Chapter 7.1, Problem 40E

Problem

Let X and Y be sets, let A and B be any subsets of X, and let F be a function from X to Y. Fill in the blanks in the following proof that $F(A) \cup \quad \cup F(B) \subseteq F(A \cup B)$.

Proof: Let y be any element in $F(A) \cup F(B)$. *[We must show that y is in $F(A \cup B)$.]* By definition of union, (a).

***Case 1, $y \in F(A)$*:** In this case, by definition of $F(A)$, $y = F(x)$ for (b)$x \in A$. Since $A \subseteq A \cup B$, it follows from the definition of union that $x \in$ (c). Hence, $y = F(x)$ for some $x \in A \cup B$, and thus, by definition of $F(A \cup B)$, $y \in$ (d).

***Case 2, $y \in F(B)$*:** In this case, by definition of $F(B)$, (e) $x \in B$. Since $B \subseteq A \cup B$ it follows from the definition of union that (f).

Therefore, regardless of whether $y \in F(A)$ or $y \in F(B)$, we have that $y \in F(A \cup B)$ *[as was to be shown]*.

Step-by-step solution

Step 1 of 2

The function F is defined from X to Y and $A, B \subseteq X$.

The objective is to complete the proof of $F(A) \cup F(B) \subseteq F(A \cup B)$.

Let $y \in F(A) \cup F(B)$.

By definition of union, either $y \in F(A) \text{ or } y \in F(B)$

Case 1, $y \in F(A)$:

By the definition of the image set $F(A) = \{f(x) | x \in A\}$, there is a $y = F(x)$ for some $x \in A$

In this case, by the definition of union and $A \subseteq A \cup B$, it implies that $x \in B$.

Hence, $y = F(x)$ for some $x \in A \cup B$.

Thus, $y \in F(A \cup B)$.

Step 2 of 2

Case 2, $y \in F(B)$:

By the definition of the image set $F(B) = \{f(x) | x \in B\}$, there is a $y = F(x)$ for some $x \in B$.

In this case, by the definition of union and $B \subseteq A \cup B$, it implies that $x \in A$.

Hence, $y = F(x)$ for some $x \in A \cup B$.

Thus, $y \in F(A \cup B)$.

Therefore, regardless of whether $y \in F(A)$ or $y \in F(B)$, the element $y \in F(A \cup B)$.

The blue highlighted expressions fill the given blanks.