# Fourier Series

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# Fourier Series:

- Fourier Series and Fourier Transform aim at determining spectral representation of signal and system models.
- Fourier Series is used to determine the spectral representation of periodic signals. It can be also used to determine the spectral representation of a time limited signal that is extended to periodic signal, the spectral representation is limited to the time interval of the signal.

Forms of Fourier series:

Sinusoidal form: A periodic signal or system model $x(t)$with period $T_0$ has a sinusoidal Fourier series (FS) spectral representation ↔ ∃ a set of real parameters $a_{0,}\, a_n, and\ b_n$ so that the following series form converges to $x(t)$ :

$$\mathrm{x(t)} = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

Complex Exponential Form: A periodic signal or system model $x(t)$with period $T_0$ has a Complex Fourier series (FS) spectral representation ↔ ∃ a set of complex parameters $X_n$, so that the following series form converges to $x(t)$ :

$$\mathrm{x(t)} = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

Important Observations:

- The sinusoidal form represents a single sided spectral representation, whereas, the complex form represents a double sided spectral representation.
- The series, if converges, converges to a periodic signal since it is the sum of sinusoidal signals and phasors with **a rational number** frequency ratio between any two signals.
- The real and complex series forms are mathematically related through the Euler Formula and the single sided and double sided spectral representation relations with $Re(X_n) = \frac{a_n}{2}$ and $Im(X_n) = -\frac{b_n}{2}$

Proof:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\, cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n\, sin(n\omega_0 t) = a_0 + \sum_{n=1}^{\infty} a_n \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} + \sum_{n=1}^{\infty} b_n \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$$

*Collecting the positive exponents in one sum and the negative exponent terms in the second we obtain:*

$$x(t) = a_0 + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-jn\omega_0 t}$$

*Making the substitution m=-n in the second summation* $x(t) = a_0 + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{jn\omega_0 t} + \sum_{m=-1}^{-\infty} \frac{a_{-m} + jb_{-m}}{2} e^{jm\omega_0 t}$

*Reordering the second summation and changing symbol m to n (dummy index) we obtain*

$$x(t) = a_0 + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} \frac{a_{-n} + jb_{-n}}{2} e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \; with \begin{cases} X_0 = a_0 \\ X_n = \frac{a_n - jb_n}{2} & for\; n \geq 1 \\ X_{-n} = \frac{a_{-n} + jb_{-n}}{2} & for\; n \leq -1 \end{cases}$$

Determination of the Sinusoidal Fourier Series Coefficients:

The coefficients are determined using the alternation and the signal orthogonality conditions.

Determination of $a_0$:

Integrating both sides of the Series on one signal period we get

$$\int_{T_0} x(t)dt = \int_{T_0} a_0 dt + \sum_{n=1}^{\infty} a_n \int_{T_0} \cos(n\omega_0 t)dt + \sum_{n=1}^{\infty} b_n \int_{T_0} \sin(n\omega_0 t)dt$$

Since the integrals in the summation are integrals of alternating signals over an integer number of periods their result is zero, $a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$

Determination of $a_n$:

Multiplying both sides of the series with $\cos(m\omega_0 t)$ which is orthogonal to $\cos(n\omega_0 t)$ $n \neq m$ and orthogonal to $\sin(n\omega_0 t)$ for any $n$ and integrating over $T_0$ all the integrations of the orthogonal signals give zero

Moreover, $a_0 \int_{T_0} \cos(m\omega_0 t)\, dt = 0$ because it is alternating and integrated over $T_0$.

The term of the first summation for $n = m$ gives, $\int_{T_0} x(t) \cos(m\omega_0 t)\, dt = a_m \cdot \frac{T_0}{2} \rightarrow a_m = \frac{2}{T_0} \int_{T_0} x(t) \cos(m\omega_0 t)\, dt$

Determination of $b_n$:

Similarly multiplying both parts with $\sin(m\omega_0 t)$ and taking into account the orthogonality conditions we obtain:

$$b_m = \frac{2}{T_0} \int_{T_0} x(t) \sin(m\omega_0 t)\, dt$$

$$\text{Summarizing} \begin{cases} a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt \quad \textit{called average value or DC component} \\ a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos(n\omega_0 t)\, dt \\ b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin(n\omega_0 t)\, dt \end{cases}$$

Determination of the Complex Fourier Series Coefficients $X_n$:

Following a similar procedure based on orthogonality by multiplying both sides with $e^{-jm\omega_0 t}$, applying the Euler formula, and integrating all the terms with $n \neq m$ becomes zero. For $n = m$ we obtain:

$$\int_{T_0} x(t) e^{-jn\omega_0 t} dt = X_n \cdot T_0 \rightarrow X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

Spectral Components Terminology:

We will adopt the following terminology:

The component with *n=0* is called the DC component, the component with *n=1* is called the fundamental component, and the components with *n=2,3,…* are called first, second, third …. Harmonics, respectively.

Observation:

Note the Fourier Series coefficients are defined by an integration operator, therefore if the series converges, it converges to an infinite number of generally continuous periodic signals, that is signals that differs with a countable set of discontinuity points.

**Theorem:**

It is sufficient for the periodic signal $x(t)$ to have a Fourier Series representation that it satisfies the following Dirichlet conditions:

- $x(\mathrm{t})$: generally continuous in the period and,
- $x(\mathrm{t})$: absolutely integrable in the period $\int_{T_0} |x(t)|dt < \infty$ (since $\int_{T_0} x(t)\cos(n\omega_0 t)\,dt \le \int_{T_0} |x(t)|dt$)

Properties of Fourier Series Coefficients:

- If $x(t)$ is real then $X_n = X_{-n}^*$ that is $\begin{cases} |X_n| = |X_{-n}| & \textit{Amplitude has even symmetry} \\ < X_n = -< X_{-n} & \textit{Phase has odd symmetry} \end{cases}$
- If $x(t)$ is real and even (time domain), then $X_n$ is real and even (frequency domain)
- If $x(t)$ is real and odd (time domain), then $X_n$ is imaginary and odd (frequency domain)

Proof: $X_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt = \frac{1}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt - \frac{j}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$

$X_{-n} = \frac{1}{T_0}\int_{T_0} x(t)e^{-j(-n)\omega_0 t}dt = \frac{1}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt + \frac{j}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$

- $x(t)$ is real then the term with the cosine is real and that of the sine is imaginary, thus the two terms have a conjugation relation.
- if $x(t)$ is real and even then the integral term with the sine=0, because the product of an odd sine and even $x(t)$ is odd $X_n = \frac{a_n}{2}, b_n = 0.$
- if $x(t)$ is real and odd then the integral term with the cosine=0, because the product of an odd $x(t)$ and even cosine is odd $X_n = -j\frac{b_n}{2}, a_n = 0$
- If $x(t)$ is alternating then $a_0 = 0.$
- If $x(t)$ has a half-wave odd symmetry then $X_n = 0$ for *n* even

Example1: Compute the Fourier Series coefficients of the periodic signal defined in its basic period by:

$$\begin{cases} A & for\ 0 < t < \frac{T_0}{2} \\ -A & for\ -\frac{T_0}{2} < t < 0 \end{cases}$$

Solution:

The selection between using the sinusoidal or the complex format depends of the number of integrations to do (generally higher in the sinusoidal form) and the complexity of the interpretation of the results (generally higher in the complex exponential form).

In our case the signal is alternating, has odd symmetry, and has half-wave odd symmetry, thus $a_0 = 0, a_n = 0,$ and $b_n = 0$ for *n* even.

We have to compute only $b_n$ for odd values of *n*.

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt = \frac{-2A}{T_0}\int_{-\frac{T_0}{2}}^{0}\sin(n\omega_0 t)\,dt + \frac{2A}{T_0}\int_{0}^{\frac{T_0}{2}}\sin(n\omega_0 t)\,dt$$

$$\frac{-2A}{T_0}[\frac{-\cos(n\omega_0 t)}{n\omega_0}]_{-\frac{T_0}{2}}^{0} + \frac{2A}{T_0}[\frac{-\cos(n\omega_0 t)}{n\omega_0}]_{0}^{\frac{T_0}{2}} = \frac{2A}{T_0}\left[\frac{1-\cos\left(-n\omega_0\frac{T_0}{2}t\right)}{n\omega_0}\right] - \frac{2A}{T_0}[\frac{\cos\left(n\omega_0\frac{T_0}{2}t\right)-1}{n\omega_0} = \frac{2A}{2n\pi}[2-2\cos(n\pi)]$$

$$= \frac{2A}{n\pi}\cdot 2sin^2(\frac{n\pi}{2}) \rightarrow b_n = \begin{cases} 0 & for\ n\ even\ (as\ expected) \\ \frac{4A}{n\pi} & nfor\ n\ odd \end{cases}$$

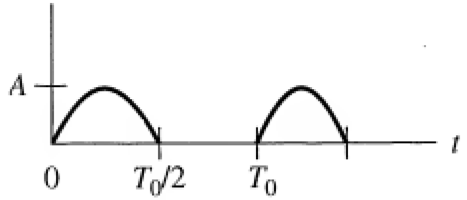
The complex Fourier coefficients $X_n = -j\frac{b_n}{2} = -j\frac{2A}{n\pi}$ for $n$ odd

Exercise1: Compute the integrals to show that$a_0 = 0, a_n = 0$

Exercise2: Compute $X_n$ using the complex Fourier directly.

Example2: Determine the Fourier series coefficients of the signal defined in its basic period as:

$$x(t) = \begin{cases} A \sin \omega_0 t, & 0 \le t \le T_0/2 \\ 0, & T_0/2 \le t \le T_0 \end{cases}$$




Solution:

The signal has no symmetry properties, it is preferable to compute $X_n$ in order to reduce the number of integrations

$$X_n = \frac{1}{T_0}\int_0^{T_0/2} A \sin \omega_0 t \, e^{-jn\omega_0 t}\, dt$$

$$= \frac{A}{2jT_0}\left[\int_0^{T_0/2} (e^{j\omega_0 t} - e^{-j\omega_0 t})e^{-jn\omega_0 t}\, dt\right]$$

$$= \frac{A}{2jT_0}\left[\int_0^{T_0/2} e^{j\omega_0(1-n)t}\, dt - \int_0^{T_0/2} e^{-j\omega_0(1+n)t}\, dt\right]$$

$$= -\frac{A}{4\pi}\left[\frac{e^{j(1-n)\pi} - 1}{1-n} + \frac{e^{-j(1+n)\pi} - 1}{1+n}\right], \qquad n \ne 1 \quad \text{or} \quad -1$$

To simplify the results we should compute the value of the phasor with $e^{j(1\pm n)\pi} = \cos(1 \pm n)\pi + j \sin(1 \pm n)\pi = -(-1)^n,$

Thus: $X_n = 0, \qquad n \text{ odd}, \quad n \ne \pm 1$ $\qquad X_n = \frac{A}{\pi}\frac{1}{1-n^2}, \quad n \text{ even}$

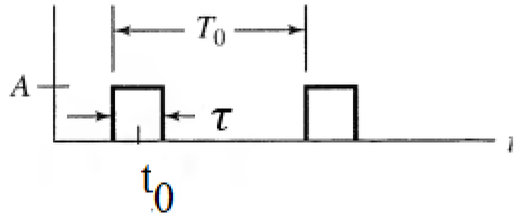
To complete the solution, it is sufficient to compute one of the cases $n = 1 \; and \; n = -1$ because of the conjugation relation

$$X_1 = \frac{A}{2jT_0}\int_0^{T_0/2} (e^{j\omega_0 t} - e^{-j\omega_0 t})e^{-j\omega_0 t}\, dt \;= \frac{A}{2jT_0}\int_0^{T_0/2} (1 - e^{-j2\omega_0 t})\, dt = \frac{A}{4j}$$

Example3: Determine the Fourier series coefficients of the signal defined in its basic period as:

$$x(t) = \sum_{m=-\infty}^{\infty} A\Pi\left(\frac{t - t_0 - mT_0}{\tau}\right), \qquad \tau < T_0$$




Solution: $$X_n = \frac{1}{T_0}\int_{t_0-\tau/2}^{t_0+\tau/2} Ae^{-jn\omega_0 t}\,dt = \frac{-A}{jn\omega_0 T_0} e^{-jn\omega_0 t}\Bigg|_{t_0-\tau/2}^{t_0+\tau/2}$$

$$= \frac{2A}{n\omega_0 T_0} e^{-jn\omega_0 t_0}\left(\frac{e^{jn\omega_0\tau/2} - e^{-jn\omega_0\tau/2}}{2j}\right), \qquad n \neq 0$$

$$= \frac{2A}{n\omega_0 T_0} e^{-jn\omega_0 t_0} \sin\frac{n\omega_0\tau}{2}, \qquad n \neq 0$$

Dividing and multiplying by $\tau$ $and$ rearranging the equation we obtain: $X_n = \frac{A\tau}{T_0} e^{-j2\pi n f_0 t_0} \frac{\sin \pi n f_0 \tau}{\pi n f_0 \tau} = \frac{A\tau}{T_0} e^{-j2\pi n f_0 t_0}\ sinc(nf_0\tau)$

For $n = 0$ $X_0 = \frac{A\tau}{T_0}$

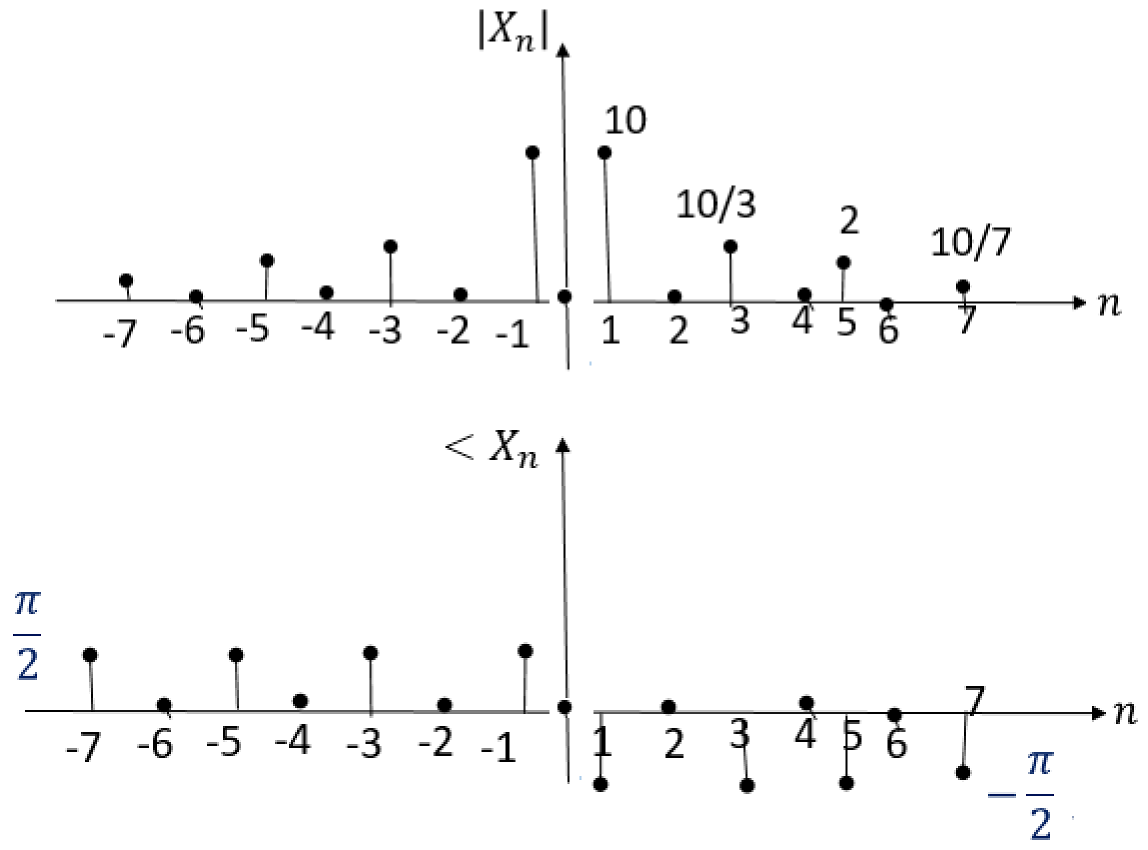
## Discrete Line Spectral Representation:

The amplitude and phase-shift spectral functions and plots of the Fourier Series phasors that is $|X_n| and < X_n$ for the double sided spectral representation or its corresponding single sided spectral representation with amplitude $|A_n| = 2|X_n| and$ $< X_n$ for $n \geq 0$ . The name line-spectra is due to the fact that the frequency is defined in a countable set of points(discrete) and the values are extended by lines to show their position. In general the frequency variable is normalized to the signal fundamental frequency, that is $\frac{n\omega_0}{\omega_0} = \mathrm{n}$.

Example1: Determine and plot the double sided spectral representation of the coefficients of Fourier Series defined by:

$$X_n = \begin{cases} 0 & for\ n\ even \\ -j\frac{2A}{n\pi} & for\ n\ odd \end{cases} \quad \text{with } A = 5\pi$$

$$|X_n| = \frac{10}{n}, < X_n = \begin{cases} 0 & for\ n\ even \\ -\frac{\pi}{2} & for\ n > 0\ and\ odd \\ \frac{\pi}{2} & for\ n < 0\ and\ odd \end{cases}$$

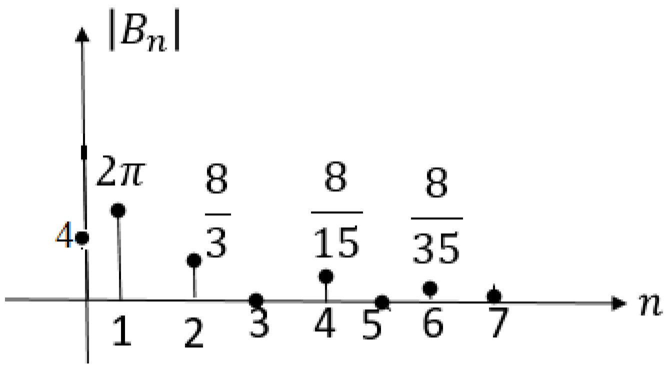


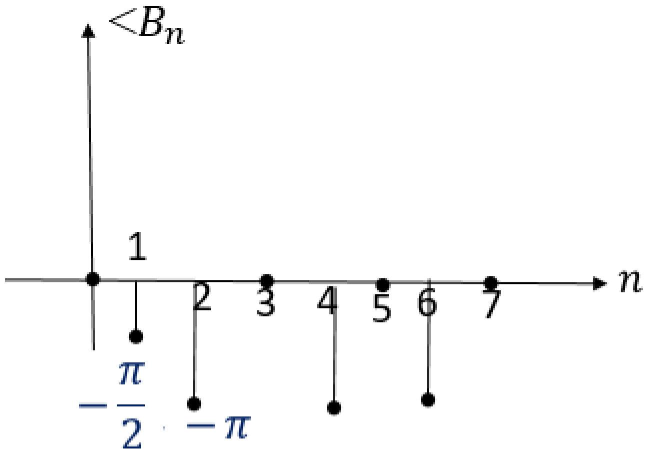

Exercise: Plot the single sided spectral representation

Example2: Determine and plot the single sided spectral representation of the coefficients of Fourier Series defined by the following and using $A = 4\pi$:

$$X_n = \begin{cases} \dfrac{A}{\pi(1-n^2)}, & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n \text{ odd and } \neq \pm 1 \\ -\frac{1}{4} jnA, & n = \pm 1 \end{cases}$$

For single sided spectral representation $|B_n| = 2|X_n| and \; < B_n = < X_n$ for $n \geq 0$




Exercise: Plot the double sided spectral representation

Signal Power Using Spectral Representation:

Aims to compute the signal average power or the average power contained in a defined frequency band using the spectral the Fourier Series Spectral representation.

Observation: remember that the signal power or energy are defined by the magnitude and they are independent of the phase-shift (signal delay).

Parseval's Theorem

Let $x(t)$ be a periodic signal with period $T_0$ , average power $P_{av}$ and Fourier Series $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$, the total average power of the signal can be computed by $P_{av} = \sum_{n=-\infty}^{\infty} |X_n|^2$.

Proof:

$$P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt = \frac{1}{T_0}\int x(t)x*(t)\,dt = \frac{1}{T_0}\int_{T_0} x(t)\left(\sum_{n=-\infty}^{\infty} X_n^* e^{-jn\omega_0 t}\right)dt = \sum_{n=-\infty}^{\infty} X_n^*\left[\frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\,dt\right] = \sum_{n=-\infty}^{\infty} |X_n|^2$$

Lemma1: If $x(t)$ is real then the conjugation relation between the even and odd indexed terms holds and we can write:

$$\sum_{n=-\infty}^{\infty} |X_n|^2 = X_0^2 + 2\sum_{n=1}^{\infty} |X_n|^2$$

Lemma2: Let $x(t)$ be a periodic signal with period $T_0$, the average power contained in the frequency band $\omega \in [\omega_1, \omega_2]$ (or similarly $f \in [f_1, f_2]$) is given by: $\sum_{n_1}^{n_2} |X_n|^2$ with $n_1 = \min(n)$ and $n_2 = \max(n)$ $so\ that$ $[n_1\omega_0, n_2\omega_0] \subseteq [\omega_1, \omega_2]$ (or similarly $[n_1 f_0, n_2 f_0] \subseteq [f_1, f_2]$)

Example1: compute the average of the real power signal with the following Fourier Series coefficients:

$$a_0 = a_n = 0, b_n = 0\ for\ n\ even, and\ b_n = \frac{4A}{\pi n}\ for\ \ n\ odd$$

Solution:

$P_{av} = \sum_{n=-\infty}^{\infty} |X_n|^2 = |X_0|^2 + 2\sum_{n=1}^{\infty} |X_n|^2$, since $X_n = -j\frac{b_n}{2} = -j\frac{2A}{\pi n}$ we have $P_{av} = 2\frac{(2A)^2}{\pi^2}\sum_{k=0}^{\infty} \left|\frac{1}{2k+1}\right|^2 = \frac{8A^2}{\pi^2}\left(\frac{\pi^2}{8}\right) = A^2$

Example2: compute the average power contained in **the frequency band** $\omega \in [0,80]$ **rad/sec. The real** power signal has frequency $\omega_0 = 15 \text{ rad/sec}$ and the following Fourier Series coefficients:

$a_0 = a_n = 0, b_n = 0 \; for \; n \; even, and \; b_n = \frac{4A}{\pi n} \; for \;\; n \; odd \; with \; A = \frac{\pi}{2}$

Solution:

$$X_n = -j\frac{b_n}{2} = -j\frac{2A}{\pi n} = -j\frac{1}{n}$$

The minimum and maximum indices defined by $\omega_0$ and the assigned frequency range are $n_{min} = 0 \; and \; n_{max} = floor\left(\frac{80}{15}\right) = 5$

Therefore the average power contained in this range is $P_{av} = \sum_{n=-5}^{5} |X_n|^2 = |X_0|^2 + 2\sum_{n=1}^{5} |X_n|^2 = 2(|X_1|^2 + |X_3|^2 + |X_5|^2)$

$$P_{av} = 2(|\frac{1}{1}|^2 + |\frac{1}{3}|^2 + |\frac{1}{5}|^2 = 2.3 \; watt$$

Example3: compute the average power contained in **the frequency band** $f \in [0,100]$ **Hz. The real** power signal has frequency $f_0 = 16 \text{ Hz}, A = \pi$, and the following Fourier Series coefficients:

and

$$X_n = \begin{cases} \dfrac{A}{\pi(1-n^2)}, & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n \text{ odd and } \neq \pm 1 \\ -\frac{1}{4} jnA, & n = \pm 1 \end{cases}$$

Solution:

The minimum and maximum indices defined by $\omega_0$ and the assigned frequency range are $n_{min} = 0 \; and \; n_{max} = floor\left(\frac{100}{16}\right) = 6$

Therefore the average power contained in this range is $P_{av} = \sum_{n=-6}^{6} |X_n|^2 = |X_0|^2 + 2\sum_{n=1}^{6} |X_n|^2 = |X_0|^2 + 2(|X_1|^2 + |X_2|^2 + |X_4|^2 + |X_6|^2)$

$$P_{av} = |1|^2 + 2(|\frac{\pi}{4}|^2 + |\frac{1}{-3}|^2 + |\frac{1}{-15}|^2 + |\frac{1}{-35}|^2 = 2.466 \; watt$$

## Steady-State Response of an LTI System to a Periodic Signal with Fourier Series Representation:

Theorem:

Given an LTI system with frequency response $H(\omega)$, the steady-state response to the periodic signal $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$ is given by $y(t) = \sum_{n=-\infty}^{\infty} |X_n||H(n\omega_0)|e^{j(n\omega_0 t + <X_n + <H(n\omega_0))}$

Proof:

The system is Linear time invariant, thus the superposition can be applied and we can write $y(t) = \sum_{n=-\infty}^{\infty} y_n(t)$, where $y_n(t)$ is the response to the generic phasor component $x_n(t) = X_n e^{jn\omega_0 t}$.

Applying the theorem of the steady-state response of the LTI system to the generic sinusoidal phasor with frequency $n\omega_0$, Amplitude $|X_n|$, and phase shift $< X_n$ we obtain:

$y_n(t) = |X_n||H(n\omega_0)|e^{j(n\omega_0 t + <X_n + <H(n\omega_0))}$

Following the superposition theorem we obtain $y(t) = \sum_{n=-\infty}^{\infty} y_n(t) = \sum_{n=-\infty}^{\infty} |X_n||H(n\omega_0)|e^{j(n\omega_0 t + <X_n + <H(n\omega_0))}$

Example

Let $H(\omega) = \frac{10}{10+j\omega}$ , compute the steady-state response of the system to the periodic signal defined in its basic period by $\begin{cases} A & for\ 0 < t < \frac{T_0}{2} \\ -A & for\ -\frac{T_0}{2} < t < 0 \end{cases}$

Solution:

As computed in previous examples the only non-zero spectral components of the signal are given by $b_n = \frac{4A}{\pi n}\ for\ \ n\ odd$. Thus, the Fourier series expressed in trigonometric form is given by $\mathrm{x(t)} = \sum_{k=0}^{\infty} \frac{4A}{\pi(2k+1)} \sin((2k+1)\omega_0 t)$.

The response of the system is given by the sin component of the complex phasor input response:

$y(t) = \sum_{k=0}^{\infty} \frac{10}{\sqrt{100+[(2k+1)\omega_0]^2}} \cdot \frac{4A}{\pi(2k+1)} \sin((2k+1)\omega_0 t - tan^{-1}(\frac{(2k+1)\omega_0}{10}))$ (note that $< b_n = 0$)>

Exercise1: Write the response for $\omega_0 = 10$ rad/sec.

Exercise2: Determine the response of the same system to the periodic input:

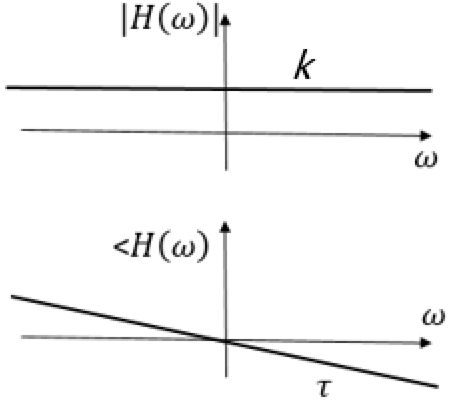
$$x(t) = 10\cos\left(10t + \frac{\pi}{6}\right) + 20\sin\left(30t - \frac{\pi}{3}\right) + 15\cos\left(50t + \frac{\pi}{4}\right)$$

## System and Signal Distortion :

Definition:

An LTI system is said to be a distortion-less system ↔ its input/output relation is defined by $y(t) = kx(t-\tau)$ with constant $k\ and\ \tau$ values.

Lemma: the frequency response of a distortionless LTI system is given by $H(\omega) = ke^{-j\omega\tau}$ with constant $k\ and\ \tau$ values. Such a system with constant amplitude and linear phase spectral representation (for any $\omega\ or\ f$) is called an *Ideal Channel*



*Ideal Channel*

Signal Distortion:

Definition: A signal that is scaled and delayed with constant fixed values values for all the time-values is said to be undistorted.

Lemma: the output signal of any distortionless system (ideal channel) is not distorted.

Types of distortion:

- Amplitude distortion: the signal amplitude is scaled with non constant *k-values.*
- Phase distortion: the signal is delayed wit different values at different time periods, or its ph frequency behavior that does not follow a constant slope linear characteristic.
- Frequency distortion: can be generated only by nonlinear systems which can generate new frequency components or cause a frequency shift.

Example1: Determine if the system with frequency response is with distortion and the type of distortion introduced.

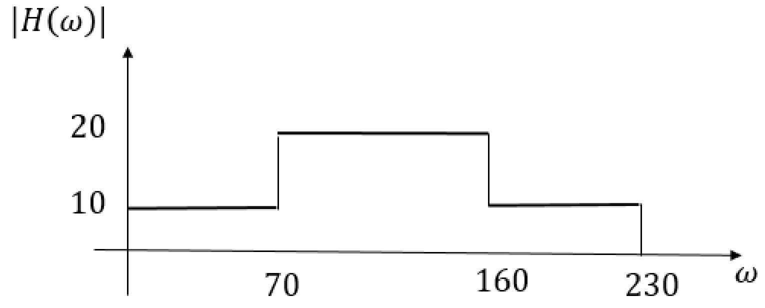
- $H(\omega) = \frac{10}{10+j\omega} \rightarrow |H(\omega)| = \frac{10}{\sqrt{100+\omega^2}}$ and phase characteristic $< H(\omega) = -tan^{-1}\left(\frac{\omega}{10}\right)$.

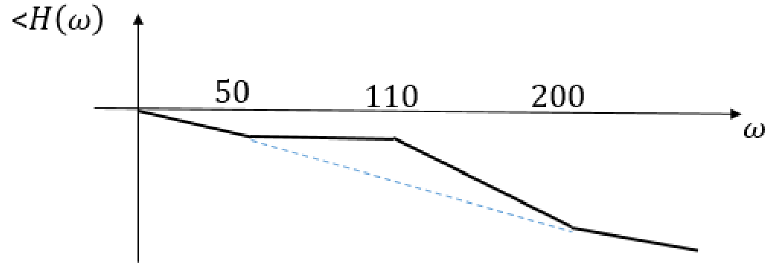
*The system is with distortion and has amplitude and phase distortions because its amplitude characteristic is not a constant function of* $\omega$ *and its phase is not a linear phase of* $\omega$

Example2:

For the system with the following single sided amplitude and phase spectral representation,

1. Determine if the system is distortionless.
2. Determine if its responses to the following signals are distorted and the type of distortion.
    - $x_1(t) = 10\cos\left(10t + \frac{\pi}{6}\right) + 20\sin\left(30t - \frac{\pi}{3}\right)$
    - $x_2(t) = 10\cos\left(10t + \frac{\pi}{6}\right) + 20\sin\left(60t - \frac{\pi}{3}\right)$
    - $x_3(t) = 10\cos\left(60t + \frac{\pi}{6}\right) + 20\sin\left(90t - \frac{\pi}{3}\right)$
    - $x_4(t) = 10\cos\left(120t + \frac{\pi}{6}\right) + 20\sin\left(140t - \frac{\pi}{3}\right)$
    - $x_5(t) = 10\cos\left(120t + \frac{\pi}{6}\right) + 20\sin\left(170t - \frac{\pi}{3}\right)$
    - $x_6(t) = 10\cos\left(40t + \frac{\pi}{6}\right) + 20\sin\left(215t - \frac{\pi}{3}\right)$




Solution

1. The system is with phase and amplitude distortion because it has non-constant Amplitude and phase characteristics
1. The responses are as follow:
    - $x_1(t)$ is not distorted because the amplitudes are scaled with the same value 10 and the signal components have the same delay since their frequencies are on the linear part of the phase response (line that goes through the origin)
    - $x_2(t)$ has a phase distortion.
    - $x_3(t)$ has both amplitude and phase distortions.
    - $x_4(t)$ has a phase distortion since the phase-line does not go through the origin (nonlinear characteristic)
    - $x_5(t)$ has both phase and amplitude distortions.
    - $x_6(t)$ is not distorted because its components are scaled with the same value and delayed by the same linear line

# Fourier Transform

Dr. Jamal Siam

**Fourier Transform:**

- Aims at determining the spectral representation of aperiodic energy signal using Fourier Integral form.
- Spectral representation for other types of signals can be determined using theorems and generalized calculus positive order singularities (delta, Kronecker, and higher order delta derivatives,…)
- As all transforms the Fourier transform has an inverse transform.

Definition:

A signal $x(t)$ is said to have a Fourier Transform spectral representation $X(f)$ in the Fourier Integral sense if and only if the following integrals converges.

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt < \infty$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df < \infty$$

Observations:

- $X(f)$ is a complex function of real variable, the frequency $f$
- The defined double sided form of the Fourier Transform represents a double sided spectral representation.
- $X(f)$ is a spectral density since its dimension is ${x(t)_{units}}/_{Hz}$
- $x(t) \leftrightarrow X(f)$ are called a pair of Fourier transform.
- The Fourier transform can be expressed as a spectral function of the rotational frequency $\omega$ using the substitution $f = \frac{\omega}{2\pi}$.
- The Fourier Transform operator has the same form the Fourier Series operator, but with continuous frequency variable instead of discrete harmonics frequency type.

Theorem:

It is sufficient for the signal $x(t)$ to have Fourier Transform in the Fourier Integral sense that it satisfies the Dirichlet conditions. The is:

- $x(t)$ is generally continuous on its domain.
- $x(t)$ is absolutely integrable on its domain

Symmetry properties of the Fourier transform:

The proofs of the properties are similar to those of the Fourier Series.

Property1:

If $x(t)$ is real then $X(f) = X^*\,(-f) \;\rightarrow \begin{cases} |X(f)| = |X(-f)| \; even\ symmetry \\ < X(f) = -< X(-f) \;\; odd\ symmetry \end{cases}$

Property2:

If $x(t)$ is real and even (time domain) then $X(f)$ is real and even (frequency domain).

Property3:

If $x(t)$ is real and odd (time domain) then $X(f)$ is imaginary and odd (frequency domain).

Example1:

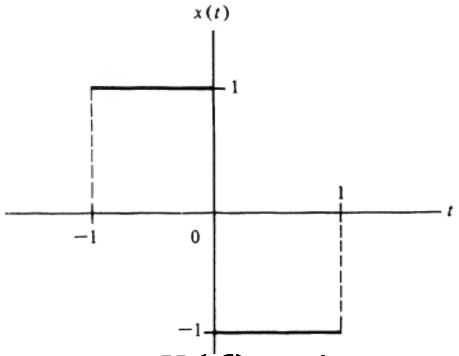
Determine the Fourier Transform of the signal $x(t) = \pi(t)$.

Solution:

The signal satisfies the Dirichlet conditions, it is also real and even, hence we know that $X(f)$ and expect $X(f)$ to be real and even.

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot e^{-j2\pi ft}\,dt = \frac{e^{-j2\pi f \cdot \frac{1}{2}} - e^{-j2\pi f \cdot \frac{-1}{2}}}{-2j\pi f} = \frac{\sin(\pi f)}{\pi f} = sinc(f)$$

Observation: $sinc(f)$ is real and even as expected.




Example 2:

Determine the Fourier Transform of the signal $x(t) = \pi\left(t + \frac{1}{2}\right) - \pi\left(t - \frac{1}{2}\right)$

Solution:

The signal satisfies the Dirichlet conditions, it is also real and odd, hence we know that $X(f)$ and expect $X(f)$ to be imaginary and odd.

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt = \int_{-1}^{0} 1 \cdot e^{-j2\pi ft}\,dt - \int_{0}^{1} 1 \cdot e^{-j2\pi ft}\,dt = \frac{1 - e^{+j2\pi f}}{-j2\pi f} + \frac{e^{-j2\pi f} - 1}{j2\pi f}$$

$$X(f) = \frac{e^{+j2\pi f} + e^{-j2\pi f} - 2}{j2\pi f} = \frac{2cos(2\pi f) - 2}{j2\pi f} = j2\pi f\,\frac{sin^2(\pi f)}{(\pi f)^2} = j2\pi f\,sinc^2(f)$$

Observation:

The signal is imaginary and odd (product of the odd $f$ and the even $sinc^2(f)$ signals) as expected.

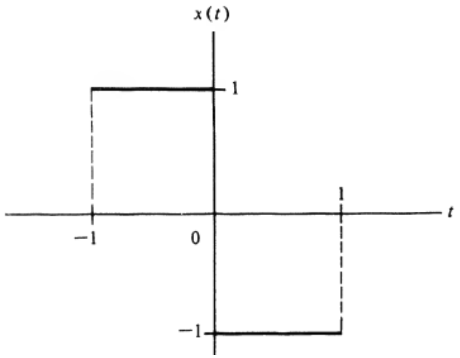
Fourier Transform Theorems:

Fourier Transform theorems assist in simplifying Fourier Transform computation and defining a generalized calculus (using delta) Fourier Transform for models that do not have Fourier Transform in the Fourier Integral sense.

Superposition theorem: Fourier transform is a linear transform, consequently the superposition can be applied. Given the pairs of transforms $x_1(t) \leftrightarrow X_1(f)$ , $x_2(t) \leftrightarrow X_2(f)$ *and the parameters* $\alpha_1$ *and* $\alpha_2$, the following transform pair is valid: $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \leftrightarrow \alpha_1 X_1(f) + \alpha_2 X_2(f) = X(f)$

Example:

Let $x_1(t) = \pi\left(t + \frac{1}{2}\right) \leftrightarrow X_1(f) = sinc(f)e^{j\pi f}$ and $x_2(t) = \pi\left(t - \frac{1}{2}\right) \leftrightarrow X_2(f) = sinc(f)e^{-j\pi f}$ determine the Fourier transform of the signal of $x(t) = \pi\left(t + \frac{1}{2}\right) - \pi\left(t - \frac{1}{2}\right)$.




Solution:

Applying superposition $X(f) = sinc(f)e^{j\pi f} - sinc(f)e^{-j\pi f} = sinc(f)\left[e^{j\pi f} - e^{-j\pi f}\right) =$ $2j\pi f sinc(f)\frac{e^{j\pi f} - e^{-j\pi f}}{2j \cdot \pi f} = 2j\pi f sinc^2(f)$ (as in the previous example)

Delay theorem: Given the pair of transforms $x_1(t) \leftrightarrow X_1(f)$

$x_2(t) = x_1(t - \tau) \leftrightarrow X_2(f) = X_1(f)e^{-j\pi f\tau}$

Example:

Let $x_1(t) = \pi(t) \leftrightarrow sinc(f)$, determine the transform of $x_2(t) = \pi\left(t - \frac{1}{2}\right)$

Solution

$$X_2(f) = \int_0^1 \pi\left(t - \frac{1}{2}\right) e^{-j2\pi f t} dt$$

Using the substitution $t' = t - 0.5 \to dt' = dt, for\ t = 0: t' = -0.5\ and\ for\ t = 1: t' = 0.5$

$$X_2(f) = \int_{-0.5}^{0.5} \pi(t') e^{-j2\pi f(t'+0.5)} dt' = \int_{-0.5}^{0.5} \pi(t') e^{-j2\pi f(t'+0.5)} dt' = e^{-j\pi f} \int_{-0.5}^{0.5} e^{-j2\pi f t'} dt' = sinc(f) e^{-j\pi f}$$

Scaling Theorem:

Given the pair of transforms $x_1(t) \leftrightarrow X_1(f)$, $x_2(t) = x_1(\alpha t) \leftrightarrow X_2(f) = \frac{1}{|\alpha|} X_1\left(\frac{f}{\alpha}\right)$.

Exercise : Prove the theorem. Hint: the prove follows the same steps of the scaling property of $\delta(t)$.

Example: Let $x_1(t) = \pi(t) \leftrightarrow sinc(f)$, determine the transform of $x_2(t) = \pi(0.5t)$ (extended by 2).

Solution $X_2(f) = \int_{-1}^{1} \pi(0.5t) e^{-j2\pi f t} dt$

Using the substitution $t' = 0.5t \to dt' = 0.5dt, for\ t = -1: t' = -0.5\ and\ for\ t = 1.5: t' = 0.5$

$\int_{-0.5}^{0.5} \pi(t') e^{-j2\pi f \cdot 2t'} \cdot 2dt' = 2\int_{-0.5}^{0.5} \pi(t') e^{-j2\pi(2f)\cdot t'} \cdot dt' = 2sinc(2f)$ which can b obtained by direct application of the theorem $X_2(f) = \frac{1}{|0.5|} sinc\left(\frac{f}{0.5}\right) = 2sinc(2f)$.

Inversion Theorem:

Given the pair of transforms $x_1(t) \leftrightarrow X_1(f)$, $x_2(t) = x_1(-t) \leftrightarrow X_2(f) = X_1(-f)$.

The proof is direct by applying the scaling theorem with $\alpha = -1$. the theorem asserts that folded signals have folded spectral representations.

Example: consider the signal $x(t) = \pi\left(t + \frac{1}{2}\right) - \pi\left(t - \frac{1}{2}\right) \leftrightarrow 2j\pi f sinc^2(f)$ , the Fourier transform of:

$y(t) = x(-t) = \pi\left(-t + \frac{1}{2}\right) - \pi\left(-t - \frac{1}{2}\right)$ applying the theorem we obtain is $Y(f) = -2j\pi f sinc^2(f)$

Exercise: plot the signal and its folded version and compute the Fourier Transform of the folded signal.

Duality Theorem:

Given the pair of transforms $x_1(t) \leftrightarrow X_1(f)$, then $X_1(t) \leftrightarrow x_1(-f)$ is also a pair of Fourier Transform.

Proof:

$x_1(t) = \int_{-\infty}^{\infty} X_1\,(f)e^{j2\pi ft}\,df,$ exchanging the name of the variable -$t \leftrightarrow f$ we obtain $\int_{-\infty}^{\infty} X_1(-t)e^{-j2\pi ft}\,dt$, which is the Fourier transform of $x_1(-f)$ by the inversion theorem.

Example:

Determine the Fourier transform of $x(t) = sinc(t)$

Solution:

The solution can be determine by integration, how ever this integration is hard to do. The duality theorem can be used to compute this transform. In fact Considering the pair of transforms $\pi(t) \leftrightarrow sinc(f)$ and applying the duality theorem we get $sinc(t) \leftrightarrow \pi(-f) = \pi(f)$ because $\pi(f)$ has even symmetry, hence $sinc(t) \leftrightarrow \pi(f)$ is a pair of Fourier transform.

Frequency-Shift Theorem:

Let $x(t) \leftrightarrow X(f)$ be a pair of Fourier Transform, then $x(t)e^{j2\pi f_0 t} \leftrightarrow X(f - f_0)$ is also a pair of Fourier Transform.

Exercise: Prove this theorem.

Hint: The proof is a direct consequence of the time delay and duality theorems.

Example:

Determine the transform of $x(t) = \pi(t)e^{j\pi t}$ if you know that $\pi(t) \leftrightarrow sinc(f)$ is a pair of Fourier Transform.

Solution:

Applying the frequency-shift theorem we obtain $X(f) = sinc\left(f - \frac{1}{2}\right)$

Convolution Theorem:

Given the pair of Fourier Transforms $x_1(t) \leftrightarrow X_1(f)$ , $x_2(t) \leftrightarrow X_2(f)$, the Fourier Transform then $x(t) = x_1(t) * x_2(t)$ $\leftrightarrow X_1(f) \cdot X_2\,(f)$

Example:

Let $x_1(t) = x_2(t) = \pi(t) \leftrightarrow sinc(f)$, The convolution $\pi(t) * \pi(t) = \Delta(t)$.

Applying the convolution theorem:

$F[\pi(t) * \pi(t)] = \sin c(f) \cdot sinc(f) = sinc^2(f)$ as we obtained in the previous examples.

Multiplication Theorem:

Given the pair of Fourier Transforms $x_1(t) \leftrightarrow X_1(f)$ , $x_2(t) \leftrightarrow X_2(f)$, the Fourier Transform then $x(t) = x_1(t) \cdot x_2(t)$ $\leftrightarrow X_1(f) * X_2\,(f)$

Example:

Let $x_1(t) = x_2(t) = sinc(t) \leftrightarrow \pi(f)$, The product $sinc(t) \cdot sinc(t) \leftrightarrow \pi(f) * \pi(f) = \Delta(f)$.

Differentiation Theorem:

Let $x(t)$ be n-time differentiable and $x(t) \leftrightarrow X(f)$ represents a pair of Fourier Transform. Then,

$$F\left[\frac{d^n x(t)}{dt^n}\right] = (j2\pi f)^{\,n} X(f)$$

Proof:

The proof is done using the induction method. The first step for $n = 1$ is done as follow:

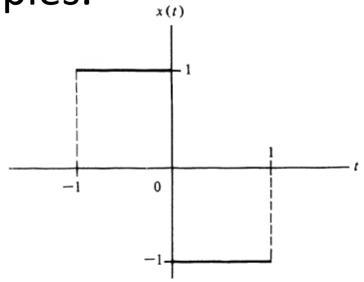
Consider $x(t) = \int_{-\infty}^{\infty} X\,(f) e^{j2\pi f t} df \rightarrow \frac{dx(t)}{dt} = \int_{-\infty}^{\infty} X\,(f) \frac{d}{dt}[e^{j2\pi f t}] df = \int_{-\infty}^{\infty} j2\pi f X\,(f) e^{j2\pi f t} df = \mathrm{F}[\frac{dx(t)}{dt}]$, hence the theorem is true for $n = 1$.

Exercise :Assume the theorem holds for $n - 1$ and use this assumption and the proved step1 for $n = 1$ to prove that the theorem is true for $n$.

Example: Determine the Fourier transform of: $x(t) = \pi\left(t+\frac{1}{2}\right) - \pi\left(t-\frac{1}{2}\right)$ if you know that $\Delta(t) \leftrightarrow sinc^2(f)$

Solution:

The signal $x(t) = \frac{d\Delta(t)}{dt}$ therefore applying $X(f) = j2\pi f \cdot F[\Delta(t)] = j2\pi f \cdot sinc^2(f)$ as in previous examples.




**Fourier Transform via Generalized Calculus** (use of density: $\delta(t)$)

Spectral representations of signal and system models that do not have Fourier transform in the Fourier Integral sense, such as periodic and power signals, can be obtained via the transform of the Dirac impulse

Transform of $\delta(t)$:

$$F[\delta(t)] = \int_{-\infty}^{t} \delta(\sigma)e^{-j2\pi f\sigma}d\sigma = 1$$

Moreover, using the duality theorem, the following two Fourier Transforms hold: $\delta(t) \leftrightarrow 1$ and $1 \leftrightarrow \delta(-f) = \delta(f)$

Integration Theorem:

Let $x(t)$ be an integrable signal or system model with $x(t) \leftrightarrow X(f)$ a pair of Fourier Transform.

The Fourier transform of $y(t) = \int_{-\infty}^{t} x(\sigma)d\sigma$ is $Y(f) = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$

Example 1: Determine the transform of $u(t) = \int_{-\infty}^{t} \delta(\sigma)d\sigma \quad (t > 0)$

Solution: $u(t)$ is a power signal and does not have a Fourier Transform in the Fourier-Integral sense. However, a spectral representation of u(t) can be obtained using the integration theorem which includes the Dirac impulse $\delta(t)$

Now applying the integration theorem we obtain $F[u(t)] = \frac{F[\delta(t)]}{j2\pi f} + \frac{1}{2}F[\delta(t)]_{f=0}\delta(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$

Exercise: Determine the Fourier transform of $\Delta(t)$ using the integration theorem and the pair

$$x(t) = \pi\left(t+\frac{1}{2}\right) - \pi\left(t-\frac{1}{2}\right) \leftrightarrow j2\pi f \cdot sinc^2(f)$$

Transform of $x(t) = \cos(2\pi f_0 t)$

The signal $x(t) = \cos(2\pi f_0 t)$ is a power signal and does not have a spectral representation in the Fourier-Integral sense, however, a spectral representation for $x(t) = \cos(2\pi f_0 t)$ can be obtained as follow:

$$\mathrm{F}[\cos(2\pi f_0 t)] = F[\frac{1 \cdot e^{+j2\pi f_0 t} + 1 \cdot e^{-j2\pi f_0 t}}{2}]$$

Applying the pair of Fourier Transforms $1 \leftrightarrow \delta(f)$ and the frequency shift theorem we obtain:

$\mathrm{F}[\cos(2\pi f_0 t)] = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$. Thus we have the generalized Fourier Transform pair:

$$\cos(2\pi f_0 t) \leftrightarrow \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

Modulation Theorem (Amplitude Modulation –AM):

Let $x(t) \leftrightarrow X(f)$ be a pair of Fourier Transform, then $x(t)\cos(2\pi f_0 t) \leftrightarrow \frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$

Proof: applying the frequency shift property we obtain:

$$\mathrm{F}[x(t)\cos(2\pi f_0 t)] = F\left[\frac{x(t) \cdot e^{+j2\pi f_0 t} + x(t) \cdot e^{-j2\pi f_0 t}}{2}\right] = \frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$$

Observation:

The operation is defined as "Amplitude Modulation" which shifts the spectra of the signal $x(t)$ called the message to the spectral band of the channel with minimum distortion (centered at $f_0$). The operation is done using a nonlinear device called the Mixer that executes the multiplication of the signal $x(t)$ called the message and the sinusoidal signal $c(t) = \cos(2\pi f_c t)$ called the carrier. The result is two frequency distorted spectral functions of $X(f)$ shifted to $\pm f_c$.

Example:

Compute the spectral representation of the modulated sinusoidal signal $x(t) = A_m \cos(2\pi f_m t)$ by the sinusoidal carrier signal $c(t) = A_c \cos(2\pi f_c t)$

Solution:

$$\mathrm{F}[A_m \cos(2\pi f_m t)] = \frac{A_m}{2}\delta(f - f_m) + \frac{A}{2}\delta(f + f_m)$$

Applying the superposition and modulation theorems we obtain the double sided spectral representation:

$$\mathrm{F}[A_m A_c \cos(2\pi f_m t)\cos(2\pi f_c t)] = \frac{A_m A_c}{2}\delta(f - f_m - f_c) + \frac{A_m A_c}{2}\delta(f + f_m - f_c) + \frac{A_m A_c}{2}\delta(f - f_m + f_c) + \frac{A_m A_c}{2}\delta(f + f_m + f_c)$$

The spectral components $\frac{A_m A_c}{2}\delta(f - f_m - f_c) + \frac{A_m A_c}{2}\delta(f + f_m - f_c)$ are called the lower-side band (LSB) and upper –side band (USB) of the modulated signal spectra.

Transformation of periodic:

Let $x(t)$ be a periodic signal with frequency $f_0$ and Fourier Series representation $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$, the Fourier Transform of $x(t)$ (which does not have spectral representation in the Fourier-Integral sense) :$X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0)$.

That is, $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t} \leftrightarrow \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0)$ is a pair of Fourier Transform.

Proof:

$$\mathrm{F}[\mathrm{x(t)}] = \mathrm{F}[\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}] = \sum_{n=-\infty}^{\infty} X_n \cdot F[e^{j2\pi n f_0 t}] = \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0)$$

Example1:

Compute the Fourier transform of the signal $x(t) = \sum_{n=-\infty}^{\infty} \pi(t - nT_0)$ knowing that the Fourier Series of $x(t)$ is:

$x(t) = \sum_{n=-\infty}^{\infty} sinc(nf_0) e^{j2\pi n f_0 t}$

Solution: Applying the periodic signal pair: $X(f) = \sum_{n=-\infty}^{\infty} sinc(nf_0)\delta(f - nf_0)$.

Example1:

Compute the Fourier transform of the train of $\delta(t)$, $x(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$ knowing that the Fourier Series of $x(t)$ is: $x(t)=\sum_{n=-\infty}^{\infty}f_0 e^{j2\pi n f_0 t}$

Solution: Applying the periodic signal pair: $X(f)=\sum_{n=-\infty}^{\infty}f_0\delta(f-nf_0)=f_0\cdot\sum_{n=-\infty}^{\infty}\delta(f-nf_0)$.

That is the transform of the train of $\delta(t)$ is a scaled train of $\delta(f)$ and we have the pair:

$$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)\leftrightarrow f_0\cdot\sum_{n=-\infty}^{\infty}\delta(f-nf_0)$$

Exercise: Prove that the Fourier series coefficients of $x(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$ are $X_n=\frac{1}{T_0}=f_0$

Transform of Periodic Signals via Convolution Theorem:

Let $x(t)\leftrightarrow X(f)$ be a pair of Fourier Transform with x(t) a time limited signal with interval extension $\tau$, the signal $x_p(t)=x(t)*\sum_{n=-\infty}^{\infty}\delta(t-nT_0)=\sum_{n=-\infty}^{\infty}x(t)*\delta(t-nT_0)=\sum_{n=-\infty}^{\infty}x(t-nT_0)$ is a periodic signal.

Computing $F[x_p(t)]=F[x(t)*\sum_{n=-\infty}^{\infty}\delta(t-nT_0)]$, applying the multiplication theorem:

$$X_p(f)=X(f)\cdot f_0\cdot\sum_{n=-\infty}^{\infty}\delta(f-nf_0)=f_0\cdot\sum_{n=-\infty}^{\infty}X(f)\cdot\delta(f-nf_0)=f_0\cdot\sum_{n=-\infty}^{\infty}X(nf_0)\cdot\delta(f-nf_0)$$

Corollary: The coefficients of the Fourier Series can be computed by $X_n=f_0\,X(nf_0)$

Example 1:

Use the pair $\delta(t)\leftrightarrow 1$ to determine the Fourier series coefficients of $x(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$.

Solution: $X_n=f_0\,X(nf_0)=f_0\cdot 1=f_0=\frac{1}{T_0}$ as expected.

Energy Spectral Density Function:

Parseval's Theorem:

Given an Energy signal $x(t)$ with Fourier Transform $X(f)$, The signal energy $E$ can be computed by:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

Proof:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t) \cdot x^*(t) \, dt = \int_{-\infty}^{\infty} x(t) \cdot [\int_{-\infty}^{\infty} X^*(f) e^{-j2\pi f t} df] \, dt$$

Changing the order of integration: $E = \int_{-\infty}^{\infty} X^*(f) \cdot [\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt] \, df = \int_{-\infty}^{\infty} X^*(f) \cdot X(f) df = \int_{-\infty}^{\infty} |X(f)|^2 \, df$

Corollary:

The signal energy contained in the energy band $B$ is given by: $E_B = \int_{-B}^{B} |x(f)|^2 \, df$

Example: Compute the energy of the signal $x(t) = Ae^{-\alpha t} u(t) \; with \; \alpha > 0$ in the band $B$ and the total energy of the signal.

Solution:

Parseval's theorem has to be used because the energy contained in a frequency band is requested. Knowing the pair $Ae^{\alpha t} u(t)$ $\leftrightarrow \frac{A}{\alpha + j2\pi f}$ , $E_B = \int_{-B}^{B} |x(f)|^2 \, df \rightarrow E_B = \int_{-B}^{B} \frac{A^2}{\alpha^2 + 4\pi^2 f^2} df = \frac{1}{\alpha^2} \int_{-B}^{B} \frac{A^2}{1 + \frac{4\pi^2 f^2}{\alpha^2}} df$

Using the substitution $v = \frac{2\pi f}{\alpha} \rightarrow df = \frac{\alpha}{2\pi} dv, \; f = -B \rightarrow v = \frac{-2\pi B}{\alpha}$,, $f = B \rightarrow v = \frac{2\pi B}{\alpha}$

$$E_B = \frac{\alpha A^2}{2\pi \alpha^2} \int_{-\frac{2\pi B}{\alpha}}^{\frac{2\pi B}{\alpha}} \frac{1}{1 + v^2} dv = \frac{A^2}{2\pi \alpha} tan^{-1}(v) \Big|_{-\frac{2\pi B}{\alpha}}^{-\frac{2\pi B}{\alpha}} = \frac{A^2}{2\pi\alpha} [tan^{-1}\left(\frac{2\pi B}{\alpha}\right) - tan^{-1}\left(\frac{-2\pi B}{\alpha}\right)] = \frac{A^2}{\pi \alpha} tan^{-1}\left(\frac{2\pi B}{\alpha}\right)$$

Total Energy: the total energy is computed as $\lim_{B \to \infty} \frac{A^2}{\pi\alpha} tan^{-1}\left(\frac{2\pi B}{\alpha}\right) = \frac{A^2}{\pi\alpha} \cdot \frac{\pi}{2} = \frac{A^2}{2\alpha}$ as expected from the time domain computation.

Corollary:
The energy spectral density of the energy signal $x(t)$ is $G(f) = |X(f)|^2$.
System Response:
Given an LTI system with frequency response $h(t) \leftrightarrow H(f)$ , the spectral representation of the system response $y(t) \leftrightarrow Y(f)$ to the input $x(t) \leftrightarrow X(f)$ is $Y(f) = X(f) \cdot H(f)$.
Proof:
Using the response convolution theorem of LTI systems $y(t) = x(t) * h(t)$, then applying the Fourier Transform convolution theorem we obtain $Y(f) = X(f) \cdot H(f)$

LTI System-Response Energy Spectral Density function:
Given an LTI system with frequency response $H(f)$ and input energy density function $G_x(f) = |X(f)|^2$, the system output response energy spectral density function $G_y(f) = |Y(f)|^2 = G_x(f) \cdot |H(f)|^2$
Proof:

$$Y(f) = H(f) \cdot X(f) \rightarrow |Y(f)|^2 = G_y(f) = |H(f)|^2 \cdot |X(f)|^2 = G_x(f) \cdot |H(f)|^2$$

LTI System Frequency Response via Fourier Transform Theorems:
Given The dynamic system representation $\sum_{i=n}^{0} \alpha_i \frac{d^i y(t)}{dt^i} = \sum_{i=m}^{0} \beta_i \frac{d^i x(t)}{dt^i}$ , the frequency response of the system is given by:

$$H(f)` = \frac{Y(f)}{X(f)} = \frac{\sum_{i=m}^{0} \beta_i \cdot (j2\pi f)^i}{\sum_{i=n}^{0} \alpha_i (j2\pi f)^i}$$

Exercise: Derive the formula of $H(f)$ computation using the Fourier differentiation theorem.
Hint: the proof is a direct result of the application of the Fourier Transform differentiation and superposition theorems and the pairs $x(t) \leftrightarrow X(f), y(t) \leftrightarrow Y(f)$, and $\delta(t) \leftrightarrow 1$.

Example: Compute the Frequency-Response of the system $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 2\frac{dx(t)}{dt} + 4x(t)$

Applying the differentiation and superposition theorems we obtain $[(j2\pi f)^2 + 5 \cdot j2\pi f + 6]Y(f) = [2 \cdot j2\pi f + 4]X(f)$

Applying $x(t) = \delta(t) \leftrightarrow 1$ to which $y(t) = h(t) \leftrightarrow H(f)$ or computing $H(f) = \frac{Y(f)}{X(f)}$ *we obtain:*

$$H(f) = \frac{Y(f)}{X(f)} = \frac{j4\pi f + 4}{6 - 4\pi^2 f^2 + j10\pi f} \rightarrow H(\omega) = \frac{j2\omega + 4}{6 - \omega^2 + j5\omega}$$

Frequency Response via Laplace Transform:
Given the impulse response $h(t)$ and the transfer function $T(s)$ of an LTI system, the frequency response $H(\omega)$ of the system can be obtained by $H(\omega) = T(s)|_{s=j\omega}$.

Example:

Determine the frequency response of the system with transfer function $T(s) = \frac{3s+1}{s^2+5}$

Solution:

$$H(\omega) = T(s)|_{s=j\omega} = \frac{j3\omega+1}{5-\omega^2}$$

LTI System Steady-State Response to Periodic Signals via Fourier Transform:

Let $x(t)=\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$ be the Fourier Series representation of a periodic signal with frequency $f_0$, the response of the steady state response of the LTI system with frequency response $H(f)$ is given as follow:

$$Y(f)=H(f)\cdot X(f)$$

$$X(f)=F\left[\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}\right]=\sum_{n=-\infty}^{\infty} X_n \delta(f-nf_0) \rightarrow$$

$Y(f)=H(f)\cdot\sum_{n=-\infty}^{\infty} X_n\delta(f-nf_0)$=$\sum_{n=-\infty}^{\infty} X_n H(nf_0)\delta(f-nf_0)$

The steady state response $y(t)$ is obtained by the inverse Fourier Transform, that is:

$$y(t)=F^{-1}\left[\sum_{n=-\infty}^{\infty} X_n H(nf_0)\delta(f-nf_0)\right]=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} X_n H(nf_0)\delta(f-nf_0)e^{j2\pi f t}df=\sum_{n=-\infty}^{\infty} X_n H(nf_0)e^{j2\pi n f_0 t} \rightarrow$$

$y(t)=\sum_{n=-\infty}^{\infty}|X_n||H(nf_0)|e^{j(2\pi n f_0 t+<X_n+<H(nf_0))}$ as was obtained using the Fourier series

$y(t)$ can also be expressed using the relation $X_n$=$f_0\, X(nf_0)$, with $X(f)$ the transform of the energy signal of the basic period of the periodic signal, as:

$$y(t)=\sum_{n=-\infty}^{\infty} f_0\,|X(nf_0)|H(nf_0)|e^{j(2\pi n f_0 t+<X(nf_0)+<H(nf_0))}$$

Example: Compute the steady state response of the system with frequency response $H(f)=\frac{10}{3+j2\pi f}$ to the periodic signal $x(t)=\sum_{n=-\infty}^{\infty} sinc(t-nT_0)$ knowing the pair $sinc(t)\leftrightarrow \pi(f)$

Solution: based on the previous analysis

$$y(t)=\sum_{n=-\infty}^{\infty} f_0\,\pi(nf_0)\frac{10}{\sqrt{9+(2\pi n f_0)^2}}e^{j\left(2\pi n f_0 t+0-tan^{-1}\left(\frac{2\pi n f_0}{3}\right)\right)}$$

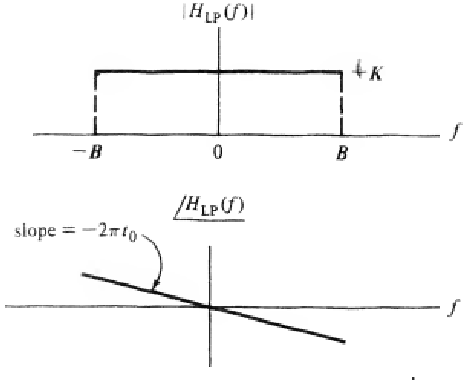
## Ideal Filters:

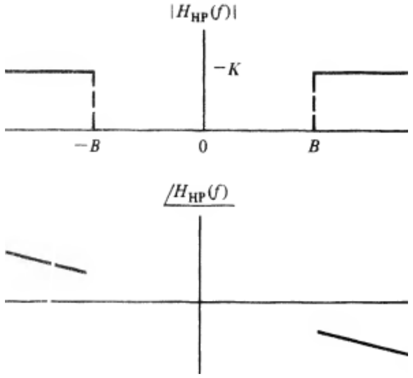
Ideal-Filters are system components considered in the frequency domain and classified according to the part of frequency they pass. That is they pass a defined band of frequency called  in the so called pass-band and cut the other range called  rejection-band.

Observation: real filters has three regions: pass-band, transition-band, and rejection-band defined by certain conditions on their frequency response magnitude spectra.

## Ideal-Filters classification:

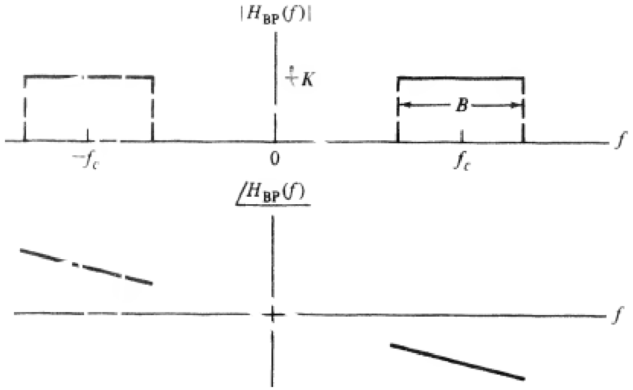
- Low-Pass Filter (LPF): passes a certain set of frequencies band around the origin with a cut-off frequency $f_c = B$ .
- High-Pass Filter (BPF): passes a certain set of frequencies band after a cut-off frequency $f_c$.




- Band-Pass Filter (BPF): passes a certain set of frequencies band around a certain frequency called the central band frequency. It has two cut-off frequencies $f_{c1} = f_c - \frac{B}{2}$ and $f_{c2} = f_c + \frac{B}{2}$ .

- Band-Pass Filter (RPF): rejects a certain set of frequencies band around a certain frequency called the central band frequency. It has two cut-off frequencies $f_{c1} = f_c - \frac{B}{2}$ and $f_{c2} = f_c + \frac{B}{2}$



Hilbert Transform:
Aims to generate orthogonal signals using same time base reference. (extensively used in communication systems).
Definition: The pair of signal $x(t) \leftrightarrow x^H(t)$ are said to be a Hilbert pair $\leftrightarrow \exists\ x^H(t) = x(t) * \frac{1}{\pi t} = \int_{-\infty}^{\infty} \frac{x(\lambda)}{\pi(t-\lambda)} d\lambda$
Theorem:
The spectral components of $X(f)$and $X^H(f)$ are orthogonal.
Proof:
Consider the signal $sign(t) = \begin{cases} 1 & for\ \ t > 0 \\ -1 & for\ t < 0 \end{cases} = 2u(t) - 1$
The Fourier transform $\mathrm{F}[2u(t) - 1] = 2\left[\frac{1}{j2\pi f} - \frac{1}{2}\delta(f)\right] - \delta(f) = \frac{1}{j\pi f}$
Using duality and linearity $\frac{1}{\pi t} \leftrightarrow jsign(-f) = -jsign(f)$, consequently using the Fourier convolution theorem

$$X^H(f) = F[x^H(t)] = F\left[x(t) * \frac{1}{\pi t}\right] = X(f) \cdot F\left[\frac{1}{\pi t}\right] = -jsign(f) \cdot X(f) \text{ that is :}$$

$$X^H(f) \leftrightarrow -jsign(f) \cdot X(f) \text{ is a pair of Fourier transform}$$

The amplitude and phase spectra of $X^H(f)$ are given by:

$$|X^H(f)| = |X(f)|$$

$$< X^H\ (f) = \begin{cases} < X(f) - \frac{\pi}{2}\ \ for\ f > 0 \\ < X(f) + \frac{\pi}{2}\ for\ \ f < 0 \end{cases} \ which\ proves\ the\ assert$$

Example: Compute the Hilbert transform of $x(t) = Acos(\omega_0 t)$

$$X(f) = \frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0) \to X^H(f) = \frac{A}{2}\delta\left(f - f_0 - \frac{\pi}{2}\right) + \frac{A}{2}\delta\left(f + f_0 + \frac{\pi}{2}\right) \to x^H(t) = Asin(\omega_0 t)$$

## Ideal-Filter Frequency and Impulse responses:

LPF frequency response: $|H(f)| = K\Pi\left(\frac{f}{2B}\right) = \begin{cases} K, & |f| \leq B \\ 0, & \text{otherwise} \end{cases}$ $\quad \underline{/H(f)} = -2\pi t_0 f$

LPF Impulse response: $h_{LP}(t) = \int_{-\infty}^{\infty} H_{LP}(f) e^{j2\pi ft}\, df = \int_{-B}^{B} K e^{-j2\pi f t_0} e^{j2\pi ft}\, df = \int_{-B}^{B} K e^{j2\pi f(t-t_0)}\, df = 2BK \operatorname{sinc}[2B(t-t_0)]$

HPF Impulse response: $h_{HP}(t) = K\delta(t-t_0) - 2BK \operatorname{sinc}[2B(t-t_0)]$

BPF Impulse response: $h_{BP}(t) = 2KB \operatorname{sinc}[B(t-t_0)] \cos[2\pi f_0(t-t_0)]$

Exercise:

Use the superposition, modulation, and time delay theorems to determine $H_{HP}(f)$ and $H_{HP}(f)$