Ch3: Differentiation

Def The derivative of a function f at point x_0 is defined by

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$ provided this limit exists

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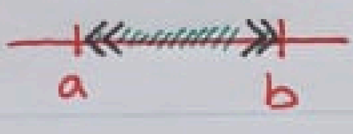
Notes: • If $f'(x_0)$ exists, then we say f is differentiable at x_0

• f is differentiable on closed interval $[a,b]$ if

→ f is differentiable on the open interval (a,b)
"f is diff on every point in (a,b)"

and

→ The right-hand derivative of f at a exists



$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

and

→ The left-hand derivative of f at b exists

$$f'_-(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

• diff means differentiable

• $f'(x_0)$ means f prime of x_0
$f''(x_0)$ means f double prime of x_0
$f'''(x_0)$ means f triple prime of x_0
$f''''(x_0)$ means f super four of x_0
⋮
$f^{(n)}(x_0)$ means f super n of x_0

Remarks. f is diff at $x = c$ iff

1. $f'_+(c)$ exists and
2. $f'_-(c)$ exists and
3. $f'_+(c) = f'_-(c)$

- If f is diff at $x = c$ then f is cont. at $x = c$

The converse is not true

Exp $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

f is cont. at $x = 0$ but $\leftarrow$ since

$\Downarrow$

$\lim\limits_{x \to 0} f(x) = 0 = f(0)$

$\lim\limits_{x \to 0^+} |x| = \lim\limits_{x \to 0^-} |x| = 0$

f is not diff at $x = 0$

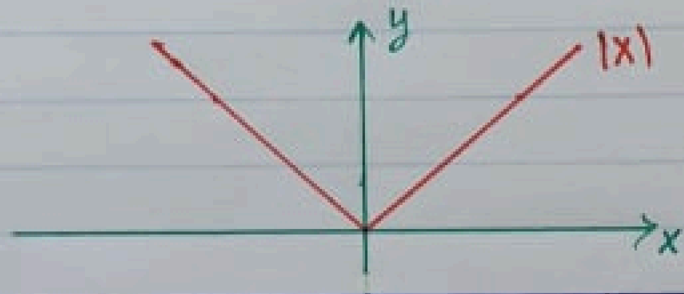
$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h - 0}{h} = 1$$

$$f'_-(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h - 0}{h} = -1$$

Hence, $f'_+(0) \neq f'_-(0) \Rightarrow f$ is not diff at $x = 0$

$f'(0)$ DNE



Th (Differentiation Rules)

Assume $f(x)$ and $g(x)$ are diff at x. Then

1. $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$
2. $\left(f(x)\, g(x)\right)' = f(x)\, g'(x) + g(x)\, f'(x)$
3. $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{g(x) f'(x) - f(x) g'(x)}{g^2(x)}$
4. $(f \circ g)'(x) = f'\left(g(x)\right) g'(x)$ Chain Rule

Note
- $f'(x) = \dfrac{df}{dx} = \left(f(x)\right)' = Df = \dfrac{d}{dx} f$
- $(f \circ g)(x) = f\left(g(x)\right)$

Exp Find $\left.\dfrac{d}{dx}\left(\dfrac{x^2 - 3}{1 - x}\right)\right|_{x=-1}$

$$= \left.\frac{(1-x)(2x) - (x^2-3)(-1)}{(1-x)^2}\right|_{x=-1}$$

$$= \frac{(2)(-2) - (-2)(-1)}{(2)^2}$$

$$= \frac{-4 - 2}{4} = \frac{-6}{4} = -\frac{3}{2}$$

Derivatives of Trigonometric functions

1. $(\sin x)' = \cos x$
2. $(\cos x)' = -\sin x$
3. $(\tan x)' = \sec^2 x$
4. $(\cot x)' = -\csc^2 x$
5. $(\sec x)' = \sec x \tan x$
6. $(\csc x)' = -\csc x \cot x$

- $(x^n)' = n x^{n-1}$ where $n \in \mathbb{R}$
- $(c)' = 0$ where c is constant

Exp Let $f(x) = \tan(\sqrt{x})$. Find tangent line at $x = \pi^2$

- Tangent line $y - y_0 = m(x - x_0)$ where the point is

$$(x_0, y_0) = (\pi^2, f(\pi^2)) = (\pi^2, \tan\sqrt{\pi^2}) = (\pi^2, \tan\pi) = (\pi^2, 0)$$

and the slope is

$$m = f'(\pi^2) = \sec^2(\sqrt{x}) \frac{1}{2\sqrt{x}} \Big|_{x=\pi^2} = \underbrace{\sec^2\pi}_{(-1)^2} \frac{1}{2\pi} = \frac{1}{2\pi}$$

- Hence, the tangent line becomes

$$y - 0 = \frac{1}{2\pi}(x - \pi^2)$$

$$\boxed{y = \frac{1}{2\pi}x - \frac{\pi}{2}}$$
