6.2 Solution of Initial Value Problems

Method of Laplace Transforms

To solve an initial value problem:

(a) Take the Laplace transform of both sides of the equation.

(b) Use the properties of the Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and then solve this equation for the transform.

(c) Determine the inverse Laplace transform of the solution by looking it up in a table or by using a suitable method (such as partial fractions) in combination with the table.

Inverse Laplace Transform

Definition 4. Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies

$$\mathcal{L}\{f\} = F, \tag{2}$$

then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notation $f = \mathcal{L}^{-1}\{F\}$.

Linearity of the Inverse Transform

Theorem 7. Assume that $\mathcal{L}^{-1}\{F\}$, $\mathcal{L}^{-1}\{F_1\}$, and $\mathcal{L}^{-1}\{F_2\}$ exist and are continuous on $[0, \infty)$ and let c be any constant. Then

$$\mathcal{L}^{-1}\{F_1 + F_2\} = \mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\}, \tag{3}$$

$$\mathcal{L}^{-1}\{cF\} = c\mathcal{L}^{-1}\{F\}. \tag{4}$$

Example 1 Determine $\mathscr{L}^{-1}\{F\}$, where

(a) $F(s) = \dfrac{2}{s^3}$. **(b)** $F(s) = \dfrac{3}{s^2+9}$. **(c)** $F(s) = \dfrac{s-1}{s^2-2s+5}$.

Solution To compute $\mathcal{L}^{-1}\{F\}$, we refer to the Laplace transform table

(a) $\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}(t) = t^2$

(b) $\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}(t) = \sin 3t$

(c) $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\}(t) = e^t\cos 2t$

In part (c) we used the technique of completing the square to rewrite the denominator in a form that we could find in the table. ◆

Example 2 Determine $\mathscr{L}^{-1}\left\{\frac{5}{s-6}-\frac{6s}{s^2+9}+\frac{3}{2s^2+8s+10}\right\}$.

Solution We begin by using the linearity property. Thus,

$$\mathscr{L}^{-1}\left\{\frac{5}{s-6}-\frac{6s}{s^2+9}+\frac{3}{2(s^2+4s+5)}\right\}$$

$$=5\mathscr{L}^{-1}\left\{\frac{1}{s-6}\right\}-6\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\}+\frac{3}{2}\mathscr{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\}.$$

Referring to the Laplace transform tables, we see that

$$\mathscr{L}^{-1}\left\{\frac{1}{s-6}\right\}(t)=e^{6t} \qquad \text{and} \qquad \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}(t)=\cos 3t\,.$$

This gives us the first two terms. To determine $\mathscr{L}^{-1}\{1/(s^2+4s+5)\}$, we complete the square of the denominator to obtain $s^2+4s+5=(s+2)^2+1$. We now recognize from the tables that

$$\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2+1^2}\right\}(t)=e^{-2t}\sin t\,.$$

Hence,

$$\mathscr{L}^{-1}\left\{\frac{5}{s-6}-\frac{6s}{s^2+9}+\frac{3}{2s^2+8s+10}\right\}(t)=5e^{6t}-6\cos 3t+\frac{3e^{-2t}}{2}\sin t\,.$$ ◆

Example 3 Determine $\mathscr{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}$.

Solution The $(s+2)^4$ in the denominator suggests that we work with the formula

$$\mathscr{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\}(t) = e^{at}t^n .$$

Here we have $a = -2$ and $n = 3$, so $\mathscr{L}^{-1}\{6/(s+2)^4\}(t) = e^{-2t}t^3$. Using the linearity property, we find

$$\mathscr{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}(t) = \frac{5}{6}\mathscr{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\}(t) = \frac{5}{6}e^{-2t}t^3 . \blacklozenge$$

Example 4 Determine $\mathcal{L}^{-1}\left\{\dfrac{3s+2}{s^2+2s+10}\right\}$.

Solution By completing the square, the quadratic in the denominator can be written as

$$s^2 + 2s + 10 = s^2 + 2s + 1 + 9 = (s+1)^2 + 3^2 .$$

The form of $F(s)$ now suggests that we use one or both of the formulas

$$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\}(t) = e^{at}\cos bt ,$$

$$\mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\}(t) = e^{at}\sin bt .$$

$$\mathcal{L}^{-1}\left\{\frac{3s+2}{s^2+2s+10}\right\}(t) = 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\}(t) - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+3^2}\right\}(t)$$

$$= 3e^{-t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t . \ \blacklozenge$$

Example 5 Determine $\mathcal{L}^{-1}\{F\}$, where

$$F(s) = \frac{7s-1}{(s+1)(s+2)(s-3)}.$$

Solution We begin by finding the partial fraction expansion for $F(s)$. The denominator consists of three distinct linear factors, so the expansion has the form

$$\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}, \tag{6}$$

where A, B, and C are real numbers to be determined.

$$7s - 1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2), ^{\dagger} \tag{7}$$

which reduces to

$$7s - 1 = (A + B + C)s^2 + (-A - 2B + 3C)s + (-6A - 3B + 2C).$$

Equating the coefficients of s^2, s, and 1 gives the system of linear equations

$$\begin{aligned} A + B + C &= 0, \\ -A - 2B + 3C &= 7, \\ -6A - 3B + 2C &= -1. \end{aligned}$$

Solving this system yields $A = 2$, $B = -3$, and $C = 1$. Hence,

$$\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3}. \tag{8}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{2}{s+1}-\frac{3}{s+2}+\frac{1}{s-3}\right\}(t) \\ &= 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)-3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) \\ &\quad +\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}(t) \\ &= 2e^{-t}-3e^{-2t}+e^{3t}. \ \blacklozenge\end{aligned}$$

Example 6 Determine $\mathscr{L}^{-1}\left\{\dfrac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$.

Solution Since $s-1$ is a repeated linear factor with multiplicity two and $s+3$ is a nonrepeated linear factor, the partial fraction expansion has the form

$$\frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}.$$

We begin by multiplying both sides by $(s-1)^2(s+3)$ to obtain

$$s^2+9s+2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2. \quad (9)$$

$$s^2+9s+2 = (A+C)s^2 + (2A+B-2C)s + (-3A+3B+C).$$

Then, equating the corresponding coefficients of s^2, s, and 1 and solving the resulting system, we again find $A=2$, $B=3$, and $C=-1$.

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{2}{s-1}+\frac{3}{(s-1)^2}-\frac{1}{s+3}\right\}(t) \\
&= 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t)+3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t) \\
&\quad -\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}(t) \\
&= 2e^t+3te^t-e^{-3t}\,. \quad \blacklozenge
\end{aligned}
$$

Example 7 Determine $\mathscr{L}^{-1}\left\{\dfrac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$.

Solution We first observe that the quadratic factor $s^2 - 2s + 5$ is irreducible (check the sign of the discriminant in the quadratic formula). Next we write the quadratic in the form $(s - \alpha)^2 + \beta^2$ by completing the square:

$$s^2 - 2s + 5 = (s - 1)^2 + 2^2 .$$

Since $s^2 - 2s + 5$ and $s + 1$ are nonrepeated factors, the partial fraction expansion has the form

$$\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} = \frac{A(s - 1) + 2B}{(s - 1)^2 + 2^2} + \frac{C}{s + 1} .$$

When we multiply both sides by the common denominator, we obtain

(11) $$2s^2 + 10s = [A(s - 1) + 2B](s + 1) + C(s^2 - 2s + 5) .$$

Hence, $A = 3$, $B = 4$, and $C = -1$ so that

$$\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} = \frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1} .$$

$$\mathscr{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}(t) = \mathscr{L}^{-1}\left\{\frac{3(s-1)+2(4)}{(s-1)^2+2^2} - \frac{1}{s+1}\right\}(t)$$

$$= 3\mathscr{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\}(t)$$

$$+ 4\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}(t) - \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)$$

$$= 3e^t\cos 2t + 4e^t\sin 2t - e^{-t}\,. \quad \blacklozenge$$

Laplace Transform of the Derivative

Theorem 4. Let $f(t)$ be continuous on $[0, \infty)$ and $f'(t)$ be piecewise continuous on $[0, \infty)$, with both of exponential order α. Then, for $s > \alpha$,

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0) . \qquad (2)$$

Proof. Since $\mathcal{L}\{f'\}$ exists, we can use integration by parts [with $u = e^{-st}$ and $dv = f'(t)dt$] to obtain

$$\text{(3)} \qquad \mathcal{L}\{f'\}(s) = \int_0^\infty e^{-st} f'(t)\, dt = \lim_{N\to\infty} \int_0^N e^{-st} f'(t)\, dt$$

$$= \lim_{N\to\infty} \left[e^{-st} f(t) \Big|_0^N + s \int_0^N e^{-st} f(t)\, dt \right]$$

$$= \lim_{N\to\infty} e^{-sN} f(N) - f(0) + s \lim_{N\to\infty} \int_0^N e^{-st} f(t)\, dt$$

$$= \lim_{N\to\infty} e^{-sN} f(N) - f(0) + s\mathcal{L}\{f\}(s) .$$

To evaluate $\lim_{N\to\infty} e^{-sN} f(N)$, we observe that since $f(t)$ is of exponential order α, there exists a constant M such that for N large,

$$|e^{-sN} f(N)| \le e^{-sN} M e^{\alpha N} = M e^{-(s-\alpha)N} .$$

Hence, for $s > \alpha$,

$$0 \le \lim_{N\to\infty} |e^{-sN} f(N)| \le \lim_{N\to\infty} M e^{-(s-\alpha)N} = 0 ,$$

Laplace Transform of Higher-Order Derivatives

Theorem 5. Let $f(t), f'(t), \ldots, f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$, with all these functions of exponential order α. Then, for $s > \alpha$,

(4) $$\mathscr{L}\{f^{(n)}\}(s) = s^n \mathscr{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0).$$

$$\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0).$$

$$\mathscr{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0).$$

Example 1 Solve the initial value problem

$$(1) \qquad y'' - 2y' + 5y = -8e^{-t}; \qquad y(0) = 2, \qquad y'(0) = 12.$$

Solution The differential equation in (1) is an identity between two functions of t. Hence equality holds for the Laplace transforms of these functions:

$$\mathcal{L}\{y'' - 2y' + 5y\} = \mathcal{L}\{-8e^{-t}\}\,.$$

Using the linearity property of $\mathcal{L}$ and the previously computed transform of the exponential function, we can write

$$\text{(2)} \qquad \mathcal{L}\{y''\}(s) - 2\mathcal{L}\{y'\}(s) + 5\mathcal{L}\{y\}(s) = \frac{-8}{s+1}\,.$$

Now let $Y(s) := \mathcal{L}\{y\}(s)$. From the formulas for the Laplace transform of higher-order derivatives (see Section 7.3) and the initial conditions in (1), we find

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 2\,,$$
$$\mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 12\,.$$

Substituting these expressions into (2) and solving for $Y(s)$ yields

$$[s^2Y(s) - 2s - 12] - 2[sY(s) - 2] + 5Y(s) = \frac{-8}{s+1}$$

$$(s^2 - 2s + 5)Y(s) = 2s + 8 - \frac{8}{s+1}$$

$$(s^2 - 2s + 5)Y(s) = \frac{2s^2 + 10s}{s+1}$$

$$Y(s) = \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)} .$$

Our remaining task is to compute the inverse transform of the rational function $Y(s)$. This was done in Example 7

(3) $$y(t) = 3e^t \cos 2t + 4e^t \sin 2t - e^{-t} ,$$

which is the solution to the initial value problem (1). ◆

Example 2 Solve the initial value problem

$$(4) \qquad y'' + 4y' - 5y = te^t; \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution Let $Y(s) := \mathscr{L}\{y\}(s)$. Taking the Laplace transform of both sides of the differential equation in (4) gives

$$\mathscr{L}\{y''\}(s) + 4\mathscr{L}\{y'\}(s) - 5Y(s) = \frac{1}{(s-1)^2}. \tag{5}$$

Using the initial conditions, we can express $\mathscr{L}\{y'\}(s)$ and $\mathscr{L}\{y''\}(s)$ in terms of $Y(s)$. That is,

$$\mathscr{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 1,$$
$$\mathscr{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s.$$

Substituting back into (5) and solving for $Y(s)$ gives

$$[s^2Y(s) - s] + 4[sY(s) - 1] - 5Y(s) = \frac{1}{(s-1)^2}$$
$$(s^2 + 4s - 5)Y(s) = s + 4 + \frac{1}{(s-1)^2}$$
$$(s+5)(s-1)Y(s) = \frac{s^3 + 2s^2 - 7s + 5}{(s-1)^2}$$
$$Y(s) = \frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3}.$$

The partial fraction expansion for $Y(s)$ has the form

$$\frac{s^3+2s^2-7s+5}{(s+5)(s-1)^3} = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}. \tag{6}$$

Solving for the numerators, we ultimately obtain $A = 35/216$, $B = 181/216$, $C = -1/36$, and $D = 1/6$. Substituting these values into (6) gives

$$Y(s) = \frac{35}{216}\left(\frac{1}{s+5}\right) + \frac{181}{216}\left(\frac{1}{s-1}\right) - \frac{1}{36}\left(\frac{1}{(s-1)^2}\right) + \frac{1}{12}\left(\frac{2}{(s-1)^3}\right),$$

where we have written $D = 1/6 = (1/12)2$ to facilitate the final step of taking the inverse transform. From the tables, we now obtain

$$y(t) = \frac{35}{216}e^{-5t} + \frac{181}{216}e^t - \frac{1}{36}te^t + \frac{1}{12}t^2e^t \tag{7}$$

as the solution to the initial value problem (4). ◆

EXAMPLE 2

Find the solution of the differential equation

$$y'' + y = \sin 2t \tag{19}$$

satisfying the initial conditions

$$y(0) = 2, \qquad y'(0) = 1. \tag{20}$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 2/(s^2 + 4),$$

where the transform of $\sin 2t$ has been obtained from line 5 of Table 6.2.1. Substituting for $y(0)$ and $y'(0)$ from the initial conditions and solving for $Y(s)$, we obtain

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}. \tag{21}$$

Using partial fractions, we can write $Y(s)$ in the form

$$Y(s) = \frac{as + b}{s^2 + 1} + \frac{cs + d}{s^2 + 4} = \frac{(as + b)(s^2 + 4) + (cs + d)(s^2 + 1)}{(s^2 + 1)(s^2 + 4)}. \tag{22}$$

By expanding the numerator on the right side of Eq. (22) and equating it to the numerator in Eq. (21), we find that

$$2s^3 + s^2 + 8s + 6 = (a + c)s^3 + (b + d)s^2 + (4a + c)s + (4b + d)$$

for all s. Then, comparing coefficients of like powers of s, we have

$$a + c = 2, \qquad b + d = 1,$$

$$4a + c = 8, \qquad 4b + d = 6.$$

Consequently, $a = 2, c = 0, b = \frac{5}{3}$, and $d = -\frac{2}{3}$, from which it follows that

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}. \tag{23}$$

From lines 5 and 6 of Table 6.2.1, the solution of the given initial value problem is

$$y = \phi(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t. \tag{24}$$

EXAMPLE 3

Find the solution of the initial value problem

$$y^{(4)} - y = 0, \tag{25}$$

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 0. \tag{26}$$

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - Y(s) = 0.$$

Then, using the initial conditions (26) and solving for $Y(s)$, we have

$$Y(s) = \frac{s^2}{s^4 - 1}. \tag{27}$$

A partial fraction expansion of $Y(s)$ is

$$Y(s) = \frac{as + b}{s^2 - 1} + \frac{cs + d}{s^2 + 1},$$

and it follows that

$$(as + b)(s^2 + 1) + (cs + d)(s^2 - 1) = s^2 \tag{28}$$

for all s. By setting $s = 1$ and $s = -1$, respectively, in Eq. (28), we obtain the pair of equations

$$2(a + b) = 1, \qquad 2(-a + b) = 1,$$

and therefore $a = 0$ and $b = \frac{1}{2}$. If we set $s = 0$ in Eq. (28), then $b - d = 0$, so $d = \frac{1}{2}$. Finally, equating the coefficients of the cubic terms on each side of Eq. (28), we find that $a + c = 0$, so $c = 0$. Thus

$$Y(s) = \frac{1/2}{s^2 - 1} + \frac{1/2}{s^2 + 1}, \tag{29}$$

and from lines 7 and 5 of Table 6.2.1, the solution of the initial value problem (25), (26) is

$$y = \phi(t) = \frac{\sinh t + \sin t}{2}. \tag{30}$$