Discrete Signals and Systems

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Sampling Theorem:

Let $X(f)$ be a band limited spectral representation on $[-f_{max}, f_{max}]$ of $x(t)$, the following is a pair of Fourier Transform:

$$x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} x(nT_0)\, \delta(t - nT_0) \leftrightarrow f_0 \sum_{n=-\infty}^{\infty} X(f - nf_0)$$

Proof: (by duality theorem):

Applying the Fourier transform convolution theorem:

$$\mathrm{F}[x(t) \cdot \textstyle\sum_{n=-\infty}^{\infty} \delta(t - nT_0)] = X(f) * F[\sum_{n=-\infty}^{\infty} \delta(t - nT_0)] = X(f) * f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0) =$$

$$f_0 \sum_{n=-\infty}^{\infty} X(f) * \delta(f - nf_0) = f_0 \sum_{n=-\infty}^{\infty} X(f - nf_0)$$

Thus,

$$x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \leftrightarrow f_0 \sum_{n=-\infty}^{\infty} X(f - nf_0)$$

That is the spectral representation of the sampled signal with an ideal uniform sampling scheme is a scaled repetition of the spectral representation of the original signal.

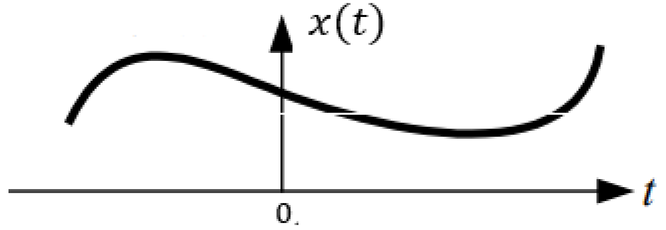
Nyquist Sampling Theorem:

Given a low-pass signal with maximum frequency f_{max} , The minimum ideal and uniform sampling frequency with which the signal can reconstructed is the Nyquist sampling frequency: $f_N = 2f_{max}$.

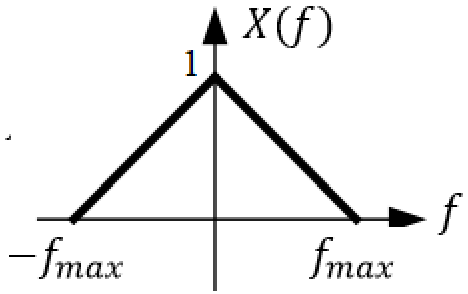
Proof:

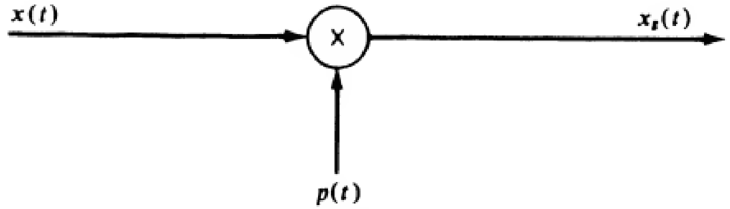
Consider the following signal, It is clear that to avoid spectral components overlap the following condition must be satisfied: $f_0 - f_{max} \geq f_{max} \rightarrow f_0 \geq 2f_{max} = f_N$ which proves the assert

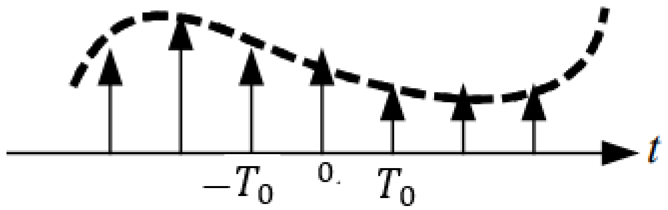
Proof:



$\leftrightarrow$







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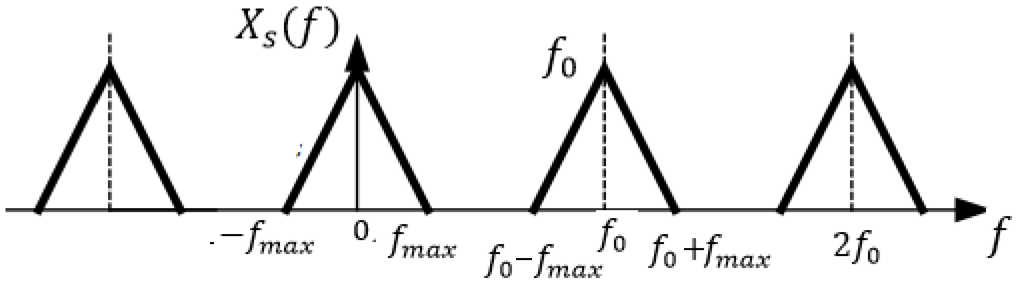
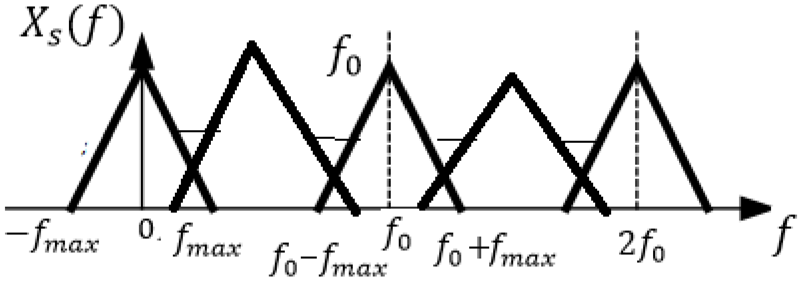


$$x_s(t) = x(t)\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_0)\,\delta(t-nT_0)$$

$$f_0\sum_{n=-\infty}^{\infty} X(f-nf_0)$$

Aliasing



Observation1:

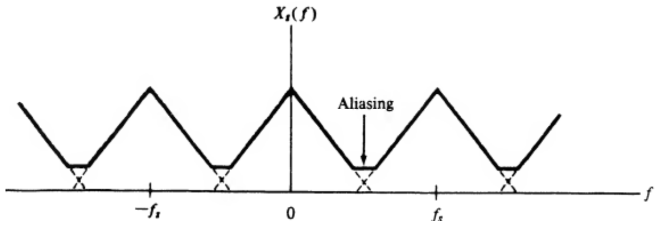
The Nyquist frequency is the theoretical limit that permits the signal reconstruction with an ideal LPF. Hence, in real application the reconstruction requires the use of a sampling frequency $f_0 > 2f_{max}$

Observation2:

The sampler is a nonlinear device that distort the spectra of original signal.

Observation3:

When the sampling frequency $f_0 < f_N$ the repetitions of the spectra overlap and thus can not be separated or reconstructed. This phenomenon is a special case of frequency distortion called **Aliasing**.



The LPF Reconstruction filter:

The LPF reconstruction filter has amplitude $T_s = \frac{1}{f_s}$ *and cut-off frequency* $f_c = \frac{f_s}{2}$

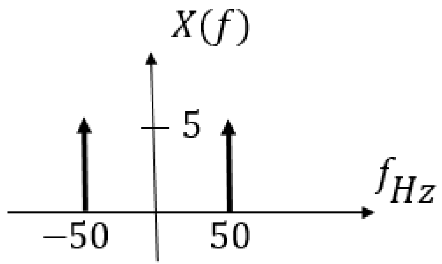
$$H_{LP}(f) = \frac{1}{f_s}\pi(\frac{f}{f_s})e^{-j2\pi f\tau}$$

Example:

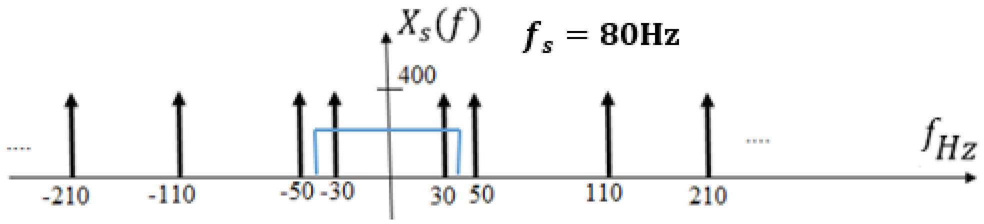
Consider the signal $x(t) = 10\cos(100\pi t)$ and:

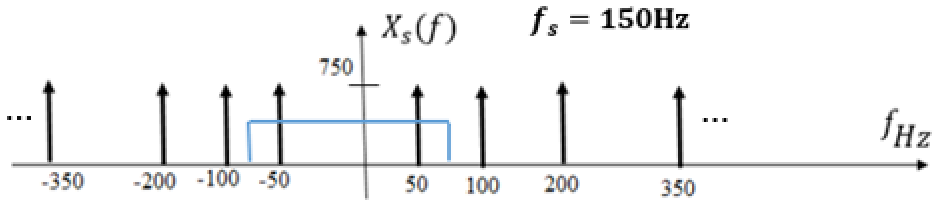
- Compute and plot the spectral representation of the signal
- Compute and plot the spectral representation of the sampled signal with ideal uniform sampling frequency $f = 80Hz$.
- Compute and plot the spectral representation of the sampled signal with ideal uniform sampling frequency $f = 150Hz$.
- Determine and plot (impose on the sampled signal spectra) the ideal LPF reconstruction filter in both cases.
- Determine which of the two cases has aliasing.

- $X(f) = 5\delta(f-50) + 5\delta(f+50)$



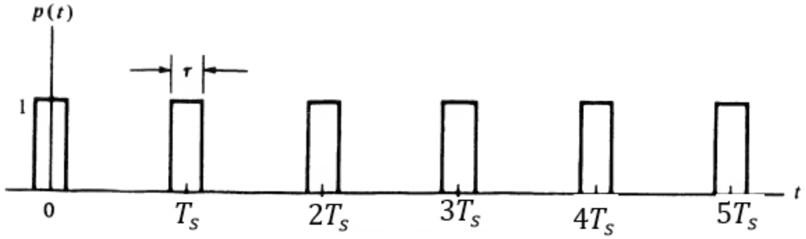
- $f_s = 80 \rightarrow X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) = 400 \sum_{n=-\infty}^{\infty} \delta(f - 50 - 80n) + \delta(f + 50 - 80n)$
- $f_s = 150 \rightarrow X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) = 750 \sum_{n=-\infty}^{\infty} \delta(f - 50 - 150n) + \delta(f + 50 - 150n)$
- $H_{LP-80}(f) = \frac{1}{80}\pi\left(\frac{f}{80}\right)e^{-j2\pi f\tau} \rightarrow f_c = 40\text{Hz}$
- $H_{LP-150}(f) = \frac{1}{150}\pi(\frac{f}{50})e^{-j2\pi f\tau} \rightarrow f_c = 75\text{Hz}$
- The sampled signal with $\rightarrow f_s = 80\text{Hz}$ has aliasing because the reconstructed signal is sinusoidal with frequency $f_{rec} = 30\text{Hz}$ whereas the original signal has $f_0 = 50\text{Hz}$. The reconstructed signal with sampling frequency $f_s = 150\text{Hz}$ is identical to the original signal.

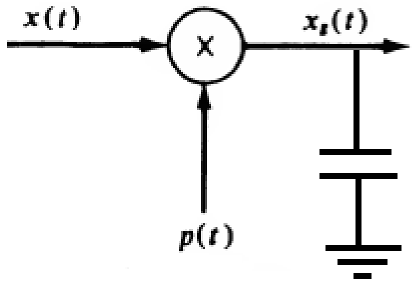




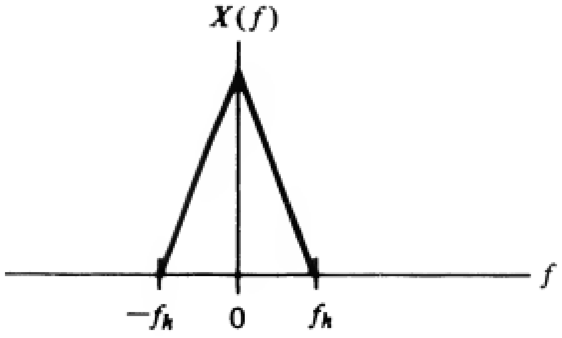
Effect of Non-Ideal Sampling:

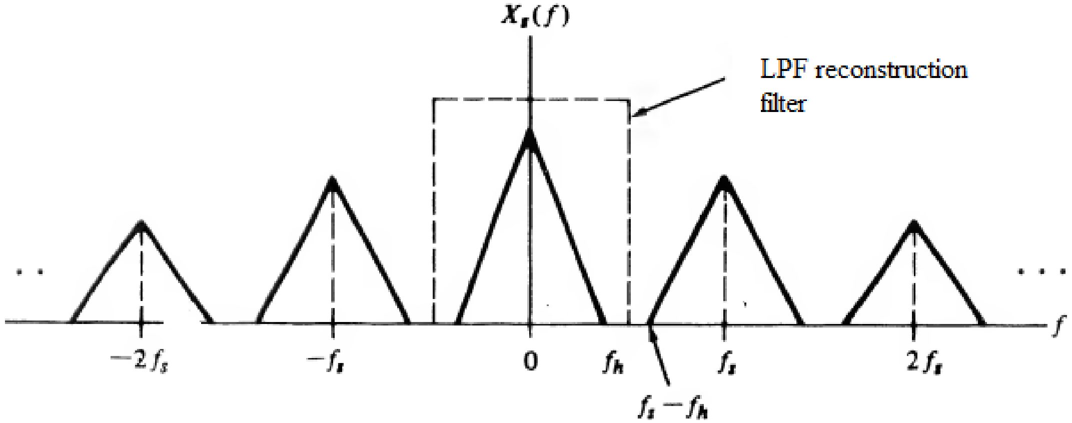
Since the Dirac impulse can not be realized in practice and ideal sampling requires the use of infinite speed switch, real sampler uses a train of finite pulses to control the sampler switch. The used device is called a *sample and hold* system composed of a switch and a memory element that stores the average value of the sampled values.





Since the amplitude of Fourier transform of train of finite pulses is $= P(f) = \tau f_s \sum_{n=-\infty}^{n=\infty} |sinc(\tau f_s)\delta(f - nf_s)|$, the result of the sampling of x(t) becomes $X_s(f) = \tau f_s \sum_{n=-\infty}^{n=\infty} X(f) \ * sinc(\tau f_s)\delta(f - nf_s) = \ \tau f_s \sum_{n=-\infty}^{n=\infty} sinc(n\tau f_s) X(f - nf_s)$ Thus using $f_s > f_N$ each repetition of the signal spectra is multiplied by the corresponding value of the amplitude spectra which leads to amplitude distortion in the spectra of $X_s(f)$ as in figure.

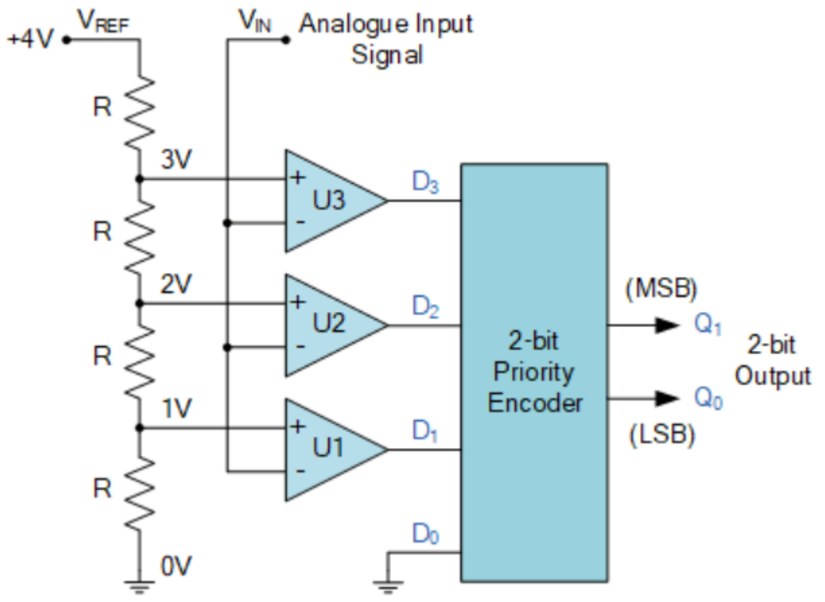




Analog to Digital Conversion:

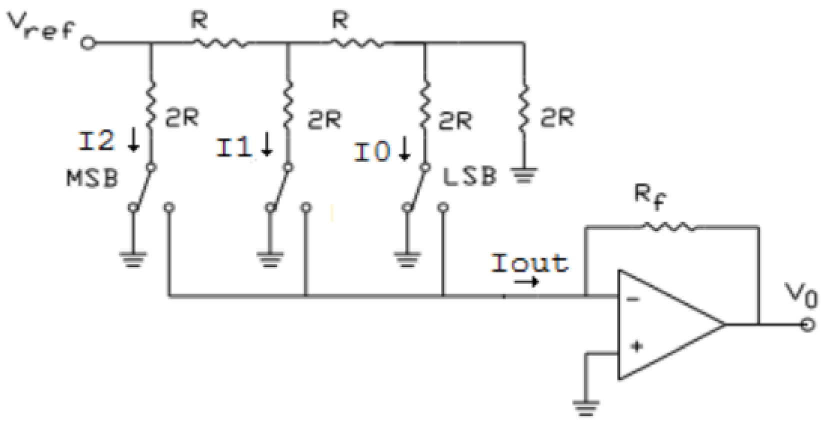
To convert an analog signal to a digital one three steps are required:

- Sampling $x(t) with\ t \in R \ \rightarrow x_s(n) \ \in R\ with\ n \in N$ $and\ time\ normalized\ with\ respect\ to\ T_s$
- Quantization $x_s(n) \in R \rightarrow \quad x_q(n) = floor\big(x_s(n)\big)$ ∈ R_level
- Encoding where each quantized signal level is mapped to a binary code.



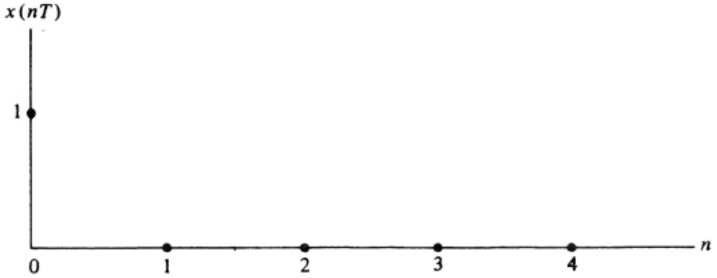
Digital to Analog Conversions

To convert a digital signal to an analog one, The signal is introduces to a sum circuit that adds the contribution of each bit and a smoothing filter (LPF)

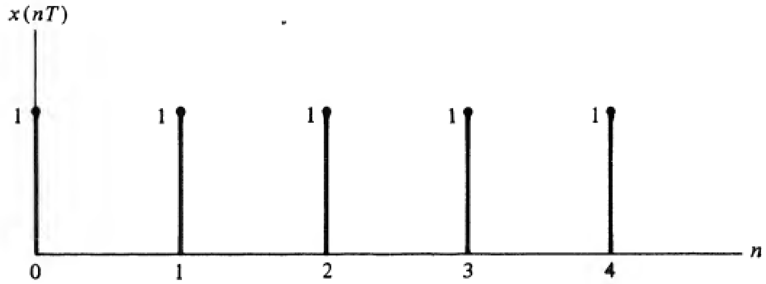


Elementary Discrete Signals:

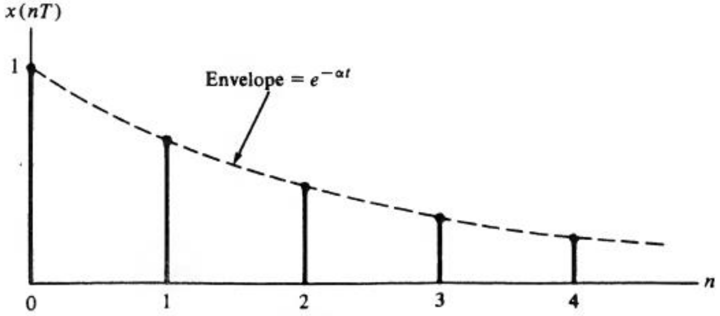
- Unit impulse: $\delta(n) = \begin{cases} 1 & for\ n = 0 \\ 0 & for\ n \neq 0 \end{cases}$



- Unit Step: $u(n) = \begin{cases} 1 & for\ n \geq 0 \\ 0 & for\ < 0 \end{cases}$



- Discrete Exponential $x(n) = \mathrm{A}e^{\alpha n}$



- Sinusoidal $x(n) = \mathrm{Acos}(\mathrm{n}\omega_0 + \theta)$

Classification of Discrete Systems:

A system discrete model is defined by: $y(n) = T[x(n)]$ with $x(n)$ the excitation and $y(n)$ the system response.

- The classification follows the same concepts illustrated in the continuous time case, taking into account that the differentiation and integration operations (which are not valid in discrete analysis) are transformed to difference and sequences computation.

- *Linear and nonlinear system*:
  A discrete system is said to be linear if it satisfies the additivity and proportionality properties which can be included in the superposition form:
  The system is linear
  $\leftrightarrow\ \forall\ two\ inputs\ x_1(n) and\ x_2(n)\ \ with\ responses\ y_1(n) and\ y_2(n), respectively\ and\ \forall \alpha_1(n) and\ \alpha_2(n)$, the response to the input $x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n)$ is $y(n) = \alpha_1 y_1(n) + \alpha_2 y_2(n)$.
  Example:
  Determine if the system $y(n) = kx(n)$ is linear.
  Solution:
  Additivity test: the responses to the first input $x_1(n)$ and $x_2(n)$ are $y_1(n) = kx_1(n)\ and\ y_2(n) = \mathrm{k}x_2(n) \rightarrow$
  $y_1(n) + y_2(n) = k(x_1(n) + x_2(n))$
  Consider $x(n) = x_1(n) + x_2(n)$, the response is $y(n) = kx(n) = kx(n) = k(x_1(n) + x_2(n))$
  $\rightarrow the\ additivity\ applies$
  Proportionality test:
  The response of the system to $\bar{x}(n) = \alpha x(n)$ is $\bar{y}(n) = k[\alpha x(n)] = \alpha[kx(n)] = \alpha y(n)$
  $\rightarrow The\ proportionality\ applies$
  The system is linear.
  Exercise: Determine if the systems $y(n) = kx(n) + h$ and $y(n) = \sqrt{x(n)}$ are linear/nonlinear.

- *Time-invariant/Time-variant systems:*

  A discrete system is said to be time-invariant $\leftrightarrow$ $\forall x(n) and\ \forall n_0$, the response for $x(n)$ is $y(n)$ $\rightarrow$ the response for the input $\bar{x}(n) = x(n - n_0)$ is $\bar{y}(n) = y(n - n_0)$.

  Example1: Determine if the system $y(n) = x^2(n)$ is time variant/invariant.

  Solution:

  Consider the input $\bar{x}(n) = x(n - n_0)$, its response $\bar{y}(n) = x^2(n - n_0)$ which is equal to the response to $x(n)$ shifted by n_0 that is $y(n - n_0) = x^2(n - n_0)$. Therefore, the system is time-invariant.

  Example2: Determine if the system $y(n) = x(n^2)$ is time variant/invariant.

  Solution:

  Consider the input $\bar{x}(n) = x(n - n_0)$, its response $\bar{y}(n) = x((n - n_0)^2)$ which is not equal to the response to $x(n)$ shifted by n_0 that is $y(n - n_0) = x(n^2 - n_0)$. Therefore, the system is time-variant.

- *Causal and non-causal systems:*

  A discrete system is said to be causal $\leftrightarrow \forall x_1(n), x_2(n) and\ \forall n_0,\ x_1(n) = x_2(n)\ \forall n \leq n_0 \rightarrow y_1(n) = y_2(n)\ \forall n \leq n_0$. *Or similarly* $\leftrightarrow \forall x(n) and\ \forall n_0,\ x(n) = 0\ \forall n \leq n_0 \rightarrow y(n) = 0\ \forall n \leq n_0$. *That is to be causal the response time should not precede the input time, that is* $t_{excit} \leq t_{resp}$

  Example:

  Determine if the systems $y(n) = x^2(n - 4)$ and $y(n) = x^2(n + 4)$ are causal/non-causal.

  Solution:

  The system $y(n) = x^2(n - 4)$ is causal because $n - 4 < n\ \forall n$

  The system $y(n) = x^2(n + 4)$ is non-causal because $n + 4 > n\ \forall n$

- *Static and Dynamic systems:*

A discrete system is said to be static ↔ the response to the excitation at any time n depends only on the value of the excitation at the same time n.

A discrete system is said to be dynamic ↔ the response to the excitation at a time instance n depends on the input or system response history or future values (it has memory or future-stores).

<u>Example1:</u>

Determine if the system $y(n) = 10x^2(n)$ is dynamic/static

Solution:

The system is static since the response at any time *n* depends only the value of the input at the input value at time *n*.

<u>Example2:</u>

Determine if the discrete systems $y(n) = 10x(n-1)\ and\ y(n) = 2y(n-1) + x(n) and$ are dynamic/static

<u>Solution:</u>

The two systems are dynamic, the first needs a memory cell to store the previous value of the input, the second needs to store the previous value of the response.

Linear Shift-Invariant Dynamic Discrete Time system:

Definition: A linear shift-invariant dynamic system of order N is defined by a linear constant-coefficients difference equation of the type:

$$\sum_{k=0}^{N} \alpha_k y(n-k) = \sum_{k=0}^{M} \beta_k x(n-k)$$

Finite Impulse Response (FIR) and Infinite Impulse-Response (IIR) systems:

FIR System: the FIR system has its response defined by the weighted combination of its input sequence. It does not include a feedback path from the output to the input of the system. The FIR system characteristic difference equation is:

$$y(n) = \sum_{k=0}^{M} \beta_k x(n-k)$$

The name is due to the fact that the summation can be considered as the convolution sum of the input with a finite $\mathrm{M}+1$ constant-value sequence $\beta_k \quad k = 0,1, \dots M$.

IIR System:

The response of the system is defined from the input and output sequences, hence it has a feedback path from the output to the inputs (called secondary inputs/present state). Its characteristic equation can be written as:

$$y(n) = -\sum_{k=1}^{N} \alpha_k y(n-k) + \sum_{k=0}^{M} \beta_k x(n-k)$$

Solution of Difference Equations and Impulse Response:

The difference equations can be solved using:

- The time domain iterative process.
- the Z and Z^{-1} transforms

Solution of the difference equation in time domain:

The solution is obtained by recursion.

Example:

Determine the impulse response of the causal system: $y(n) = 2y(n-1) + x(n)$

Solution:

problem data:

the impulse response $h(n)$ is required→ $x(n) = \delta(n) = 1\ for\ n = 0\ and\ 0\ otherwise$

The system is causal→ $h(-1) = 0$

Procedure:

$n = 0\ \rightarrow h(0) = 2h(-1) + \delta(0) = 1$

$n = 1\ \rightarrow h(1) = 2h(0) + \delta(1) = 2 \cdot 1 + 0 = 2$

$n = 2\ \rightarrow h(2) = 2h(1) + \delta(2) = 2 \cdot 2 + 0 = 2^2$

$n = 3\ \rightarrow h(3) = 2h(2) + \delta(3) = 2 \cdot 2^2 + 0 = 2^3$

…

$for\ the\ generic\ n\ index$: $h(n) = 2h(n-1) + 0 = 2 \cdot 2^{n-1} + 0 = 2^n$

Thus the response of this IIR system can be written as $h(n) = 2^n u(n)$

Example:

Determine the response of the causal system: $y(n) = 0.5y(n-2) + x(n-2)\ for\ x(n) = \ \delta(n)$

Solution:

problem data:

the impulse response $h(n)$ is required→ $x(n) = \delta(n) = 1\ for\ n = 0, 0\ otherwise.$

The system is causal→ $h(n) = 0\ \forall n < 0$

Procedure:

$n = 0\ \rightarrow h(0) = 0.5h(-2) + \delta(-2) = 0$

$n = 1\ \rightarrow h(1) = 0.5h(-1) + \delta(-1) = 0$

$n = 2\ \rightarrow h(2) = 0.5h(0) + \delta(0) = 1$

$n = 3\ \rightarrow h(3) = 0.5h(1) + \delta(1) = 0$

$n = 4\ \rightarrow h(4) = 0.5h(2) + \delta(2) = 0.5$

$n = 5\ \rightarrow h(5) = 0.5h(3) + \delta(3) = 0.5 \cdot 0 = 0$

$n = 6\ \rightarrow h(6) = 0.5h(4) + \delta(4) = 0.5 \cdot 0.5 = 0.5^2$

$n = 7\ \rightarrow h(7) = 0.5h(5) + \delta(5) = 0.5 \cdot 0 = 0$

$n = 8\ \rightarrow h(8) = 0.5h(6) + \delta(6) = 0.5^2 \cdot 0.5 = 0.5^3$

…

$$for\ the\ generic\ n\ index:\ h(n) = \begin{cases} 0.5h(n-2) + 0 = 0.5 \cdot 0 = 0 & \text{odd index} \\ 0.5h(n-2) = (0.5)^{\frac{n}{2}-1}u(n-2) & even\ index \end{cases}$$

Z-transform and Z-inverse

Definition: Given a discrete time sequence $x(n)$, we define as the single-sided z-transform of $x(n)$ the complex function of complex variable $X(z)$ to which the following series converges:

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Which also satisfies the inverse z-transform Integral $x(n) = \oint_{Analytic-region} X(z) z^{n-1} dz$

Observation:

The inverse z-transform of a rational $X(z)$ function can be computed avoiding the integration form by using the development of partial-fractions and the inverse transform of some elementary signals.

Mapping between the s and z domains:

The mapping is done based on the single-sided Laplace transform of the sampled signal. In fact for a given a signal *x(t),* its sampled signal of $x(t)$ can be written as $x_s(t) = \sum_{n=0}^{\infty} x(t) \cdot \delta(t - nT_0) = \sum_{n=0}^{\infty} x(nT_0) \cdot \delta(t - nT_0) \rightarrow$

$$X_s(s) = \int_0^{\infty} \sum_{n=0}^{\infty} x(nT_0) \cdot \delta(t - nT_0) e^{-st} dt = \sum_{n=0}^{\infty} x(nT_0) \cdot \int_0^{\infty} \delta(t - nT_0) e^{-st} dt = \sum_{n=0}^{\infty} x(nT_0) e^{-snT_0}$$

Consider the mapping $z = e^{sT_0} = e^{\alpha T_0} \cdot e^{j\omega T_0}$ which maps vertical lines of the s-plane into circles in the z-plane, we obtain the following :

$$\begin{cases} \alpha \in]-\infty, 0[\rightarrow represents\ in\ the\ z-plane\ \ the\ internal\ of\ a\ circle\ with\ radius\ r = 1 \\ \alpha = 0\ represents\ a\ circle\ with\ radius\ r = 1 \\ \alpha \in]0, \infty[\rightarrow represents\ in\ the\ z-plane\ \ the\ external\ of\ a\ circle\ with\ radius\ r = 1 \end{cases}$$

Thus, the left semi plane of the s-plane is mapped into the internal of the circle of radius=1 in the z-plane, the imaginary axe is mapped into the circle of radius =1, and finally the right semi plane of the s-plane is mapped into the external of the circle of radius =1

Observation:

The true conclusions in the s-domain left semi-plane, right semi-plane, and imaginary axe for continuous time signals and systems are true for the discrete time signals and systems in the internal, external, and on the circle of radius =1, respectively.

For example, a continuous-time linear time invariant system is asymptotically stable if and only if all the system poles are in the left semi-plane, BIBO stable if it has no roots on the right semi-plane an non-repeated roots on the imaginary axe, unstable if there are right side system poles, or repeated imaginary poles. This is transferred to discrete systems as:

a discrete time linear time invariant system is asymptotically stable if and only if all the system poles are in inside the circle of radius =1 in the z-plane, BIBO stable if it has no roots out of the circle of radius =1 and non-repeated roots on the circle, unstable if there are system poles out of the circle of radius =1 , or repeated poles on the circle.

Z-Transform of elementary signals:

- Transform of $\delta(n)$:

  $X(z) = \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1 \cdot z^0 = 1$

- Transform of $u(n)$:

  $X(z) = \sum_{n=0}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n$, which is a geometric series with base = z^{-1} and converges for $|z^{-1}| < 1 \rightarrow |z| > 1$ and its sum is $X(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$

- Transform of $k^n u(n)$:

  $X(z) = \sum_{n=0}^{\infty} k^n u(n) z^{-n} = \sum_{n=0}^{\infty} k^n z^{-n} = \sum_{n=0}^{\infty} (kz^{-1})^n$, which is a geometric series with base = kz^{-1} and converges for $|kz^{-1}| < 1 \rightarrow |z| > k$ and its sum is $X(z) = \frac{1}{1-kz^{-1}} = \frac{z}{z-k}$

- Transform of $x(n) = cos(n\omega_0)u(n)$:

$$X(z) = \sum_{n=0}^{\infty} \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} z^{-n} = \frac{1}{2}\left(\frac{z}{z - e^{j\omega_0 n}} + \frac{z}{z - e^{-j\omega_0 n}}\right) = \frac{z\left(z - e^{-j\omega_0 n}\right) + z\left(z - e^{+j\omega_0 n}\right)}{\left(z - e^{j\omega_0 n}\right)\left(z - e^{-j\omega_0 n}\right)} =$$

Exercise: Compute the z-transform of $x(n) = sin(n\omega_0)u(n)$ and $x(n) = a^n cos(n\omega_0)u(n)$

Z-Transform Theorems:

Assume: $x_1(n)$ and $x_2(n)$ two unilateral sequences with $x_1(n) \leftrightarrow X_1(z)$ and $x_2(n) \leftrightarrow X_2(z)$

- Superposition-Linearity $Z[\alpha_1 x_1(n) + \alpha_2 x_2(n)] = \alpha_1 X_1(z) + \alpha_2 X_2(z)$
- Right-shift property: $Z[x(n-m)] = z^{-m}X(z) = X_{n-sh}(z)$

Proof:

$X_{n-sh}(z) = \sum_{n=0}^{\infty} x(n-m)z^{-n}$, using substitution $k = n - m$, $n = 0 \to k = -m, n = \infty \to k = \infty$, Consequently,

$X_{n-sh}(z) = \sum_{k=-m}^{\infty} x(k)z^{-(k+m)} = z^{-m} \sum_{k=0}^{\infty} x(k)z^{-k} = z^{-m}X(z),$

observe that the value of the sample $x(-m) to\ x(-1) are\ equal\ zero\ because\ the\ sequence\ is\ a\ unilateral\ sequence.$

- Multiplication by an exponential sequence: $Z[a^n x(n)] = X\left(\frac{z}{a}\right)$

Proof:

$$X_{sc}(z) = \sum_{n=0}^{\infty} a^n x(n) z^{-n} = \sum_{n=0}^{\infty} x(n)(a^{-1}z)^{-n} = X(\frac{z}{a})$$

- Reversal theorem: $Z[x(-n)] = X\left(\frac{1}{z}\right)$

<u>Proof:</u> $X_{rev}(z) = \sum_{n=0}^{\infty} x(-n)z^{-n}$, by substitution $k = -n, n = 0 \rightarrow k = 0, n = \infty \rightarrow k = -\infty$

$$X_{rev}(z) = \sum_{n=0}^{\infty} x(-n)z^{-n} = \sum_{k=-\infty}^{0} x(k)z^{k} = \sum_{k=-\infty}^{0} x(k)(z^{-1})^{-k} = X(z^{-1}) = X(\frac{1}{z})$$

- Z-domain differentiation: $Z[n^k x(n)] = (-1)^k z^k \frac{d^k X(z)}{dz^k}$

<u>Exercise:</u> Proof by induction method.

<u>Example:</u>

Knowing that $u(n) \leftrightarrow \frac{z}{z-1}$ determine the z-transform of $x(n) = nu(n)$

<u>Solution:</u>

$$Z[nu(n)] = (-1)^1 z^1 \frac{d}{dz}\left[\frac{z}{z-1}\right] = -\text{z}\frac{(z-1)-z}{(z-1)^2} = \frac{z}{(z-1)^2}$$

- Convolution theorem: $Z[x_1(n) * x_2(n)] = X_1(z) \cdot X_2(z)$

- Initial value theorem: $x(0) = \lim_{z\to\infty} X(z)$
- Final value theorem: $\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)\,X(z)$

The inverse z-transform of rational functions:

- Sequence samples generation using long division
- Closed-form inverse z-transform using partial fraction and transforms of elementary signals.

Samples generation using long division

Example:

$$X(z) = \frac{z^2}{z^2 - 1.2z + 0.2}$$

The first four elements of the sequence $x(n) = 1, 1.2, 1.24, 1.248 \ldots.$

1

$$1+1.2z^{-1} + 1.24z^{-2} + 1.248z^{-3}\ldots$$

$$z^2 - 1.2z + 0.2 \;\overline{)\; z^2}$$

$$z^2 - 1.2z+0.2$$

$$1.2z - 0.2$$

$$1.2z - 1.44 + 0.24z^{-1}$$

$$1.24 - 0.24z^{-1}$$

$$1.24 - 0.1488z^{-1} + 0.248z^{-2}$$

$$1.248z^{-1} - 0.248z^{-2}$$

Computation of the inverse z-transform by partial fractions:

Example1:

$$X(z) = \frac{1}{1 - 1.2z^{-1} + 0.2z^{-2}} = \frac{z^2}{(z-1)(z-0.2)}$$

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.2)} = \frac{K_1}{z-1} + \frac{K_2}{z-0.2}$$

$$K_1 = \lim_{z\to 1}(z-1)\frac{X(z)}{z} = 1.25$$

$$K_2 = \lim_{z\to 0.2}(z-0.2)\frac{X(z)}{z} = -0.25$$

$$X(z) = \frac{1.25z}{z-1} - \frac{0.25z}{z-0.2}$$

$$x(n) = [1.2 - 0.25(0.2)^n]u(n)$$

Example2:

$$Y(z) = \frac{1}{z^2 - 1.2z + 0.2} = \frac{1}{(z-1)(z-0.2)}$$

$$\frac{Y(z)}{z} = \frac{1}{z(z-1)(z-0.2)} = \frac{K_1}{z} + \frac{K_2}{z-1} + \frac{K_3}{z-0.2}$$

$$K_1 = \lim_{z\to 0} z\frac{Y(z)}{z} = \frac{1}{0.2} = 5$$

$$K_2 = \lim_{z\to 1}(z-1)\frac{Y(z)}{z} = \frac{1}{0.8} = 1.25$$

$$K_3 = \lim_{z\to 0.2}(z-0.2)\frac{Y(z)}{z} = -6.25$$

$$Y(z) = 5 + 1.25\frac{z}{z-1} - 6.25\frac{z}{z-0.2}.$$

$$y(n) = 5\delta(n) + [1.25 - 6.25(0.2)^n]u(n)$$

LSI-System representation in z-domain:

To determine the LSI discrete system transfer function we apply the Z-transform to the difference equation that represents the system model taking into account the superposition and time shift theorems of the Z-transform.

IIR-system

$$Z[\sum_{k=0}^{N} \alpha_k y(n-k)] = Z[\sum_{k=0}^{M} \beta_k x(n-k)] \rightarrow \mathrm{Y(z)} \sum_{k=0}^{N} \alpha_k z^{-k} = \mathrm{X(z)} \sum_{k=0}^{M} \beta_k z^{-k} \rightarrow$$

$$H(z^{-1}) = \frac{Y(z^{-1})}{X(z^{-1})} = \frac{\sum_{k=0}^{M} \beta_k z^{-k}}{\sum_{k=0}^{N} \alpha_k z^{-k}} \quad \text{The system transfer function in the } z^{-1} - form$$

To obtain the transfer function in the $z - form$ we extract z^{-N} $from\ the\ denominator$ which leads to

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} \beta_k z^{N-k}}{\sum_{k=0}^{N} \alpha_k z^{N-k}}$$

Example:

Determine the transfer function of the system:

$$y(n) = 2y(n-1) + 3y(n-2) + 5x(n) - 4x(n-1)$$

Solution:

Rearranging the equation and applying the time delay theorem we obtain the transfer function in $z^{-1} - form$:

$$Y(z)[1 - 2z^{-1} + 3z^{-2}] = X(z)[5 - 4z^{-1}] \rightarrow H(z^{-1}) = \frac{5 - 4z^{-1}}{1 - 2z^{-1} + 3z^{-2}}$$

To determine the $z - form$ we extract z^{-2} from the denominator which leads to:

$$H(\mathrm{z}) = \frac{5 - 4z^{-1}}{z^{-2}(z^2 - 2z + 3)} = \frac{z^2(5 - 4z^{-1})}{(z^2 - 2z + 3)} = \frac{(5z^2 - 4z)}{(z^2 - 2z + 3)}$$

FIR-system

Following the same procedure we obtain:

$$Z[y(n)] = \sum_{k=0}^{M} \beta_k Z[x(n-k)] \rightarrow \frac{Y(z^{-1})}{X(z^{-1})} = H(z^{-1}) = \sum_{k=0}^{M} \beta_k z^{-k}$$

Which can be written in $z - form$ as $H(z) = \frac{\sum_{k=0}^{M} \beta_k z^{M-k}}{z^M}$

Example:

Determine the transfer function of the system:

$$y(n) = 3x(n-2) + 5x(n-1) - 4x(n)$$

$$H(z^{-1}) = \frac{Y(z^{-1})}{X(z^{-1})} = 3z^{-2} + 5z^{-1} + 4 \rightarrow H(z) = \frac{4z^2 + 5z + 3}{z^2}$$

Observation:

The FIR system can have system poles only in the origin. Any pole out of the origin indicates that the system is an IIR one.

Computation of system response using z-domain:

$Y(z) = X(z) \cdot H(z) \rightarrow y(n) = Z^{-1}[X(z) \cdot H(z)]$

The inverse Z-transform of the system response transform can be computed by partial fractions since the transfer function of an LSI system is a rational function in $z - variable$.

Example1:

Compute the unit step response of the system $H(z) = \frac{z}{(z-0.8)(z-0.2)}$

Solution:

The response $Y(z) = H(z) \cdot X(z)$ with $X(z) = Z[u(n)] = \frac{z}{z-1}$

$$Y(z) = \frac{z}{(z-0.8)(z-0.2)} \cdot \frac{z}{z-1} \rightarrow y(n) = Z^{-1}[Y(z)]$$

Since the function is a rational function, the development in partial fractions and the inverse of the elementary signals can be used.

We apply the partial fraction to $\frac{Y(z)}{z} = \frac{1}{(z-0.8)(z-0.2)} \cdot \frac{z}{z-1}$in order to evaluate the fraction parameters in z-form following the same procedure used in the s-domain.

$$\frac{Y(z)}{z} = \frac{k_1}{z-0.8} + \frac{k_2}{z-0.2} + \frac{k_3}{z-1}$$

$$k_1 = \frac{0.8}{(0.8-0.2)(0.8-1)} = -6.67, k_2 = \frac{0.2}{(0.2-0.8)(0.2-1)} = 0.42, k_3 = \frac{1}{(1-0.8)(1-0.2)} = 0.625 \rightarrow$$

$$Y(z) = \frac{-6.67\boldsymbol{z}}{z-0.8} + \frac{0.42\boldsymbol{z}}{z-0.2} + \frac{0.625\boldsymbol{z}}{z-1} \rightarrow y(n) = [0.625 - 6.67(0.8^n) + 0.42(0.2^n)]u(n)$$

Exercise

Compute the unit impulse response of the system $H(z) = \frac{z}{(z-0.8)^2(z+0.2)}$

Example1:

Compute the response of the system $H(z) = \frac{1}{(z-0.1)(z+0.2)}$ to $x(n) = 10\delta(n-2)$

Solution:

The system is LSI so we can compute the impulse response and apply superposition and time-shift.

$\bar{x}(n) = \delta(n) \rightarrow \bar{X}(z) = 1$

$h(n) = z^{-1}[H(z)]$, using partial fractions:

$$\frac{H(z)}{z} = \frac{1}{z(z-0.1)(z+0.2)} = \frac{k_1}{z} + \frac{k_2}{z-0.1} + \frac{k_3}{z+0.2}$$

$k_1 = \frac{1}{-0.1\cdot 0.2} = -50,\ k_2 = \frac{1}{0.1\cdot 0.3} = 33.33,\ k_1 = \frac{1}{-0.2\cdot -0.3} = 16.67 \rightarrow$

$H(z) = -50 + \frac{33.3z}{z-0.1} + \frac{16.7z}{z+0.2} \rightarrow h(n) = [-50\delta(n) + 33.33(0.1^n) + 16.67(-0.2^n)]u(n)$

Using the linearity and time-invariance system conditions, the required response is:

$h(n) = [-500\delta(n-2) + 333.3(0.1^{n-2}) + 166.7(-0.2^{n-2})]u(n-2)$

Computation of the system response using convolution sum:

Theorem:

Given an LSI system with impulse response $h(n)$ the zero state response $y(n)$ of the system to any input $x(n)$ is given by:

$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k)$

Proof:

Assume the LSI system is defined as $y(n) = T[x(n)]$, since $x(n)$ is a sequence it can be defined as

$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \rightarrow y(n) = T[x(n)] = T[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)] \rightarrow \sum_{k=-\infty}^{\infty} x(k)T[\delta(n-k)]$

$$\rightarrow y(n) \sum_{k=-\infty}^{\infty} x(k)h(n-k)]$$

Corollary 1:

If $x(n)$is a right-unilateral signal then $y(n)\sum_{k=0}^{\infty} x(k)h(n-k)$

Corollary 2:

If $x(n)$is a right-unilateral signal and the system is causal then $y(n) = \sum_{k=0}^{n} x(k)h(n-k)$

Proof:

If the system is causal then $h(n-k) = 0$ for $n-k < 0 \rightarrow h(n-k) = 0\ for\ k > n$.

Example1:

Compute the response of the system with $h(n) = 2^n u(n)$ to the input $x(n) = 3^n u(n)$

Solution:

The system is causal and the input signal is unilateral, thus:

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k)] = \sum_{k=0}^{n} 3^k \cdot 2^{n-k} = 2^n \sum_{k=0}^{n} (\frac{3}{2})^k$$

$\sum_{k=0}^{n}(a)^k$ is a geometric sum and its sum is equal to $S = \frac{1-a^{n+1}}{1-a}$. Thus,

$$y(n) = 2^n \sum_{k=0}^{n} (\frac{3}{2})^k = 2^n \cdot \frac{1-\left(\frac{3}{2}\right)^{n+1}}{1-\frac{3}{2}} u(n) = \frac{2^n - \frac{1}{2}\cdot 3^{n+1}}{-\frac{1}{2}} u(n) = (3^{n+1} - 2^{n+1})u(n)$$

Example2:

Compute the response of the system with $h(n) = 2^n u(n)$ to the input $x(n) = (\frac{1}{3})^{n-2} u(n-2)$

Solution:

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k)] = \sum_{k=0}^{n} 2^{n-k} \cdot \left(\frac{1}{3}\right)^{k-2} u(k-2) = \sum_{k=2}^{n} 2^{n-k} \cdot \left(\frac{1}{3}\right)^{k-2}$$

Applying the substitution $m = k - 2,\ k = 2 \rightarrow m = 0, k = n \rightarrow m = n - 2 \rightarrow$

$$y(n) = \sum_{m=0}^{n-2} 2^{n-(m+2)} \cdot \left(\frac{1}{3}\right)^m = 2^{n-2} \sum_{m=0}^{n-2} 2^{-m} \cdot \left(\frac{1}{3}\right)^m = 2^{n-2} \cdot \sum_{m=0}^{n-2} (\frac{1}{6})^m = 2^{n-2} \cdot \frac{1 - \left(\frac{1}{6}\right)^{n-2+1}}{1 - \frac{1}{6}} = 6\,\frac{2^{n-2} - \frac{1}{2}(\frac{1}{3})^{n-1}}{5}$$

$$= 3\,\frac{2^{n-1} - (\frac{1}{3})^{n-1}}{5}$$

Discrete Systems Stability:

Definition: A discrete LSI system is said to be asymptotically stable if its transient response goes to zero and a steady state response is reached for n goes to infinity.

Theorem1: an LSI system with impulse response $h(n)$ is asymptotically stable ↔ $\lim_{n\to\infty} h(n) = 0$.

Theorem2: a dynamic LSI system is asymptotically stable↔ all the poles of its transfer function have $|P_i| < 1$ (located inside the circle of radius =1)

Theorem3: an LSI system is unstable if it has at least a pole with $|P_i| > 1$ (located outside the circle of radius =1) or a repeated poles with $|P_i| = 1$ (located on the circle of radius =1) .

Definition:(BIBO stability) an LSI system is said to be BIBO (Bounded Input/Bounded Output)↔ $\forall\, input\ x(n) with\ |x(n)| \le N,\ \ \exists M < \infty$ so that the respons $|y(n)| \le M, \forall t$ (weak stability)

Theorom4: a system is BIBO stable ↔ $\sum_{-\infty}^{\infty} |h(n)| < \infty$ that is if its impulse response is absolutely integrable.

Theorem3: an LSI system is BIBO stable if it has no poles located outside the circle of radius =1 and poles with multiplicity =1 (not repeated) on the circle with radius =1 ($|P_i| = 1$).

Examples

$$H(z) = \frac{z}{(z-0.8)^2(z+0.2)} \to Asymptotically\ stable$$

$$H(z) = \frac{z}{(z-0.8)^2(z-1.8)} \to Unstable$$

$$H(z) = \frac{z}{(z-0.6+j0.8)^2(z-0.6-j0.8)^2(z-0.7)} \to Unstable$$

$$H(z) = \frac{z}{(z-0.6+j0.8)(z-0.6-j0.8)(z-0.7)} \to BIBO\ stable.$$

Example : Determine if the system with with $h(n) = (\frac{1}{4})^n u(n)$ is BIBO stable.

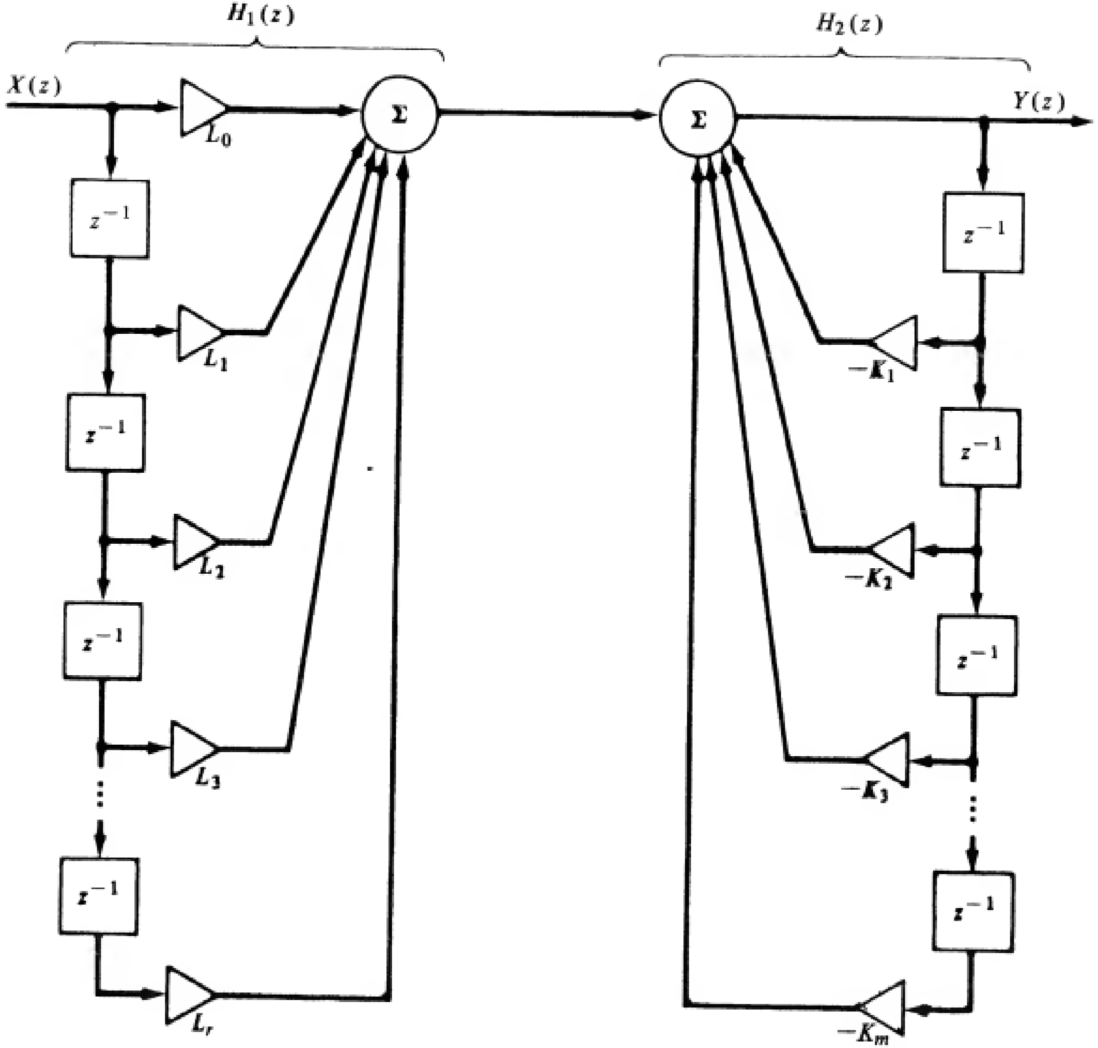
Solution: $\sum_{-\infty}^{\infty} |h(n)| = \sum_{-\infty}^{\infty} \left| \left(\frac{1}{4}\right)^n u(n) \right| = \sum_0^{\infty} | \left(\frac{1}{4}\right)^n)| = \frac{1}{1-0.25} = \frac{4}{3} < \infty \to BIBO\ stable$ (but also asymptotically stable $\lim_{n\to\infty} (\frac{1}{4})^n u(n) = 0$)

Direct Form-I Modeling and Realization:

Discrete systems modeling and realization is used to simulate and implement the systems. An example of this is the Direct Form-I realization which is derived from the direct form of the difference equation:

$$y(n) = -\sum_{i=1}^{m} k_i y(n-i) + \sum_{i=0}^{r} L_i x(n-i)$$

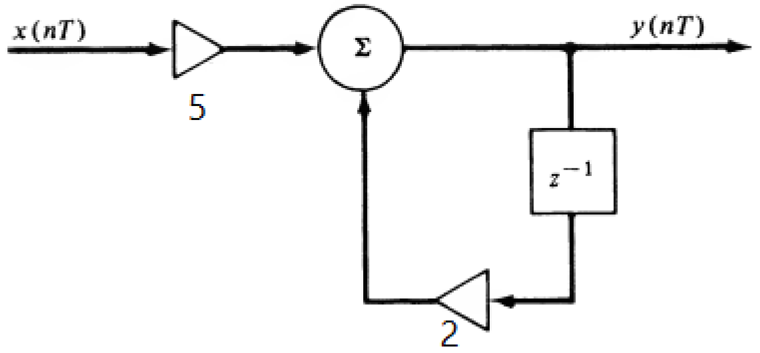
$$\mathrm{Y}(z^{-1}) = -\sum_{i=1}^{m} K_i z^{-i} + \mathrm{X}(z^{-1}) \sum_{i=0}^{r} L_i z^{-i}$$



Example1:

Determine the direct Form-I realization of the system

$$H(z^{-1}) = \frac{Y(z^{-1})}{X(z^{-1})} = \frac{5}{1 - 2z^{-1}}$$

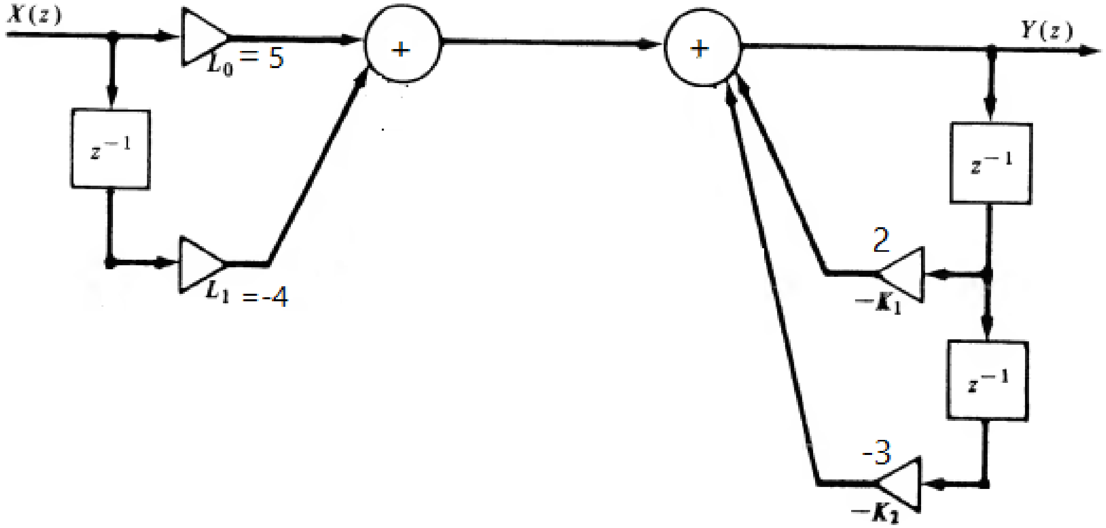


Example2 :

Determine the direct Form-I realization of the system

$$H(z^{-1}) = \frac{Y(z^{-1})}{X(z^{-1})} = \frac{5 - 4z^{-1}}{1 - 2z^{-1} + 3z^{-2}} \rightarrow$$

$$Y(z^{-1}) = [2z^{-1}Y(z^{-1}) - 3z^{-2}Y(z^{-1})] + [5X(z^{-1}) - 4z^{-1}X(z^{-1})]$$



Frequency-Response:

The frequency response of a discrete system can be obtained by applying:

- The Discrete Fourier Transform (DFT) or its optimized form called Fast Fourier Transform (FFT).

$$X(\omega) = \sum_{k=0}^{\infty} x(k)e^{-j\omega k}$$

- Computing the $Z - Transform$ at the unit circle with $z = e^{j\omega}$, that is $H(\omega) = H(z)|_{z=e^{j\omega}}$

Definition:

Given a causal LSI system with impulse response h(n), the frequency response of the system can be computed using the DFT as $H(\omega) = \sum_{k=0}^{n} h(n)e^{-j\omega k}$

Example1:

Compute the frequency response of the system with $h(n) = (\frac{1}{3})^n$

Solution:

$$H(\omega) = \sum_{k=0}^{\infty} h(n)e^{-j\omega n} = \sum_{n=0}^{\infty} (\frac{1}{3})^n e^{-j\omega n} = \sum_{n=0}^{\infty} (3e^{j\omega})^{-n} = \frac{1}{1 - \frac{1}{3e^{j\omega}}} = \frac{1}{1 - \frac{1}{3}e^{-j\omega}}$$

Example2:

Compute the frequency response of the system with $H(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$

Solution: $H(\omega) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$

Sinusoidal Steady State Response:

Theorem: Given a discrete LSI system with impulse response h(n), the response of the system to a sinusoidal input $x(n) = Xcos(\omega_0 n + \varphi)$ is sinusoidal with the same input frequency $y(n) = Ycos(\omega_0 n + \theta)$ with:

$$Y = X \cdot |H(\omega)|_{\omega_0}, \qquad \theta = \varphi + < H(\omega)|_{\omega_0}$$

Proof:

let $x(n) = Xe^{j(\omega_0 n + \varphi)}$ then $y(n) = \sum_{k=0}^{\infty} h(n) Xe^{j(\omega_0 (n-k+\varphi)} = Xe^{j(\omega_0 n + \varphi)} \sum_{k=0}^{\infty} h(n) e^{-j\omega_0 k} = Xe^{j(\omega_0 t + \varphi)} \cdot H(\omega)|_{\omega_0}$

Where $H(\omega)$ is the frequency response of the system that characterizes the spectral response of the linear time invariant system. $H(\omega)$ is a complex function of the real variable ω that represents the DFT of the impulse response $h(n)$.

$Re(x(n)) = Xcos(\omega_0 n + \varphi) \rightarrow Re(y(n)) = Ycos(\omega_0 n + \theta)$ *which proves the assertion of the theorem*

Example (sinusoidal steady-state response):

compute the steady-state response of the system with frequency response $H(\omega) = \frac{1}{1 - \frac{1}{3} e^{-j\omega}}$ to the input signal

$x(n) = 2\cos(\pi n + \frac{\pi}{3})$

Solution:

$$H(\omega) = \frac{1}{1 - \frac{1}{3}\cos(\omega) + \frac{1}{3}\sin(\omega)} \rightarrow |H(\omega)| = \frac{1}{\sqrt{(1 - \frac{1}{3}\cos(\omega))^2 + (\frac{1}{3}\sin(\omega))^2}}, \qquad < H(\omega) = -tan^{-1}(\frac{\frac{1}{3}\sin(\omega)}{1 - \frac{1}{3}\cos(\omega)})$$

$$y(n) = 2 \cdot \frac{1}{\sqrt{(1 - \frac{1}{3}\cos(\pi))^2 + (\frac{1}{3}\sin(\pi))^2}} \cos(\pi n + \frac{\pi}{3} - tan^{-1}\left(\frac{\frac{1}{3}\sin(\pi)}{1 - \frac{1}{3}\cos(\pi)}\right) = \frac{2}{\sqrt{(\frac{4}{3})^2}} \cos(\pi t + \frac{\pi}{3} - tan^{-1}(0)) = \frac{3}{2}\cos(\pi n + \frac{\pi}{3})$$