Discussion $\boxed{7.4}$

(30) Growth of Bacteria

A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours there are 10,000 bacteria. At the end of 5 hours there are 40,000 bacteria. How many bacteria were present initially?

- Assume $B(t)$ is the Bacteria size at time t
- $B(t) = B_0 e^{Kt}$, $K > 0$ is the growth rate
- we need to find B_0 given that $B(3) = 10{,}000$, $B(5) = 40{,}000$
- $B(3) = B_0 e^{3K}$ and $B(5) = B_0 e^{5K}$

$10{,}000 = B_0 e^{3K}$ ... (1) $\qquad 40{,}000 = B_0 e^{5K}$ ... (2)

$$\frac{40{,}000}{10{,}000} = \frac{\cancel{B_0} e^{5K}}{\cancel{B_0} e^{3K}} \iff 4 = e^{2K}$$

$$\ln 4 = \ln e^{2K}$$

$$\cancel{2} \ln 2 = \cancel{2} K$$

$$\boxed{K = \ln 2}$$

Using for example (1) $\Rightarrow$ $10{,}000 = B_0 e^{3\ln 2}$

$$10{,}000 = B_0 e^{\ln 2^3}$$

$$10{,}000 = B_0 e^{\ln 8}$$

$$10{,}000 = B_0 (8) \Rightarrow B_0 = \frac{10{,}000}{8} = 1250$$

Population Growth : $y(t) = y_0 e^{kt}$

Exp 3 $y(t)$: The number of infected people by a disease at time t (years)

Assume the number of people cured (تماثلوا للشفاء) is proportional to the number y.

Suppose in one year the number is reduced by 20% . If there are 10,000 case today, how many years will it take to reduce the number to 1000 case ?

• $y_0 = 10{,}000 \Rightarrow y(t) = y_0 e^{kt} = 10{,}000\, e^{kt}$

• To find $k \Rightarrow y(1) = 80\%\, y_0$

$$10{,}000\, e^{k} = 0.8\,(10{,}000)$$
$$e^{k} = 0.8$$
$$\boxed{k = \ln(0.8)}$$

we need to find time t^* such that $y(t^*) = 1000$

$$10{,}000\, e^{kt^*} = 1000$$

$$10\, e^{kt^*} = 1 \Rightarrow e^{kt^*} = 0.1$$

$$kt^* = \ln(0.1)$$

$$t^* = \frac{\ln(0.1)}{k} = \frac{\ln(0.1)}{\ln(0.8)} \approx 10.32 \text{ years}$$

Radioactivity $y(t) = y_0 e^{-kt}$

Exp 4 Carbon-14 has half-life time of 5700 years. Find age of a sample in which 10 % of the radioactive material has decayed.

$$k = \frac{\ln 2}{\tau} = \frac{\ln 2}{5700}$$

we need to find time t^* such that $y(t^*) = 90\%\, y_0$

$$\cancel{y_0} e^{-kt^*} = 0.9\, \cancel{y_0}$$

$$e^{-kt^*} = 0.9$$

$$-kt^* = \ln(0.9)$$

$$t^* = -\frac{\ln(0.9)}{k}$$

$$= \frac{-\ln(0.9)}{\frac{\ln 2}{5700}}$$

$$= -\frac{5700 \ln(0.9)}{\ln 2}$$

$$\simeq 866 \text{ years}$$

Exp (Radioactivity)

The half-life of a radioactive material is ln8 years. If 10 gm of this material is released into atmosphere, how many years will it take for 80% of the material to decay.

Decay Equation: $y(t) = y_0 e^{-kt} = 10 e^{-kt}$

$$k = \frac{\ln 2}{\tau} = \frac{\ln 2}{\ln 8} = \frac{\ln 2}{\ln 2^3} = \frac{\ln 2}{3 \ln 2} = \frac{1}{3}$$

We need to find time t^* such that

$$y(t^*) = 20\% \, y_0$$

$$10 e^{-kt^*} = 0.2 \, (10)$$

$$e^{-kt^*} = 0.2$$

$$-k t^* = \ln(0.2)$$

$$t^* = -\frac{\ln(0.2)}{k}$$

$$= -\frac{\ln(0.2)}{\frac{1}{3}}$$

$$= -3 \ln(0.2)$$

$$= 3 \ln 5 \text{ years}$$

$$\ln(0.2) = \ln \frac{2}{10} = \ln \frac{1}{5} = \ln 1 - \ln 5 = -\ln 5$$