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Ch. 3

Q.1:-

* Z_{12} : {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

$|Z_{12}| = 12$

$|0| = 1$, $|1| = |5| = |7| = |11| = 12$, $|2| = |10| = 6$

$|3| = |9| = 4$, $|4| = |8| = 3$, $|6| = 2$

* $U(10)$:- {1, 3, 7, 9}

$|U(10)| = 4$

$|1| = 1$, $|3| = |7| = 4$, $|9| = 2$

* $U(20)$: {1, 3, 7, 9, 11, 13, 17, 19}.

$|U(20)| = 8$

$|1| = 1$, $|3| = |7| = |13| = |17| = 4$, $|9| = |11| = |19| = 2$

* D_4 : { ρ_0, ρ_{90}, ρ_{180}, ρ_{270}, V, H, D_1, D_2 }

$|D_4| = 8$

$|\rho_0| = 1$, $|\rho_{90}| = |\rho_{270}| = 4$

$|\rho_{180}| = |V| = |H| = |D_1| = |D_2| = 2$

* In each case, notice that the order of the element divides the order of the group.

Q.2:- $\langle \frac{1}{2} \rangle = \{ \ldots, -\frac{3}{2}, -1, 0, 1, \frac{3}{2}, \ldots \}$
$= \{ n(\frac{1}{2}) ; n \in Z \}$, $\Rightarrow$ In Q.

$\langle \frac{1}{2} \rangle = \{ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \}$ $\Rightarrow$ In Q^*
$= \{ (\frac{1}{2})^n ; n \in Z \}$

Q.4:- let $|a| = n$, $|a^{-1}| = m$, $m, n > 0$:-
$\Rightarrow a^n = e$, $(a^{-1})^m = (a^m)^{-1} = (e)^{-1} = e$

$\Rightarrow e = e.e = (a^n)(a^m)^{-1} = a^{n-m} = |a| = n-m$ -X
by ~~...~~ contradiction.

Q.5:- Z_{30} :- $\{2, 28\} \Rightarrow 2+28=0$, $2^{-1}=28$, $28^{-1}=2$
$\{8, 22\} \Rightarrow 8+22=0$, $8^{-1}=22$, $22^{-1}=8$

$U(15)$:- $\{2, 8\} \Rightarrow 2.8=1$, $2^{-1}=8$, $8^{-1}=2$
$\{7, 13\} \Rightarrow 7.13=1$, $7^{-1}=13$, $13^{-1}=7$

Q.6:- $a^6 = e$ ✓ $\boxed{6}$ ✓ / $|a|$ = divides 6 $\rightarrow$ 1, 2, 3, 6
$a^6 = e \Rightarrow a^5 a = e \Rightarrow a = e$ X $\boxed{5}$ X
$a^6 = e \Rightarrow a^4 a^2 = e \Rightarrow a^2 = e$ X $\boxed{4}$ X
$a^6 = e \Rightarrow a^3 a^3 = e \Rightarrow e = e$ $\boxed{3}$ ✓
$a^6 = e \Rightarrow a^2 a^2 a^2 = e \Rightarrow e = e$ $\boxed{2}$ ✓
$a^6 = e \Rightarrow aaaaaa = e \Rightarrow e = e$ $\boxed{1}$ ✓

$\Rightarrow$ Possibilities : $\{1, 2, 3, 6\}$. $\{4, 5\}$ not possible.

Q.7:- By contradiction:-

suppose $m \neq n$ and $a^m = a^n$:-

$a^m a^{-n} = a^n a^{-n}$

$a^{m-n} = a^0$

$m-n = 0 \Rightarrow m = n$ ※

Q.8:- suppose $x^4 = e$:-

$e = (x^4)^2 \Rightarrow e = x^8 = x^6 \cdot x^2 \Rightarrow e = x^2$

suppose $x^5 = e$:-

$e = (x^5)^2 = x^{10} = x^6 \cdot x^4 \Rightarrow e = x^4$

So $|x| = 3$ or 6.

Q.9:- * If a has infinite order, $e, a, a^n, \ldots$ distinct and belongs to G.

* If $|a| = n \rightarrow a^i = a^j$, $0 < i < j < n$

$a^{j-i} = e \Rightarrow$ ※ $(i-j=0, i=j)$ ※

thus $e, a, a^2, \ldots, a^{n-1}$ are all distinct and belong to G

So G has at least n elements.

Q.10:- $U(14) = \{1, 3, 5, 9, 11, 13\}$

$\langle 3 \rangle = \{3, 3^2, 3^3, 3^4, 3^5, 3^6\} = \{3, 9, 13, 11, 5, 1\} = U(14)$ ✓

$\langle 5 \rangle = \{5, 5^2, 5^3, 5^4, 5^5, 5^6\} = \{5, 11, 13, 9, 3, 1\} = U(14)$ ✓

$\langle 11 \rangle = \{11, 9, 1\} \neq U(14)$!

Q.11:- $U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$

$|U(20)| = 8$

for $U(20) = \langle K \rangle$, for some K it must the case that $|K| = 8$.

but $1^1 = 1$, $3^4 = 1$, $7^4 = 1$, $9^2 = 1$, $11^2 = 1$, $13^4 = 1$, $17^4 = 1$

$19^2 = 1$ $\Rightarrow$ the max order of any element is $\underline{4}$.

Q.12:- suppose a, b are 2 elements of order 2.

then $\{a, b, e, ab\}$ closed and subgroup of order 4.

Q.14:- H contains 18, 30 and 40 :-

$18^{-1} = -18$, $30^{-1} = -30$ [subgroups are closed under Inverse]

[Closed] $-18 + 30 = 12$ and $-30 + 40 = \underline{10}$

$-10 + 12 = \boxed{2}$

since H contains 2, its must all integral multiples of 2 (all even).

H is exactly the subgroup generated by 2, $\underline{\langle 2 \rangle}$

Q.17:- $U_4(20) = \{1, 9, 13, 17\}$

$U_5(20) = \{1, 11\}$

$U_5(30) = \{1, 11\}$.

$\Rightarrow$ * $U_K(n)$ is closed $\Rightarrow (ab) \bmod K = (a \bmod K)(b \bmod K) = 1.1 = 1$

* H is not closed since $7 \in H$ but $7.7 = 9$ is not in H

$H \rightarrow$ not subgroup.

Q.18:- * $H \cap K \neq \phi$, since $e \in H \cap K$.

① let $x, y \in H \cap K$. then since H and K are subgroups
$\Rightarrow xy^{-1} \in H$ and $xy^{-1} \in K \Rightarrow xy^{-1} \in H \cap K$.

Q.19:- If $x \in Z(G)$ then $x \in C(a)$ for all a, so $x \in \bigcap_{a \in G} C(a)$.

If $x \in \bigcap_{a \in G} C(a)$ then $xa = ax$ for all a in G, so $x \in Z(G)$

Q.20:- suppose $x \in C(a) \Rightarrow xa = ax$
$a^{-1}(\cancel{x}a) = a^{-1}(ax) = x$
Thus, $(a^{-1}x)a = x$ and $a^{-1}x = xa^{-1}$, $x \in C(a^{-1})$.

Q.23:- a) $C(1) = C(5) = G$
$C(2) = C(6) = \{1, 2, 5, 6\}$
$C(3) = C(7) = \{1, 3, 5, 7\}$.
$C(4) = C(8) = \{1, 4, 5, 8\}$.

b) $Z(G) = \{1, 5\}$

c) $|1| = 1$, $|2| = |4| = |5| = |6| = |8| = 2$, $|3| = |7| = 4$
They divide the order of the group.

Q.28:- Yes, elements in the center commute with all elements.

Q.27:- NO, In D_4: $C(R_{180}) = D_4$.

Q.26:- $*$ $C(H) \neq \phi$ since $eh = he = e \in C(H)$.

① let $x, y \in C(H) \Rightarrow xh = hx$

$yh = hy$

$(xy)h = x(yh) = x(hy) = (xh)y = h(xy) \Rightarrow xy \in C(H)$.

② let $x \in C(H) \Rightarrow x^{-1}(xh = hx)x^{-1}$

$hx^{-1} = x^{-1}h \Rightarrow x^{-1} \in C(H)$

Q.32:- let A be the subset of even members of Z_n.

and B " " " " odd " " " .

If $x \in B$ then $x + A = \{x + a : a \in A\} \subseteq B$.

so $|A| \leq |B|$

and $x + B = \{x + b : b \in B\} \subseteq A$ so $|B| \leq |A|$

Q.36:- $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$|A| = 4$ $A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$A^4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}}_e$

$B = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, $B^2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$

$|B| = 3$ $B^3 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}}_e$

$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ------ $|AB| = \infty$

$*$ if two distinct elements have a finite order, their product still may have an infinite order.

Q.42 :- $U(15) = \{1, 2, 4, 7, 8, 11, 13, 14\}$.

$\langle 1 \rangle = \{1\}$, $\langle 2 \rangle = \{2, 4, 8, 1\}$, $\langle 4 \rangle = \{4, 1\}$

$\langle 7 \rangle = \{7, 4, 13, 1\}$, $\langle 8 \rangle = \{8, 4, 2, 1\} = \langle 2 \rangle$, $\langle 11 \rangle = \{11, 1\}$.

$\langle 13 \rangle = \{4, 7, 1, 13\} = \langle 7 \rangle$, $\langle 14 \rangle = \{14, 1\}$.

$\Rightarrow$ 6 cycle :- $\langle 1 \rangle, \langle 2 \rangle, \langle 4 \rangle, \langle 7 \rangle, \langle 11 \rangle, \langle 14 \rangle$.

Q.43 :- For every nonidentity element a of odd order a^{-1} is distinct from a and has the same order as a.
Thus nonidentity elements of odd order come in pairs
So there must be some element a of even order,
$|a| = 2m$ then $|a^m| = 2$

Q.52 :- ① $\det A = 2^m$ and $\det B = 2^n$
$\det(AB) = 2^{m+n} \in H$.
② $\det A^{-1} = 2^{-m} \in H$
H subgroup.

Q.54 :- let $f, g \in H$,
then $(f \cdot g^{-1})(2) = f(1)\, g^{-1}(2) = 1 \cdot 1 = 1$
the 2 can be replaced by any number.