Ch. 7.1 Inverse function and their derivatives:

One - to - One function :-

Def: A function $f: D \to R$ is called 1-1 function if No two different element of D has same image in R

That is: if $x_1 \neq x_2 \longrightarrow$ then $f(x_1) \neq f(x_2)$

Equivelantly If $f(x_1) = f(x_2) \longrightarrow$ then $x_1 = x_2$

Graphically :- A function is 1-1 iff no horizantal line intersect its graph more than once

Remark: Any increasing / Decreasing function on $\mathbb{R}$ $\longrightarrow$ Then its 1-1 function

Ex: $f(x) = (x+1)^3 + 2$ Are the function $y = f(x)$ one to one??

$f(x)$ is 1-1 fun. becaus let $f(x_1) = f(x_2)$

$$(x_1+1)^3 + 2 = (x_2+1)^3 + 2$$

$$x_1 + 1 = x_2 + 1$$

$$x_1 = x_2$$

Exp $g(x) = \frac{x+1}{x}$

$g(x)$ is 1-1 func because $g'(x) = \frac{-1}{x^2}$

$\longrightarrow$ $g(x)$ is decreasing on $\mathbb{R}$

$\longrightarrow$ $g'(x)$ exist

Another solution: Assume $g(x_1) = g(x_2)$

$$\frac{x_1+1}{x_1} = \frac{x_2+1}{x_2}$$

$$x_1 x_2 + x_2 = x_1 x_2 + x_1$$

$$\boxed{x_2 = x_1}$$

$h(x) = x + \frac{1}{x}$

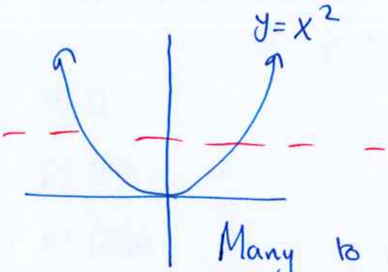
$h(x)$ is not 1-1 because

$x_1 = 2 \longrightarrow f(2) = 2 + \frac{1}{2} = \frac{5}{2}$

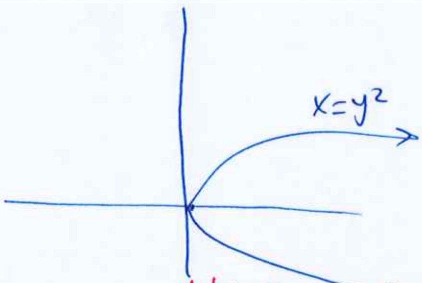
$x_2 = \frac{1}{5} \longrightarrow f\left(\frac{1}{2}\right) = \frac{1}{2} + 2 = \frac{5}{2}$

$g(x)$ not 1-1 fon. because ; $x_1 \neq x_2$

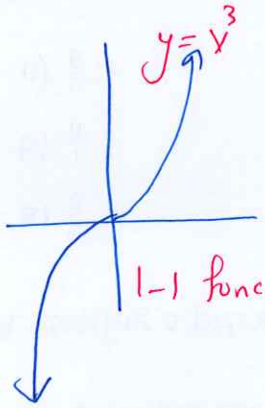
but $f(x_1) = f(x_2)$



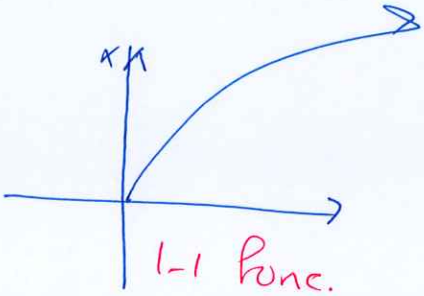
Many to one function



Not Function



1-1 func.



1-1 Func.

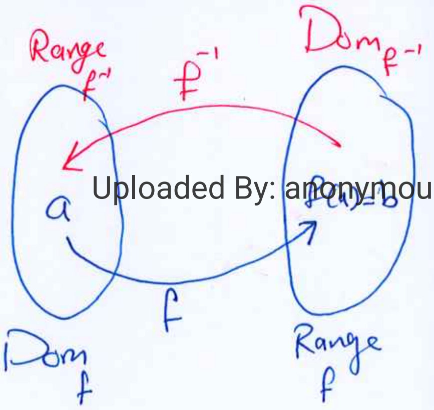
Def Inverse function

Suppose that f is 1-1 function

$\rightarrow$ The inverse function $f^{-1}(x)$ of $f(x)$ defined by $f^{-1}(b)=a$ if $f(a)=b$

* The Domain of $f^{-1}(x)$ = Range of $f(x)$

* The Range of $f^{-1}(x)$ = Domain of $f(x)$



Properties:-

① If f^{-1} is the inverse fn., then $f^{-1} \neq \frac{1}{f}$

② $f(a)=b \longrightarrow f^{-1}(f(a)) = f^{-1}(b)$

$$f^{-1}\circ f(a) = f^{-1}(b)$$

$$a = f^{-1}(b)$$

③ $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$, if $x \in Dom_f$

④ $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$, if $x \in Dom_{f^{-1}}$

⑤ $f(x)$ has an inverse iff f is one to one fn.

Exp: Let $f(x) = 8x^3 + 3$, Show that

$\boxed{1}$ $f^{-1}(x)$ exist

Show that $\boxed{2}$ $f^{-1}(x) = \frac{1}{2}\sqrt[3]{x-3}$

Sol $\boxed{1}$ $f'(x) = 24x^2 > 0$

$f(x)$ Inc on $\mathbb{R}$

f is 1-1 fn.

$f^{-1}(x)$ exist

Sol $\boxed{2}$ $f \circ f^{-1}(x) = f(f^{-1}(x))$

$= 8\left(\frac{1}{8}(x-3)\right) + 3$

$= x$

$f^{-1} \circ f(x) = f^{-1}(f(x))$

$= \frac{1}{2}\sqrt[3]{8x^3}$

$= \frac{1}{2}(2x) = x$

Exp. find the value of a if $f^{-1}(a) = 2$

$f(x) = x^3 + 2x - 3$

Sol $f(f^{-1}(a)) = f(2)$

$a = f(2)$

$a = 9$

How to find $f^{-1}(x)$ from the function $y = f(x)$?

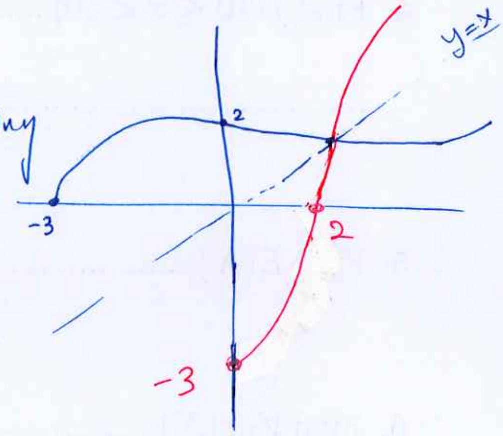
$\boxed{1}$ Solve the equation $y = f(x)$ to get x as a function of y.

$\boxed{2}$ Interchange x and y, the resulting equation is

$$y = f^{-1}(x)$$

Note: The graph of $f(x)$ and $f^{-1}(x)$ are symmetric about $y = x$

* If (a,b) is a point on the graph $y = f(x)$ then the point (b,a) is a point on the graph $y = f^{-1}(x)$



Exp: Determine whether or not the following functions are 1-1

If so, find it's inverse.

$\boxed{1}$ $f(x) = 3x^2 + 1$

$f(2) = 13$

$f(-2) = 13$

So f is not 1-1 because $f(2) = f(-2)$

but $2 \neq -2$

f^{-1} doesn't exist

(2) $f(x) = (2-x)^3$

$f'(x) = -3(2-x)^2 \leq 0 \quad \forall x$, f is decreasing $\longrightarrow$ f is 1-1

$y = (2-x)^3$

$\sqrt[3]{y} = 2-x$

$x = 2 - \sqrt[3]{y}$

$y = 2 - \sqrt[3]{x}$

$f^{-1}(x) = 2 - \sqrt[3]{x}$

(3) $f(x) = x^2 - 2x \quad , \quad x \leq 1$

$Dom_f = x \leq 1$

$Range_f = y \geq -1$

$f'(x) = 2x - 2 \leq 0 \quad \forall x \leq 1$

$f(x)$ dec. for all $x \leq 1$

$y+1 = x^2 - 2x + 1 \qquad x \leq 1$

$y+1 = (x-1)^2 \qquad x \leq 1$

$\sqrt{y+1} = |x-1| \qquad x \leq 1$

$x-1 = -\sqrt{y+1} \qquad x \leq 1$

$x = 1 - \sqrt{y+1} \qquad x \leq 1$

$y = 1 - \sqrt{x+1} \qquad \boxed{y \leq 1} \longrightarrow$

$1 - \sqrt{x+1} \leq 1$

$\sqrt{x+1} \geq 0$

$x \geq -1$

$\boxed{f^{-1}(x) = 1 - \sqrt{x+1} \quad , \quad \overset{Dom_{f^{-1}} = Range_f}{x \geq -1}}$

Theorem: If Dom_f is the interval I, $f^{-1}(x)$ exist and $f'(x) \neq 0$ on the interval I then $f^{-1}(x)$ is differentiable at every point in the domain. The value of $(f^{-1})'$ at a point $b = f(a)$ in the domain of f^{-1} is the following formula:

$$\left(\frac{df^{-1}}{dx}\right)\Bigg|_{x=b=f(a)} = \frac{1}{\left(\frac{df}{dx}\right)\Big|_{x=a=f^{-1}(b)}}$$

Exp.

let $f(x) = \dfrac{2x-1}{x+2}$, $x \neq -2$

① Show that $f^{-1}(x)$ exist

② find $\dfrac{df^{-1}}{dx}$ at $x=-3$

Sol.

[1] $f'(x) = \dfrac{(x+2)(2) - (2x-1)}{(x+2)^2} = \dfrac{5}{(x+2)^2} > 0$ $f(x)$ is 1-1 fn. $f^{-1}(x)$ exist

[2] $\dfrac{df^{-1}}{dx}\Big|_{x=-3} = \dfrac{1}{\frac{df}{dx}\Big|_{x=-1}} = \dfrac{1}{5}$

$\rightarrow \dfrac{2x-1}{x+2} = -3$

$2x-1 = -3x-6$

$5x = -5 \rightarrow \boxed{x=-1}$

$\dfrac{df}{dx} = \dfrac{5}{(x+2)^2}$

If g is the inverse function of $f(x)$

and

x	4	-1	4	1
$f(x)$	5	1	-3	0
$f'(x)$	2/3	-1/5	-2	-1/5

$g(x) = f^{-1}(x)$

Find $g'(5)$

$$g'(5) = \left.\frac{dg}{dx}\right|_{x=5} = \left.\frac{df^{-1}}{dx}\right|_{x=5} = \frac{1}{\left.\frac{df}{dx}\right|_{x=4}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

Find $g'(1)$

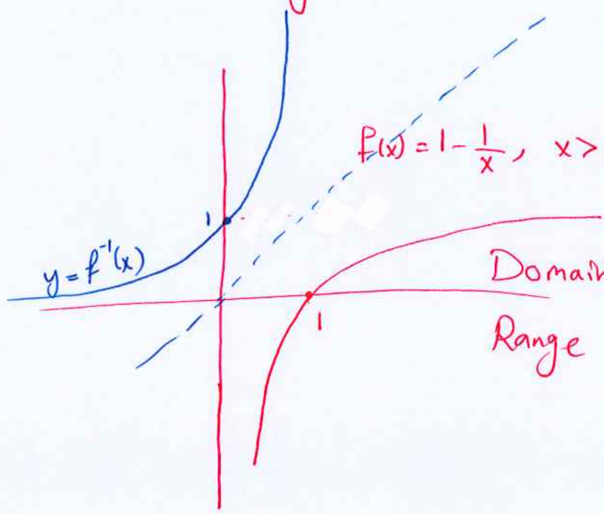
$$g'(1) = \left.\frac{dg}{dx}\right|_{x=1} = \left.\frac{df^{-1}}{dx}\right|_{x=1} = \frac{1}{\left.\frac{df}{dx}\right|_{x=-1}} = \frac{1}{\frac{-1}{5}} = -5$$

Find $g'(-3) = \frac{-1}{2}$

$g'(0) = -5$

Outline Exercises :—

Q12. Sketch the graph of $f^{-1}(x)$, and identify the domain and Range of $f^{-1}(x)$.



Domain $x > 0$

Range $y < 1$

$\text{Domain}_{f^{-1}(x)} = x < 1$

$\text{Range}_{f^{-1}(x)} = y > 0$

$\boxed{31}$ let $f(x) = \dfrac{x+3}{x-2}$, $x \neq 2$

Show that $f^{-1}(x)$ exist $f(x)$ is 1-1 fn. (Decreasing fn.)

because $f'(x) = \dfrac{-5}{(x-2)^2} < 0$

find $f^{-1}(x)$

$$y = \frac{x+3}{x-2}$$

$$xy - 2y = x + 3$$

$$x(y-1) = 2y + 3$$

$$x = \frac{2y+3}{y-1}$$

$$y = \frac{2x+3}{x-1}$$

$\text{Dom}_{f^{-1}}$ $x \neq 1$

$$f^{-1}(x) = \frac{2x+3}{x-1}$$

$\text{Range}_{f^{-1}}$ $y \neq 2$

Check $f^{-1}(x) = \frac{2x+3}{x-1}$ is the inverse fn. of $f(x) = \frac{x+3}{x-2}$

$$f\left(f^{-1}(x)\right) = \frac{\frac{2x+3}{x-1} + 3}{\frac{2x+3}{x-1} - 2} = \frac{2x+3+3x-3}{2x+3-2x+2} = \frac{5x}{5} = x$$

$$f^{-1}\left(f(x)\right) = \frac{2\left(\frac{x+3}{x-2}\right) + 3}{\left(\frac{x+3}{x-2}\right) - 1} = \frac{2x+6+3x-6}{x+3-x+2} = \frac{5x}{5} = x$$

$\boxed{41}$ let $f(x) = x^3 - 3x^2 - 1$, $x \geq 2$

find $\left.\frac{df^{-1}}{dx}\right|_{x=-1} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(-1)=3}}$

$$= \frac{1}{9}$$

$x^3 - 3x^2 - 1 = -1$

$x^2(x-3) = 0$

$\boxed{x=0}$ reject OR $\boxed{x=3}$ accept

$f'(x) = 3x^2 - 6x$