Birzeit University
Mathematics Department

COST — Math 330

- Cost: Total number of operations $(+, -, \times, \div)$
- Let A, B be $n \times n$ matrices, b an $n \times 1$ vector, α a scalar, and p a positive integer.

Expression	**Cost**
$A+B$	n^2
$A-B$	n^2
αA	n^2
AB	$2n^3 - n^2$
Ab	$2n^2 - n$
A^p	$(p-1)(2n^3 - n^2)$
$det(A)$	$n! \sum_{k=0}^{n-1} (\frac{1}{k!}) - 1$

- **Special sums**:

(i) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(ii) $\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$

(iii) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

(iv) $\sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6}$

(v) $\sum_{k=1}^{n} c = c.n$

Q. If A, B are 3×3 matrices, find the cost of evaluating $2A + |B|B^3$

Answer: Cost = 9 + 9 + 9 + 14 + 2 (45) = 131

(1) Gaussian Elimination.

Total cost = Cost of transforming $[A|b] \to [U|c]$ + Cost of Back Substitution

- transforming $[A|b] \to [U|c]$:

Step	$\pm$	$\times$	$\div$
1	$(n-1)n$	$(n-1)n$	$n-1$
2	$(n-2)(n-1)$	$(n-2)(n-1)$	$n-2$
.	.	.	.
.	.	.	.
k	$(n-k)(n-k+1)$	$(n-k)(n-k+1)$	$(n-k)$
.	.	.	.
.	.	.	.
$n-1$			

$$\text{Cost}=\sum_{k=1}^{n-1}(n-k)(n-k+1)+\sum_{k=1}^{n-1}(n-k)(n-k+1)+\sum_{k=1}^{n-1}(n-k)=\frac{4n^3+3n^2-7n}{6}$$

- Back Substitution: Cost $= n^2$

- Total cost $= \dfrac{4n^3+3n^2-7n}{6} + n^2 = \dfrac{4n^3+9n^2-7n}{6}$

Example: Solving a 4×4 linear system $Ax=b$.

- transforming $[A|b] \to [U|c]$:

Step	$\pm$	$\times$	$\div$
1	3 (4)	3 (4)	3
2	2 (3)	2 (3)	2
3	1 (2)	1 (2)	1

Cost$=20+20+6=46$

- Back Substitution: Cost $= 4^2 = 16$

- Total cost = 46 + 16 = 62

(2) LU Factorization.

Total cost = Cost of transforming $A \to U$ + Cost of Forward substitution + Cost of Backward substitution

- transforming $A \to U$:

Step	$\pm$	$\times$	$\div$
1	$(n-1)^2$	$(n-1)^2$	$n-1$
2	$(n-2)^2$	$(n-2)^2$	$n-2$
.	.	.	.
.	.	.	.
k	$(n-k)^2$	$(n-k)^2$	$(n-k)$
.	.	.	.
.	.	.	.
$n-1$			

$$\text{Cost}=\sum_{k=1}^{n-1}(n-k)^2+\sum_{k=1}^{n-1}(n-k)^2+\sum_{k=1}^{n-1}(n-k)=\frac{4n^3-3n^2-n}{6}$$

- Forward Substitution: Cost $= n^2 - n$
- Back Substitution: Cost $= n^2$
- Total cost $= \dfrac{4n^3-3n^2-n}{6} + n^2 - n + n^2 = \dfrac{4n^3+9n^2-7n}{6}$

Example: Solving a 4×4 linear system $Ax = b$.

- transforming $A \to U$:

Step	$\pm$	$\times$	$\div$
1	3 (3)	3 (3)	3
2	2 (2)	2 (2)	2
3	1 (1)	1 (1)	1

Cost$=14+14+6=34$

- Forward Substitution: Cost $= 4^2 - 4 = 12$
- Back Substitution: Cost $= 4^2 = 16$
- Total cost $= 34 + 12 + 16 = 62$

(3) Gauss-Jordan Reduction.

Total cost = Cost of transforming $[A|b] \to [I|x]$

- transforming $[A|b] \to [I|x]$:

Step	$\pm$	$\times$	$\div$
1	$(n-1)n$	$(n-1)n$	n
2	$(n-1)(n-1)$	$(n-1)(n-1)$	$n-1$
.	.	.	.
.	.	.	.
k	$(n-1)(n-k+1)$	$(n-1)(n-k+1)$	$n-k+1$
.	.	.	.
.	.	.	.
n			

- Total cost $= \sum_{k=1}^{n}(n-1)(n-k+1) + \sum_{k=1}^{n}(n-1)(n-k+1) + \sum_{k=1}^{n}(n-k+1) = \frac{2n^3+n^2-n}{2}$

Example: Solving a 4×4 linear system $Ax = b$.

- transforming $[A|b] \to [I|x]$:

Step	$\pm$	$\times$	$\div$
1	3 (4)	3 (4)	4
2	3 (3)	3 (3)	3
3	3 (2)	3 (2)	2
4	3 (1)	3 (1)	1

- Total cost = 30 + 30 + 10 = 70

(4) The Inverse Method.

Total cost $=$ Cost of transforming $[A|I] \to [I|A^{-1}]$ $+$ Cost of $A^{-1}b$

- transforming $[A|I] \to [I|A^{-1}]$:

Step	$\pm$	$\times$	$\div$
1	$(n-1)(2n-1)$	$(n-1)(2n-1)$	$2n-1$
2	$(n-1)(2n-2)$	$((n-1)(2n-2)$	$2n-2$
.	.	.	.
.	.	.	.
k	$(n-1)(2n-k)$	$(n-1)(2n-k)$	$2n-k$
.	.	.	.
.	.	.	.
n			

$$\text{Cost}=\sum_{k=1}^{n}(n-1)(2n-k)+\sum_{k=1}^{n}(n-1)(2n-k)+\sum_{k=1}^{n}(2n-k)=\frac{6n^3-5n^2+n}{2}$$

- $A^{-1}b$: Cost $= 2n^2-n$
- Total cost $=\dfrac{6n^3-5n^2+n}{2}+2n^2-n=\dfrac{6n^3-n^2-n}{2}$

Example: Solving a 4×4 linear system $Ax=b$.

- transforming $[A|I] \to [I|A^{-1}]$:

Step	$\pm$	$\times$	$\div$
1	3 (7)	3 (7)	7
2	3 (6)	3 (6)	6
3	3 (5)	3 (5)	5
4	3 (4)	3 (4)	4

Cost$=66+66+22=154$

- $A^{-1}b$: Cost $=2(4)^2-4=28$
- Total cost $=154+28=182$

(5) Cramer's Rule.

Total cost = Cost of det + Cost of $\div$

- det: Cost$= (n+1) \times \left[n! \sum_{k=0}^{n-1} (\frac{1}{k!}) - 1 \right] = (n+1)! \sum_{k=0}^{n-1} (\frac{1}{k!}) - (n+1)$
- $\div$: Cost $= n$
- Total cost $= (n+1)! \sum_{k=0}^{n-1} (\frac{1}{k!}) - (n+1) + n = (n+1)! \sum_{k=0}^{n-1} (\frac{1}{k!}) - 1$

Example: Solving a 4×4 linear system $Ax = b$.

- Total cost= 5 (cost of 4×4 determinant) $+ 4 = 5(63) + 4 = 319$