(\frac{x}{3}\right)\right]^3$ $\qquad y = \left(\frac{\ln \sin\left(\frac{x}{3}\right)}{\ln 2}\right)^3$

$$\frac{dy}{dx} = 3\left[\log_2 \sin\left(\frac{x}{3}\right)\right]^2 \cdot \left[\frac{\frac{1}{3}\cos\left(\frac{x}{3}\right)}{\ln(2)\sin\left(\frac{x}{3}\right)}\right]$$

$$= \frac{1}{\ln(2)} \cot\left(\frac{x}{3}\right)\left[\log_2 \sin\left(\frac{x}{3}\right)\right]^2$$

Evaluate the following integral

$\boxed{1}$ $\displaystyle\int_0^{\sqrt{\ln \pi}} 2x\, e^{x^2} \cos\left(e^{x^2}\right) dx$

$u = e^{x^2}$

$du = e^{x^2}(2x)\,dx$

$x = 0 \longrightarrow u = 1$

$x = \sqrt{\ln \pi} \longrightarrow u = \pi$

$$\int_1^{\pi} \cos(u)\,du = \sin(u)\Big]_1^{\pi}$$

$$= \sin(\pi) - \sin(1)$$

$$= -\sin(1)$$

$\boxed{2}$ $\int \frac{e^{-1/x^2}}{x^3}\,dx$

Q42

$u = \frac{-1}{x^2}$

$du = \frac{2}{x^3}\,dx$

$\int \frac{e^u\,du}{2} = \frac{1}{2}\int e^u\,du$

$= \frac{1}{2}e^u + C$

$= \frac{1}{2}e^{\frac{-1}{x^2}} + C$

$\boxed{3}$ $\int \frac{e^{2x}}{e^x - 1}\,dx$

$u = e^x - 1$

$du = e^x\,dx$

$\int \frac{e^{2x}}{u}\,\frac{du}{e^x}$

$\int e^x\,\frac{1}{u}\,du = \int \left(\frac{u+1}{u}\right)du = \int 1 + \frac{1}{u}\,du$

$= u + \ln|u| + C$

$= e^x - 1 + \ln|e^x - 1| + C$

$= e^x + \ln|e^x - 1| + C_1$

$C_1 = C - 1$

$\boxed{4}$ $\int \frac{1}{1+e^x}\,dx$

multiply by $\frac{e^{-x}}{e^{-x}}$

$u = e^{-x} + 1$

$du = -e^{-x}\,dx$

$-\int \frac{-e^{-x}\,dx}{e^{-x}+1} = -\ln|e^{-x}+1| + C$

Q92

$\boxed{5}$ $$\int \frac{x\,2^{x^2}}{1+2^{x^2}}\,dx$$

$u = 1 + 2^{x^2}$

$du = [2x][\ln 2]\, 2^{x^2}\, dx$

$\frac{du}{2\ln 2} = x\, 2^{x^2}\, dx$

$$\frac{1}{2\ln 2}\int \frac{du}{u}$$

$$= \frac{1}{2\ln 2} \ln|u| + C$$

$$= \frac{1}{2\ln 2} \ln\left|1 + 2^{x^2}\right| + C$$

$\boxed{6}$ $$\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan x} \sec^2 x\, dx$$

$u = \tan x$

$du = \sec^2 x\, dx$

$$\int_0^1 \left(\frac{1}{3}\right)^u du = \left.\frac{\left(\frac{1}{3}\right)^u}{\ln 3^{-1}}\right]_0^1 = \frac{-1}{\ln 3}\left[\frac{1}{3} - 1\right]$$

$$= \frac{-1}{\ln 3}\left[-\frac{2}{3}\right]$$

$$= \frac{2}{3\ln 3} = \frac{2}{\ln 27}.$$

Outline Exercises :-

26 Find the derivative y with respect x

$$\ln xy = e^{x+y}$$

$$\ln x + \ln y = e^{x+y}$$

$$\frac{1}{x} + \frac{y'}{y} = e^{x+y}(1+y')$$

$$\frac{1}{x} + \frac{y'}{y} = e^{x+y} + y' e^{x+y}$$

$$y'\left[\frac{1}{y} - e^{x+y}\right] = e^{x+y} - \frac{1}{x}$$

$$y'\left[\frac{1 - ye^{x+y}}{y}\right] = \frac{xe^{x+y} - 1}{x}$$

$$y' = \frac{y\left[xe^{x+y} - 1\right]}{x\left[1 - ye^{x+y}\right]}$$

91 $\int_2^4 x^{2x}(1+\ln x)\,dx$

$$U = x^{2x}$$

$$\ln U = 2x \ln x$$

$$\frac{U'}{U} = 2x\left(\frac{1}{x}\right) + 2\ln x$$

$$\frac{dU}{dx} = (2 + 2\ln x)U$$

$$dU = 2(1+\ln x)x^{2x}\,dx$$

$$\int_{16}^{4^8} \frac{1}{2}\,dU = \frac{1}{2}U\Big]_{16}^{4^8}$$

$$= \frac{1}{2}\left[65536 - 16\right] = 32760$$