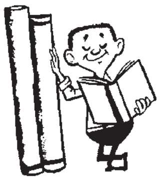
Legendre Polynomials and Functions



Reading **Problems**

Outline

Background and Definitions

The ordinary differential equation referred to as *Legendre's differential equation* is frequently encountered in physics and engineering. In particular, it occurs when solving Laplace's equation in spherical coordinates.

Adrien-Marie Legendre (September 18, 1752 - January 10, 1833) began using, what are now referred to as Legendre polynomials in 1784 while studying the attraction of spheroids and ellipsoids. His work was important for geodesy.

1. Legendre's Equation and Legendre Functions

The second order differential equation given as

$$(1-x^2)\,\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + n(n+1)\,y = 0 \qquad n > 0, \quad |x| < 1$$

is known as Legendre's equation. The general solution to this equation is given as a function of two Legendre functions as follows

$$y = A\mathrm{P}_n(x) + B\mathrm{Q}_n(x) \qquad |x| < 1$$

where

$$\mathrm{P}_n(x) = \frac{1}{2^n n!}\,\frac{d^n}{dx^n}(x^2-1)^n \qquad \text{Legendre function of the first kind}$$

$$\mathrm{Q}_n(x) = \frac{1}{2}\mathrm{P}_n(x)\ln\frac{1+x}{1-x} \qquad \text{Legendre function of the second kind}$$

2. Legendre's Associated Differential Equation

Legendre's associated differential equation is given as

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$$

If we set $m = 0$ in this equation the differential equation reduces to Legendre's equation.

The general solution to Legendre's associated equation is given as

$$y = A\ \mathrm{P}_n^m(x) + B\ \mathrm{Q}_n^m(x)$$

where $\mathrm{P}_n^m(x)$ and $\mathrm{Q}_n^m(x)$ are called the associated Legendre functions of the first and second kind given as

$$\begin{aligned} \mathrm{P}_n^m(x) &= (1-x^2)^{m/2}\,\frac{d^m}{dx^m}\,\mathrm{P}_n(x) \\ \mathrm{Q}_n^m(x) &= (1-x^2)^{m/2}\,\frac{d^m}{dx^m}\,\mathrm{Q}_n(x) \end{aligned}$$

Legendre's Equation and Its Solutions

Legendre's differential equations is

$$(1-x^2)\,\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + n(n+1)\,y = 0 \qquad n > 0,\ \ |x| < 1$$

or equivalently

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)\,y = 0 \qquad n > 0,\ \ |x| < 1$$

Solutions of this equation are called Legendre functions of order n. The general solution can be expressed as

$$y = A\mathrm{P}_n(x) + B\mathrm{Q}_n(x) \qquad\qquad |x| < 1$$

where $\mathrm{P}_n(x)$ and $\mathrm{Q}_n(x)$ are *Legendre Functions* of the first and second kind of order n.

If $n = 0, 1, 2, 3, \ldots$ the $\mathrm{P}_n(x)$ functions are called *Legendre Polynomials* or order n and are given by Rodrigue's formula.

$$\mathrm{P}_n(x) = \frac{1}{2^n n!}\,\frac{d^n}{dx^n}(x^2-1)^n$$

Legendre functions of the first kind ($P_n(x)$ and second kind ($Q_n(x)$ of order $n = 0, 1, 2, 3$ are shown in the following two plots

The first several Legendre polynomials are listed below

$$\mathbf{P_0(x) = 1} \qquad \mathbf{P_3(x) = \frac{1}{2}(5x^3 - 3x)}$$

$$\mathbf{P_1(x) = x} \qquad \mathbf{P_3(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)}$$

$$\mathbf{P_2(x) = \frac{1}{2}(3x^2 - 1)} \qquad \mathbf{P_3(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)}$$

The recurrence formula is

$$\mathbf{P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)}$$

$$\mathbf{P'_{n+1}(x) - P'_{n-1}(x) = (2n+2) P_n(x)}$$

can be used to obtain higher order polynomials. In all cases $\mathbf{P_n(1) = 1}$ and $\mathbf{P_n(-1) = (-1)^n}$

Orthogonality of Legendre Polynomials

The Legendre polynomials $\mathbf{P_m(x)}$ and $\mathbf{P_n(x)}$ are said to be orthogonal in the interval $\mathbf{-1 \leq x \leq 1}$ provided

$$\mathbf{\int_{-1}^{1} P_m(x)\ P_n(x)\ dx = 0} \qquad \mathbf{m \neq n}$$

and as a result we have

$$\mathbf{\int_{-1}^{1} [P_n(x)]^2\ dx = \frac{2}{2n+1}} \qquad \mathbf{m = n}$$

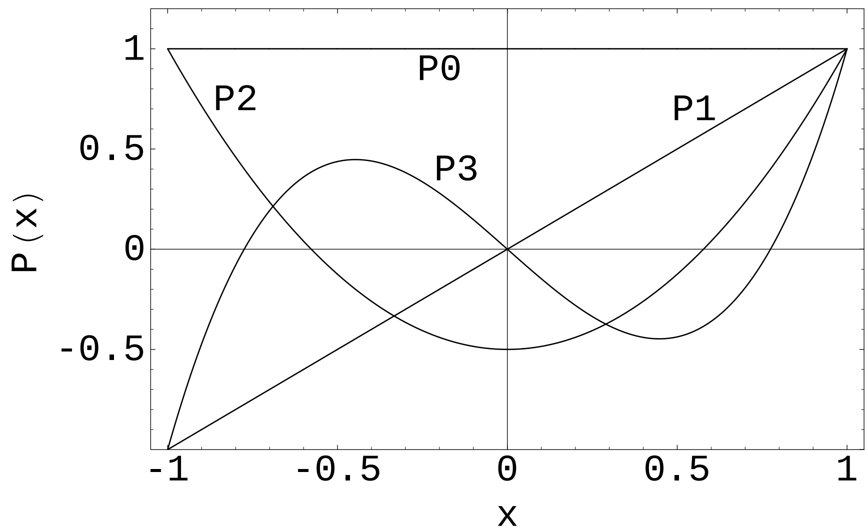


Figure 5.1: Legendre function of the first kind, $\boldsymbol{P_n(x)}$

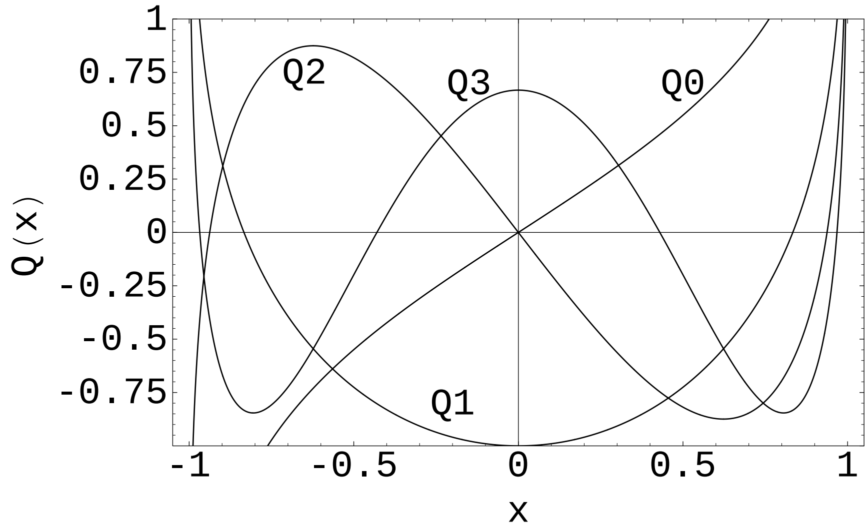


Figure 5.2: Legendre function of the second kind, $\boldsymbol{Q_n(x)}$

Orthogonal Series of Legendre Polynomials

Any function $f(x)$ which is finite and single-valued in the interval $-1 \leq x \leq 1$, and which has a finite number or discontinuities within this interval can be expressed as a series of Legendre polynomials.

We let

$$\begin{aligned} f(x) &= A_0 \mathrm{P}_0(x) + A_1 \mathrm{P}_1(x) + A_2 \mathrm{P}_2(x) + \ldots \qquad -1 \leq x \leq 1 \\ &= \sum_{n=0}^{\infty} A_n \mathrm{P}_n(x) \end{aligned}$$

Multiplying both sides by $\mathrm{P}_m(x)\ dx$ and integrating with respect to x from $x = -1$ to $x = 1$ gives

$$\int_{-1}^{1} f(x) \mathrm{P}_m(x)\ dx = \sum_{n=0}^{\infty} A_n \int_{-1}^{1} \mathrm{P}_m(x) \mathrm{P}_n(x)\ dx$$

By means of the orthogonality property of the Legendre polynomials we can write

$$A_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) \mathrm{P}_n(x)\ dx \qquad n = 0, 1, 2, 3 \ldots$$

Since $\mathrm{P}_n(x)$ is an even function of x when n is even, and an odd function when n is odd, it follows that if $f(x)$ is an even function of x the coefficients A_n will vanish when n is odd; whereas if $f(x)$ is an odd function of x, the coefficients A_n will vanish when n is even.

Thus for and even function $f(x)$ we have

$$A_n = \begin{cases} 0 & \text{n is odd} \\ (2n+1) \int_{0}^{1} f(x) \mathrm{P}_n(x)\ dx & \text{n is even} \end{cases}$$

whereas for an odd function $f(x)$ we have

$$A_n = \begin{cases} (2n+1) \int_{0}^{1} f(x) \mathrm{P}_n(x)\ dx & \text{n is odd} \\ 0 & \text{n is even} \end{cases}$$

When $x = \cos\theta$ the function $f(\theta)$ can be written

$$f(\theta) = \sum_{n=0}^{\infty} A_n \mathrm{P}_n(\cos\theta) \qquad\qquad 0 \leq \theta \leq \pi$$

where

$$A_n = \frac{2n+1}{2}\int_0^{\pi} f(\theta)\mathrm{P}_n(\cos\theta)\sin\theta\; d\theta \qquad\qquad n = 0, 1, 2, 3 \ldots$$

Some Special Results Legendre Polynomials

Integral form

$$\mathrm{P}_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x + \sqrt{x^2-1}\;\cos t\right]^n\; dt$$

Values of $\mathrm{P}_n(x)$ at $x = 0$ and $x = \pm 1$

$$\mathrm{P}_{2n}(0) = \frac{(-1)^n\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n+1)} \qquad \mathrm{P}_{2n+1}(0) = 0$$

$$\mathrm{P}'_{2n}(0) = 0 \qquad \mathrm{P}'_{2n+1}(0) = \frac{(-1)^n 2\Gamma(n+3/2)}{\sqrt{\pi}\Gamma(n+1)}$$

$$\mathrm{P}_n(1) = 1 \qquad \mathrm{P}_n(-1) = (-1)^n$$

$$\mathrm{P}'_n(1) = \frac{n(n+1)}{2} \qquad \mathrm{P}'_n(-1) = (-1)^{n-1}\,\frac{n(n+1)}{2}$$

$$|\mathrm{P}_n(x)| \leq 1$$

The primes denote differentiation with respect to x therefore

$$\mathrm{P}'_n(1) = \frac{d\mathrm{P}_n(x)}{dx} \quad \text{at} \quad x = 1$$

Generating Function for Legendre Polynomials

If $\boldsymbol{A}$ is a fixed point with coordinates $(\boldsymbol{x_1, y_1, z_1})$ and $\boldsymbol{P}$ is the variable point $(\boldsymbol{x, y, z})$ and the distance $\boldsymbol{AP}$ is denoted by $\boldsymbol{R}$, we have

$$\boldsymbol{R^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}$$

From the theory of Newtonian potential we know that the potential at the point $\boldsymbol{P}$ due to a unit mass situated at the point $\boldsymbol{A}$ is given by

$$\boldsymbol{\phi = \frac{C}{R}}$$

where $\boldsymbol{C}$ is some constant. It can be shown that this function is a solution of Laplace's equation.

In some circumstances, it is desirable to expand $\boldsymbol{\phi}$ in powers of $\boldsymbol{r}$ or $\boldsymbol{r^{-1}}$ where $\boldsymbol{r} = \sqrt{\boldsymbol{x^2 + y^2 + z^2}}$ is the distance from the origin $\boldsymbol{O}$ to the point $\boldsymbol{P}$.

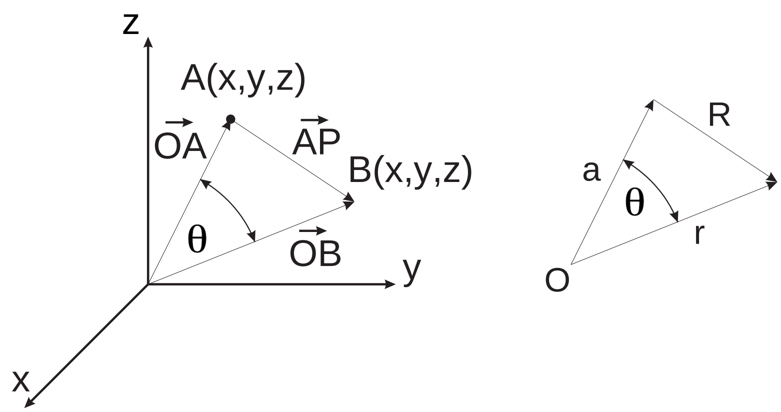


Figure 5.3: Generating Function for Legendre Polynomials

$$\begin{aligned}
\boldsymbol{a} &= \left|\boldsymbol{\vec{OA}}\right| \\
\boldsymbol{r} &= \left|\boldsymbol{\vec{OB}}\right| \\
\boldsymbol{\phi} &= \boldsymbol{\frac{C}{R} = \frac{C}{\sqrt{r^2 + a^2 - 2\cos^{-1}\theta}}}
\end{aligned}$$

Through substitution we can write

$$\phi = \frac{C}{r}\left[1 - 2xt + t^2\right]^{-1/2}$$

where

$$t = \frac{a}{r}, \qquad\qquad x = \cos\theta$$

Therefore

$$\phi \equiv \frac{C}{r}\, g(x,t)$$

We introduce the angle θ between the vectors $\vec{OA}$ and $\vec{OP}$ and write

$$R^2 = r^2 + a^2 - 2\ \cos^{-1}\theta$$

where $a = |\vec{OA}|$. If we let $r/R = t$ and $x = \cos\theta$, then

$$g(x,t) = (1 - 2xt + t^2)^{-1/2}$$

is defined as the generating function for $\mathbf{P}_n(x)$. Expanding by the binomial expansion we have

$$g(x,t) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\ n\ \frac{(2xt - t^2)^n}{n!}$$

where the symbol $(\alpha)_n$ is defined by

$$\begin{aligned}(\alpha)_n &= \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1) = \Pi_{k=0}^{n-1}(\alpha+k)\\ (\alpha)_0 &= 1\end{aligned}$$

$(\alpha)_n$ is referred to as the Pochammer symbol and (α, n) is the Appel's symbol.

Thus we have

$$g(x,t) = \sum_{n=0}^{\infty} \frac{(1/2)n}{n!} \sum_{k=0}^{n} \frac{n!(2x)^{n-k} t^{n-k} (-t^2)^k}{k!(n-k)!}$$

which can be written as

$$g(x,t) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n/2} \frac{(-1)^k (2n-2k)! x^{n-2k}}{2^n k!(n-2k)!(n-k)!} \right] t^n$$

The coefficient of t^n is the Legendre polynomial $\mathrm{P}_n(x)$, therefore

$$g(x,t) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} \mathrm{P}_n(x) t^n \qquad |x| \leq 1, \ |t| < 1$$

Legendre Functions of the Second Kind

A second and linearly independent solution of Legendre's equation for n=positive integers are called Legendre functions of the second kind and are defined by

$$\mathrm{Q}_n(x) = \frac{1}{2} \mathrm{P}_n(x) \ln \frac{1+x}{1-x} = \mathrm{W}_{n-1}(x)$$

where

$$\mathrm{W}_{n-1}(x) = \sum_{m=1}^{n} \frac{1}{m} \mathrm{P}_{m-1}(x) \mathrm{P}_{n-m}(x)$$

is a polynomial of the $(n-1)$ degree. The first term of $\mathrm{Q}_n(x)$ has logarithmic singularities at $x = \pm 1$ or $\theta = 0$ and π.

The first few polynomials are listed below

$$\begin{aligned}
Q_0(x) &= \frac{1}{2}\ln\frac{1+x}{1-x} \\
Q_1(x) &= P_1(x)Q_0(x) - 1 \\
Q_2(x) &= P_2(x)Q_0(x) - \frac{3}{2}x \\
Q_3(x) &= P_3(x)Q_0(x) - \frac{5}{2}x^2 + \frac{2}{3}
\end{aligned}$$

showing the even order functions to be odd in x and conversely.

The higher order polynomials $Q_n(x)$ can be obtained by means of recurrence formulas exactly analogous to those for $P_n(x)$.

Numerous relations involving the Legendre functions can be derived by means of complex variable theory. One such relation is an integral relation of $Q_n(x)$

$$Q_n(x) = \int_0^\infty \left[x + \sqrt{x^2-1}\ \cosh\theta\right]^{-n-1} d\theta \qquad\qquad |x| > 1$$

and its generating function

$$(1-2xt+t^2)^{-1/2}\cosh^{-1}\frac{t-x}{\sqrt{x^2-1}} = \sum_{n=0}^{\infty} Q_n(x)t^n$$

Some Special Values of $Q_n(x)$

$$Q_{2n}(0) = 0 \qquad\qquad Q_{2n+1}(0) = (-1)^{n+1}\frac{2\cdot4\cdot6\cdot\cdots\cdot2n}{1\cdot3\cdot5\cdots(2n-1)}$$

$$Q_n(1) = \infty \qquad\qquad Q_n(-x) = (-1)^{n+1}Q_n(x)$$

Legendre's Associated Differential Equation

The differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right] y = 0$$

is called Legendre's associated differential equation. If $m = 0$, it reduces to Legendre's equation. Solutions of the above equation are called associated Legendre functions. We will restrict our discussion to the important case where m and n are non-negative integers. In this case the general solution can be written

$$y = A\ \mathrm{P}_n^m(x) + B\ \mathrm{Q}_n^m(x)$$

where $\mathrm{P}_n^m(x)$ and $\mathrm{Q}_n^m(x)$ are called the associated Legendre functions of the first and second kind respectively. They are given in terms of ordinary Legendre functions.

$$\begin{aligned}
\mathrm{P}_n^m(x) &= (1-x^2)^{m/2}\,\frac{d^m}{dx^m}\,\mathrm{P}_n(x) \\
\mathrm{Q}_n^m(x) &= (1-x^2)^{m/2}\,\frac{d^m}{dx^m}\,\mathrm{Q}_n(x)
\end{aligned}$$

The $\mathrm{P}_n^m(x)$ functions are bounded within the interval $-1 \le x \le 1$ whereas $\mathrm{Q}_n^m(x)$ functions are unbounded at $x = \pm 1$.

Special Associated Legendre Functions of the First Kind

$$\mathrm{P}_n^0(x) = \mathrm{P}_n(x)$$

$$\mathrm{P}_n^m(x) = \frac{(1-x^2)^{m/2}}{2^n n!}\,\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n = 0 \qquad m > n$$

$$\mathrm{P}_1'(x) = (1-x^2)^{1/2} \qquad\qquad \mathrm{P}_3'(x) = \frac{3}{2}(5x^2-1)(1-x^2)^{1/2}$$

$$\mathrm{P}_2'(x) = 3x(1-x^2)^{1/2} \qquad\qquad \mathrm{P}_3^2(x) = 15x(1-x^2)$$

$$\mathrm{P}_2^2(x) = 3(1-x^2) \qquad\qquad \mathrm{P}_3^3(x) = 15(1-x^2)^{3/2}$$

Other associated Legendre functions can be obtained by the recurrence formulas.

Recurrence Formulas for $\mathrm{P}_n^m(x)$

$$\begin{aligned}(n+1-m)\mathrm{P}_{n+1}^m(x) &= (2n+1)x\mathrm{P}_n^m(x) - (n+m)\mathrm{P}_{n-1}^m(x) \\ \mathrm{P}_n^{m+2}(x) &= \frac{2(m+1)}{(1-x^2)^{1/2}}\, x\, \mathrm{P}_n^{m+1} - (n-m)(n+m+1)\mathrm{P}_n^m(x)\end{aligned}$$

Orthogonality of $\mathrm{P}_n^m(x)$

As in the case of Legendre polynomials, the Legendre functions $\mathrm{P}_n^m(x)$ are orthogonal in the interval $-1 \leq x \leq 1$

$$\int_{-1}^{1} \mathrm{P}_n^m(x)\mathrm{P}_k^m(x)\; dx = 0 \qquad\qquad n \neq k$$

and also

$$\int_{-1}^{1} \left[\mathrm{P}_n^m(x)\right]^2 \; dx = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$$

Orthogonality Series of Associated Legendre Functions

Any function $f(x)$ which is finite and single-valued in the interval $-1 \leq x \leq 1$ can be expressed as a series of associated Legendre functions

$$f(x) = A_m \mathrm{P}_1^m(x) + A_{m+1}\mathrm{P}_{m+1}^m(x) + A_{m+2}\mathrm{P}_{m+2}^m(x) + \ldots$$

where the coefficients are determined by means of

$$\boldsymbol{A_k = \frac{2k+1}{2}\frac{(k-m)!}{(k+m)!}\int_{-1}^{1} f(x)\mathrm{P}_k^m(x)\ dx}$$

Assigned Problems

Problem Set for Legendre Functions and Polynomials

1. Obtain the Legendre polynomial $P_4(x)$ from Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2-1)^n\right]$$

2. Obtain the Legendre polynomial $P_4(x)$ directly from Legendre's equation of order 4 by assuming a polynomial of degree 4, i.e.

$$y = ax^4 + bx^3 + cx^2 + dx + e$$

3. Obtain the Legendre polynomial $P_6(x)$ by application of the recurrence formula

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

assuming that $P_4(x)$ and $P_5(x)$ are known.

4. Obtain the Legendre polynomial $P_2(x)$ from Laplace's integral formula

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} (x + \sqrt{x^2-1}\cos t)^n \, dt$$

5. Find the first three coefficients in the expansion of the function

$$f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases}$$

in a series of Legendre polynomials $P_n(x)$ over the interval $(-1, 1)$.

6. Find the first three coefficients in the expansion of the function

$$f(\theta) = \begin{cases} \cos\theta & 0 \le \theta \le \pi/2 \\ 0 & \pi/2 \le \theta \le \pi \end{cases}$$

in a series of the form

$$f(\theta) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta) \qquad 0 \le \theta \le \pi$$

7. Obtain the associated Legendre functions $P_2^1(x),\ P_3^2(x)$ and $P_2^3(x)$.

8. Verify that the associated Legendre function $P_3^2(x)$ is a solution of Legendre's associated equation for $m = 2,\ n = 3$.

9. Verify the result

$$\int_{-1}^{1} P_n^m(x)\ P_k^m(x)\ dx = 0 \qquad n \ne k$$

for the associated Legendre functions $P_2^1(x)$ and $P_3^1(x)$.

10. Verify the result

$$\int_{-1}^{1} \left[P_n^m(x)\right]^2\ dx = \frac{2}{2n+1}\ \frac{(n+m)!}{(n-m)!}$$

for the associated Legendre function $P_1^1(x)$.

11. Obtain the Legendre functions of the second kind $Q_0(x)$ and $Q_1(x)$ by means of

$$Q_n(x) = P_n(x) \int \frac{dx}{[P_n(x)]^2(1-x^2)}$$

12. Obtain the function $Q_3(x)$ by means of the appropriate recurrence formula assuming that $Q_0(x)$ and $Q_1(x)$ are known.

Selected References

1. **Abramowitz, M. and Stegun, I.A.**, *Handbook of Mathematical Functions*, Dover, New York, 1965.

2. **Arfken, G.**, "Legendre Functions of the Second Kind," *Mathematical Methods for Physicists*, 3rd ed. Orlando, FL: Academic Press, pp. 701-707, 1985.

3. **Binney, J. and Tremaine, S.**, "Associated Legendre Functions," Appendix 5 in *Galactic Dynamics*, Princeton, NJ: Princeton University Press, pp. 654-655, 1987.

4. **Snow, C.** *Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory*, Washington, DC: U.S. Government Printing Office, 1952.

5. **Spanier, J. and Oldham, K.B.**, "The Legendre Functions", Ch.59 in *An Atlas of Functions*, Washington, DC: Hemisphere, pp 581-597, 1987.