Assignment 8 (MATH 215, Q1)

1. Evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the *flux* of $\mathbf{F}$ across S.

(a) $\mathbf{F}(x,y,z) = xy\,\mathbf{i}+yz\,\mathbf{j}+zx\,\mathbf{k}$, S is the part of the paraboloid $z = 4-x^2-y^2$ that lies above the square $-1 \le x \le 1$, $-1 \le y \le 1$, and has the upward orientation.

Solution. The surface S can be represented by the vector form

$$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (4-x^2-y^2)\,\mathbf{k}, \quad -1 \le x \le 1, -1 \le y \le 1.$$

It follows that $\mathbf{r}_x = \mathbf{i} - 2x\,\mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - 2y\,\mathbf{k}$. Consequently,

$$\mathbf{r}_x \times \mathbf{r}_y = 2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}.$$

Hence, with $Q := \{(x,y) : -1 \le x \le 1, -1 \le y \le 1\}$ we obtain

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_Q \mathbf{F}\cdot(\mathbf{r}_x \times \mathbf{r}_y)\,dA \\
&= \int_{-1}^{1}\int_{-1}^{1} \left[2x^2y + 2y^2(4-x^2-y^2) + x(4-x^2-y^2)\right] dx\,dy \\
&= \int_{-1}^{1} \left(\frac{4}{3}\,y + 16y^2 - \frac{4}{3}\,y^2 - 4y^4\right) dy \\
&= \left[\frac{2}{3}\,y^2 + \frac{44}{3}\,\frac{y^3}{3} - 4\,\frac{y^5}{5}\right]_{-1}^{1} = \frac{368}{45}.
\end{aligned}$$

(b) $\mathbf{F}(x,y,z) = -y\,\mathbf{i}+x\,\mathbf{j}+3z\,\mathbf{k}$, S is the hemisphere $z = \sqrt{16-x^2-y^2}$ with upward orientation.

Solution. The surface S has parametric equations

$$\mathbf{r}(\phi,\theta) = x(\phi,\theta)\,\mathbf{i} + y(\phi,\theta)\,\mathbf{j} + z(\phi,\theta)\,\mathbf{k} = 4\sin\phi\cos\theta\,\mathbf{i} + 4\sin\phi\sin\theta\,\mathbf{j} + 4\cos\phi\,\mathbf{k},$$

where $0 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$. We have

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = 16\sin\phi(\sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}).$$

Moreover,

$$\mathbf{F} = -4\sin\phi\sin\theta\,\mathbf{i} + 4\sin\phi\cos\theta\,\mathbf{j} + 12\cos\phi\,\mathbf{k}.$$

Consequently,

$$\mathbf{F}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) = 192\sin\phi\cos^2\phi.$$

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Therefore, with $Q := \{(\phi,\theta) : 0 \le \phi \le \pi/2, 0 \le \theta \le 2\pi\}$ we obtain

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_Q \mathbf{F}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta)\,dA \\ &= \int_0^{2\pi}\int_0^{\pi/2} 192\sin\phi\cos^2\phi\,d\phi\,d\theta \\ &= 2\pi\cdot 192\Big[-\frac{\cos^3\phi}{3}\Big]_0^{\pi/2} = 128\pi.\end{aligned}$$

2. Let S be the conical surface $z = \sqrt{x^2+y^2}$, $z \le 2$.

   (a) Find the center of mass of S, if it has constant density.

Solution. The surface has parametric equations

$$x = z\cos t, \quad y = z\sin t, \quad z = z, \qquad (t,z)\in Q,$$

where $Q := \{(t,z) : 0 \le t \le 2\pi, 0 \le z \le 2\}$. Let $\mathbf{r}(t,z) := z\cos t\,\mathbf{i} + z\sin t\,\mathbf{j} + z\,\mathbf{k}$. Then

$$\mathbf{r}_t \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -z\sin t & z\cos t & 0 \\ \cos t & \sin t & 1 \end{vmatrix} = z\cos t\,\mathbf{i} + z\sin t\,\mathbf{j} - z\,\mathbf{k}.$$

Note that $\mathbf{r}_t \times \mathbf{r}_z$ gives the *downward* orientation. Moreover,

$$|\mathbf{r}_t \times \mathbf{r}_z| = \sqrt{(z\cos t)^2 + (z\sin t)^2 + (-z)^2} = \sqrt{2}\,z.$$

Suppose the density is k. Then $M = \iint_S k\,dS$ and $M_{xy} = \iint_S kz\,dS$. The center of mass is $(0,0,\bar{z})$, where $\bar{z} = M_{xy}/M$. We have

$$M = \iint_S k\,dS = k\iint_Q |\mathbf{r}_t \times \mathbf{r}_z|\,dA = k\int_0^{2\pi}\int_0^2 \sqrt{2}\,z\,dz\,dt = 4\sqrt{2}\,\pi k.$$

Moreover,

$$M_{xy} = \iint_S kz\,dS = k\iint_Q z|\mathbf{r}_t \times \mathbf{r}_z|\,dA = k\int_0^{2\pi}\int_0^2 z\sqrt{2}\,z\,dz\,dt = \frac{16\sqrt{2}\,\pi k}{3}.$$

Therefore, the center of mass is $(0,0,4/3)$.

   (b) A fluid has density 15 and velocity $\mathbf{v} = x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k}$. Find the rate of flow **downward** through S.

Solution. We have

$$\mathbf{F} = \rho\mathbf{v} = 15(x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k}) = 15(z\cos t\,\mathbf{i} + z\sin t\,\mathbf{j} + \mathbf{k}).$$

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Consequently,

$$\mathbf{F} \cdot (\mathbf{r}_t \times \mathbf{r}_z) = 15(z^2 \cos^2 t + z^2 \sin^2 t - z) = 15(z^2 - z).$$

Hence, the rate of flow downward through S is

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_Q \mathbf{F} \cdot (\mathbf{r}_t \times \mathbf{r}_z)\, dA \\
&= 15 \int_0^{2\pi} \int_0^2 (z^2 - z)\, dz\, dt \\
&= 30\pi \Big[\frac{z^3}{3} - \frac{z^2}{2} \Big]_0^2 = 20\pi.
\end{aligned}$$

3. Use the divergence theorem to find $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.
   (a) $\mathbf{F}(x, y, z) = x^3\, \mathbf{i} + 2xz^2\, \mathbf{j} + 3y^2 z\, \mathbf{k}$; S is the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.

Solution. The divergence of $\mathbf{F}$ is

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(2xz^2) + \frac{\partial}{\partial z}(3y^2 z) = 3x^2 + 3y^2.$$

Let E be the region $\{(x, y, z) : 0 \le z \le 4 - x^2 - y^2\}$. By the divergence theorem, we have

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E (3x^2 + 3y^2)\, dV \\
&= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} 3r^2 r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^2 (4 - r^2) 3r^3\, dr\, d\theta \\
&= 2\pi \int_0^2 (12r^3 - 3r^5)\, dr = 32\pi.
\end{aligned}$$

   (b) $\mathbf{F}(x, y, z) = (x^2 + \sin(yz))\, \mathbf{i} + (y - xe^{-z})\, \mathbf{j} + z^2\, \mathbf{k}$; S is the surface of the region bounded by the cylinder $x^2 + y^2 = 4$ and the planes $x + z = 2$ and $z = 0$.

Solution. The divergence of $\mathbf{F}$ is

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^2 + \sin(yz)) + \frac{\partial}{\partial y}(y - xe^{-z}) + \frac{\partial}{\partial z}(z^2) = 2x + 1 + 2z.$$

Let E be the region $\{(x, y, z) : 0 \le z \le 2 - x, x^2 + y^2 \le 4\}$. By the divergence theorem, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E (2x + 1 + 2z)\, dV.$$

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Converting to cylindrical coordinates, we obtain

$$\iiint_E (2x+1+2z)\,dV = \int_0^{2\pi}\int_0^2\int_0^{2-r\cos\theta} (2r\cos\theta+1+2z)r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 (-r^2\cos^2\theta - r\cos\theta+6)r\,dr\,d\theta = \int_0^{2\pi} (-4\cos^2\theta - 8\cos\theta/3+12)\,d\theta$$
$$= 20\pi.$$

4. Use the divergence theorem to calculate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where

$$\mathbf{F}(x,y,z) = z^2x\,\mathbf{i} + (y^3/3+\tan z)\,\mathbf{j} + (x^2z+y^2)\,\mathbf{k}$$

and S is the top half of the sphere $x^2+y^2+z^2=1$ oriented upward. (Hint: Note that S is not a closed surface. Let S_1 be the disk $\{(x,y,0) : x^2+y^2\le 1\}$ oriented downward and let $S_2 = S\cup S_1$. The surface integral over S can be derived from integrals over S_1 and S_2.)

Solution. Let E be the semi-ball $\{(x,y,z) : x^2+y^2+z^2\le 1, z\ge 0\}$. Then S_2 is the boundary of E. Hence, the divergence theorem applies to the surface integral $\iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS$:

$$\iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{F}\,dV.$$

The divergence of $\mathbf{F}$ is given by

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(z^2x) + \frac{\partial}{\partial y}(y^3/3+\tan z) + \frac{\partial}{\partial z}(x^2z+y^2) = z^2+y^2+x^2.$$

Hence, we obtain

$$\iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E (x^2+y^2+z^2)\,dV.$$

The triple integral can be calculated by using the spherical coordinates:

$$\iiint_E (x^2+y^2+z^2)\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = \frac{2\pi}{5}.$$

For the surface integral $\iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS$ we note that $\mathbf{n}=-\mathbf{k}$ and $z=0$ on S_1. It follows that $\mathbf{F}\cdot\mathbf{n} = -y^2$. Consequently,

$$\iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{x^2+y^2\le 1} -y^2\,dA = \int_0^{2\pi}\int_0^1 -(r^2\sin^2\theta)r\,dr\,d\theta = -\frac{\pi}{4}.$$

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Therefore,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS - \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS = \frac{2\pi}{5} - \frac{-\pi}{4} = \frac{13\pi}{20}.$$

5. Use Stokes' theorem to evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.
   (a) $\mathbf{F}(x,y,z) = z^2\,\mathbf{i} + y^2\,\mathbf{j} + xy\,\mathbf{k}$, C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,2)$.

Solution. The curl of $\mathbf{F}$ is

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & xy \end{vmatrix} = x\,\mathbf{i} + (2z-y)\,\mathbf{j}.$$

The plane that passes through the points $(1,0,0)$, $(0,1,0)$, and $(0,0,2)$ has an equation $z = 2-2x-2y$. Hence, $\mathbf{r}_x \times \mathbf{r}_y = 2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$. By Stokes' theorem we obtain

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_Q \operatorname{curl}\mathbf{F}\cdot(\mathbf{r}_x\times\mathbf{r}_y)\,dA,$$

where

$$Q = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1-x\}.$$

Consequently,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1\int_0^{1-x} \big(2x + 4(2-2x-2y) - 2y\big)\,dy\,dx \\ &= \int_0^1\int_0^{1-x} (-6x-10y+8)\,dy\,dx = \int_0^1 (x^2-4x+3)\,dx = 4/3.\end{aligned}$$

   (b) $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2+y^2)\,\mathbf{k}$, C is the boundary of the part of the paraboloid $z = 1-x^2-y^2$ in the first octant.

Solution. The curl of $\mathbf{F}$ is

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & x^2+y^2 \end{vmatrix} = 2y\,\mathbf{i} - 2x\,\mathbf{j}.$$

The surface S can be represented as $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + (1-x^2-y^2)\,\mathbf{k}$, $x \ge 0$, $y \ge 0$, $x^2+y^2 \le 1$. It follows that

$$\mathbf{r}_x \times \mathbf{r}_y = 2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}.$$

Consequently,

$$\operatorname{curl}\mathbf{F}\cdot(\mathbf{r}_x\times\mathbf{r}_y) = 4xy - 4xy = 0.$$

Therefore, by Stokes's theorem, we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\operatorname{curl}\mathbf{F}\cdot\mathbf{n})\,dS = 0.$$

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