Exercises :

Q1 : let $f(x) = \begin{cases} 4x_1x_2 & , 0<x_1<1 \;, 0<x_2<1 \\ 0 & , \text{elsewhere} \end{cases}$ be the p.d.f of x_1 and x_2.

Find :

1. $Pr(0<x_1<\frac{1}{2} , \frac{1}{4}<x_2<1) = \int_{1/4}^{1}\int_{0}^{1/2} 4x_1x_2 \; dx_1 dx_2$

$$= \int_{1/4}^{1} \cancel{4}^{2} x_2 \frac{x_1^2}{\cancel{2}}\Big|_0^{1/2} dx_2$$

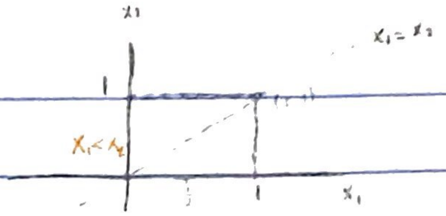
$$= \int_{1/4}^{1} 2x_2\left(\frac{1}{4}\right) dx_2$$

$$= \frac{1}{2}\left[\frac{x_2^2}{2}\Big|_{1/4}^{1}\right] = \frac{1}{2}\left[\frac{1}{2} - \frac{1/16}{2}\right]$$

$$= \frac{1}{2}\left[\frac{16\times 1}{16\times 2} - \frac{1}{32}\right]$$

$$= \frac{15}{64}.$$

2. $Pr(x_1 = x_2) =$



3. $\Pr(X_1 < X_2) = \int_0^1 \int_0^{x_2} 4x_1x_2 \, dx_1 \, dx_2$

$= \int_0^1 4\frac{x_1^2}{2} x_2 \Big|_0^{x_2} dx_2$

$= \int_0^1 2x_2^3 \, dx_2 = \frac{2x_2^4}{4}\Big|_0^1 = \frac{1}{2}$

---

4. $\Pr(X_1 \le X_2) = \Pr(X_1 < X_2) = \frac{1}{2}$.

Q3: let $F(x,y)$ be the distribution function of X and Y. Show that $\Pr(a < X \le b,\ c < Y \le d) = F(b,d) - F(b,c) - F(a,d) + F(a,c)$ for all real constants $a<b$, $c<d$.

$\Pr(a < X \le b,\ c < Y \le d) = \Pr(X \le b,\ c < Y \le d) - \Pr(X \le a,\ c < Y \le d)$

$= \Pr(X \le b, Y \le d) - \Pr(X \le b, Y \le c) - \Pr(X \le a, Y \le d) + \Pr(X \le a, Y \le c)$

$= F(b,d) - F(b,c) - F(a,d) + F(a,c)$.

Q5:

Q9: let the R.V's X_1 and X_2 have the joint p.d.f, described as follows.

(x_1, x_2)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$f(x_1, x_2)$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$

and $f(x_1, x_2)$ is equal to zero elsewhere.

a. write these probabilites in a rectangular array, recording each marginal p.d.f.

$X_2 \backslash X_1$	0	1	$f_2(x_2)$
0	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{4}{12}$
1	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{5}{12}$
2	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{3}{12}$
$f_1(x_1)$	$\frac{7}{12}$	$\frac{5}{12}$	1

b. What is $Pr(X_1 + X_2 = 1)$?

$\rightarrow A = \{(x_1, x_2) : x_1 + x_2 = 1\} \subseteq \mathcal{A}$

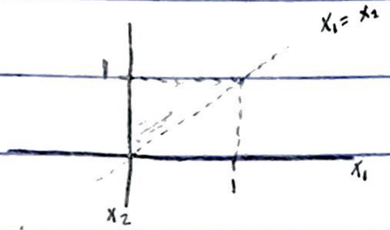
$A = \{(0,1), (1,0)\}$

$\Rightarrow Pr(X_1 + X_2 = 1) = f(0,1) + f(1,0)$

$= \frac{3}{12} + \frac{2}{12} = \frac{5}{12}$.

Q6: let X_1 and X_2 have the joint p.d.f $f(x_1, x_2) = \begin{cases} 15x_1^2 x_2 & , 0 < x_1 < x_2 < 1 \\ 0 & , \text{elsewhere.} \end{cases}$

Find each marginal p.d.f and compute $Pr(X_1 + X_2 \leq 1)$.



$$\Rightarrow f_1(x_1) = \int_{x_1}^{1} (15x_1^2 x_2)\, dx_2 = 15x_1^2 \frac{x_2^2}{2} \Big|_{x_1}^{1} = 15x_1^2 \left(\frac{1}{2}\right) - 15x_1^2 \frac{x_1^2}{2} \quad , 0 < x_1 < 1$$

$$= \frac{15x_1^2 (1 - x_1^2)}{2} \quad , 0 < x_1 < 1$$

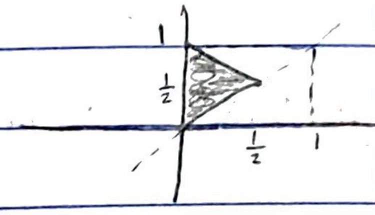
$$\Rightarrow f_1(x_1) = \begin{cases} \frac{15x_1^2 (1 - x_1^2)}{2} & , 0 < x_1 < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

$$\Rightarrow f_2(x_2) = \int_{0}^{x_2} (15x_1^2 x_2)\, dx_1 = 15 \frac{x_1^3}{3} x_2 \Big|_{0}^{x_2} = 5x_2^3 x_2 = 5x_2^4 \quad , 0 < x_2 < 1$$

$$\Rightarrow f_2(x_2) = \begin{cases} 5x_2^4 & , 0 < x_2 < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

$\rightarrow$

$$\Pr(X_1 + X_2 \leq 1) = \int_{x_1}^{1-x_1}\int_{0}^{\frac{1}{2}} 15\, x_1^2\, x_2 \; dx_1\, dx_2$$



$$= \int_{0}^{\frac{1}{2}}\int_{x_1}^{1-x_1} 15\, x_1^2\, x_2 \; dx_2\, dx_1$$

$$= \int_{0}^{\frac{1}{2}} 15 x_1^2 \frac{x_2^2}{2} \Big|_{x_1}^{1-x_1} dx_1$$

$$= \int_{0}^{\frac{1}{2}} 15 x_1^2 \frac{(1-x_1)^2}{2} - 15 x_1^2 \frac{x_1^2}{2} \; dx_1$$

$$= \int_{0}^{\frac{1}{2}} 15 x_1^2 \left(\frac{1}{2} - \frac{2x_1}{2} + \frac{x_1^2}{2}\right) - \frac{15 x_1^4}{2} \; dx_1$$

$$= \int_{0}^{\frac{1}{2}} \frac{15 x_1^2}{2} - 15 x_1^3 + \cancel{\frac{15 x_1^4}{2}} - \cancel{\frac{15 x_1^4}{2}} \; dx_1$$

$$= \frac{\overset{5}{\cancel{15}}}{2} \frac{x_1^3}{\cancel{3}} - \frac{15}{4} x_1^4 \Big|_{0}^{\frac{1}{2}}$$

$$= \frac{5}{2} x_1^3 - \frac{15}{4} x^4 \Big|_{0}^{\frac{1}{2}}$$

$$= \frac{5}{2}\left(\frac{1}{8}\right) - \frac{15}{4}\left(\frac{1}{16}\right)$$

$$= \frac{4 \times 5}{4 \times 16} - \frac{15}{64}$$

$$= \frac{20 - 15}{64} = \frac{5}{64}$$