Ch 7 Numerical Integration

* How can we estimate $\int_a^b f(x)\,dx$?

We use quadrature formula $Q[f]$.

Def (Quadrature Formula)

- Suppose that $a = x_0 < x_1 < \cdots < x_m = b$.
- A formula of the form $Q[f] = \sum_{k=0}^{m} w_k f(x_k)$

$$= w_0 f(x_0) + w_1 f(x_1) + \cdots + w_m f(x_m)$$

with the property that

$$\int_a^b f(x)\,dx = Q[f] + E[f]$$

is called a numerical integration (or quadrature formula).

- $E[f]$ is called the truncation error for integration.
- $x_0, x_1, \ldots, x_m$ are called the quadrature nodes.
- $w_0, w_1, \ldots, w_m$ are called the weights.

* We will study two types of $Q[f]$:

[1] Closed Newton-Cotes Quadrature Formula:

(a) Trapezoidal Rule

(b) Simpson's Rule

(c) Simpson's $\frac{3}{8}$ Rule

[2] Gauss-legendre Formula:

(a) $G_1(f)$

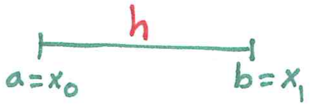
(b) $G_2(f)$

(c) $G_3(f)$

Closed Newton - Cotes Quadrature Formula

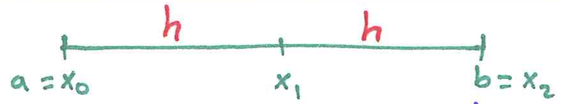
Th • Assume that $x_k = x_0 + kh$ are equally spaced nodes with $f_k = f(x_k)$

• Then [a] Trapezoidal Rule is



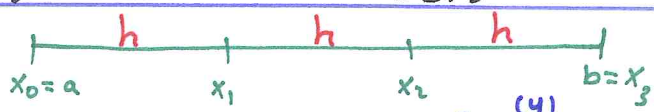
$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx \approx \underbrace{\frac{h}{2}(f_0 + f_1)}_{Q[f]} \text{ with error } \underbrace{\frac{-h^3 f''(c)}{12}}_{E[f]}$$

[b] Simpson's Rule is



$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx \approx \underbrace{\frac{h}{3}(f_0 + 4f_1 + f_2)}_{Q[f]} \text{ with error } \underbrace{\frac{-h^5 f^{(4)}(c)}{90}}_{E[f]}$$

[c] Simpson's 3/8 Rule is



$$\int_a^b f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx \approx \underbrace{\frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3)}_{Q[f]} \text{ with error } \underbrace{\frac{-3h^5 f^{(4)}(c)}{80}}_{E[f]}$$

Exp Estimate $\int_0^1 \left(1 + e^{-x}\sin(4x)\right) dx$ using

[1] Trapezoidal Rule

$$\int_0^1 (1 + e^{-x}\sin(4x))\,dx \approx \frac{h}{2}(f_0 + f_1)$$

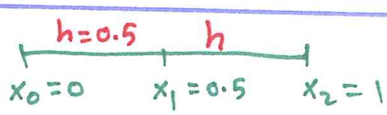
$f_0 = f(0) = 1$

$f_1 = f(1) = 0.72159$

$$= \frac{1}{2}(1 + 0.72159)$$

$$= 0.86079$$

[2] Simpson's Rule



$$\int_0^1 (1 + e^{-x}\sin(4x))\,dx \approx \frac{0.5}{3}\left(f(0) + 4f(0.5) + f(1)\right)$$

$$= \frac{1}{6}\left(1 + 4(1.55152) + 0.72159\right)$$

$$= 1.32128$$

[3] Simpson's $\frac{3}{8}$ Rule

$h = \frac{1}{3}$

$x_0 = 0, \quad x_1 = \frac{1}{3}, \quad x_2 = \frac{2}{3}, \quad x_3 = 1$

$$\int_0^1 (1 + e^{-x}\sin(4x))\,dx \approx \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3)$$

$$= \frac{3(\frac{1}{3})}{8}\left[1 + 3(1.69642) + 3(1.23447) + 0.72159\right]$$

$$= 1.31440$$

Exp Derive the Trapezoidal Rule $\int_a^b f(x) \approx \frac{h}{2}(f_0 + f_1)$

- Since $a = x_0$ and $b = x_1 \Rightarrow n = 1$

h: $x_0 = a$, $x_1 = b$

- Use Newton Interpolation $\Rightarrow f(x) \approx P_1(x)$

$$P_1(x) = a_0 + a_1(x - x_0)$$

- Hence, $\int_a^b f(x)\,dx \approx \int_a^b P_1(x)\,dx = \int_{x_0}^{x_1} (a_0 + a_1(x - x_0))\,dx$

$$= a_0(x_1 - x_0) + a_1 \left.\frac{(x - x_0)^2}{2}\right|_{x_0}^{x_1}$$

$$= a_0 h + \frac{a_1 h^2}{2}$$

or change of variables

$x - x_0 = ht$

$dx = h\,dt$

$x = x_0 \Rightarrow t = 0$

$x = x_1 \Rightarrow t = 1$

$\int_{x_0}^{x_1} (a_0 + a_1(x - x_0))\,dx = \int_0^1 (a_0 + a_1 ht)\,h\,dt$

$= h a_0 + \frac{a_1 h^2}{2}$

- But $a_0 = f[x_0] = f(x_0) = f_0$

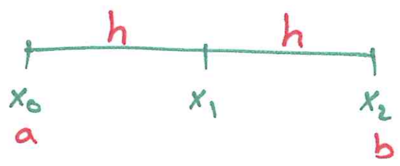
$$a_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f_1 - f_0}{h}$$

- Thus, $\int_a^b f(x)\,dx \approx f_0 h + \frac{f_1 - f_0}{2\cancel{h}} \cancel{h^2}$

$$= \frac{h}{2}(f_0 + f_1)$$

EXP Derive the Simpson's Rule $\int_a^b f(x)\,dx \simeq \frac{h}{3}\left(f_0 + 4f_1 + f_2\right)$ $\boxed{145}$

- Here $n = 2$ since we have 3 points



- Use Lagrange Poly. $\Rightarrow f(x) \approx P_2(x)$

where $P_2(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

- Hence, $\int_a^b f(x)\,dx = \int_{x_0}^{x_2} P_2(x)\,dx$

$$= \int_{x_0}^{x_2} \frac{f_0}{2h^2}(x-x_1)(x-x_2)\,dx + \int_{x_0}^{x_2} \frac{-f_1}{h^2}(x-x_0)(x-x_2)\,dx$$

$$+ \int_{x_0}^{x_2} \frac{f_2}{2h^2}(x-x_0)(x-x_1)\,dx$$

- Use the following change of variables:

$$\left.\begin{aligned} x - x_0 &= ht \\ dx &= h\,dt \end{aligned}\right\} \Rightarrow \text{when } \begin{aligned} x = x_0 &\Rightarrow t = 0 \\ x = x_2 &\Rightarrow t = 2 \end{aligned}$$

- Note that $x - x_1 = x - (x_0 + h) = (x - x_0) - h = ht - h = h(t-1)$

$x - x_2 = x - (x_0 + 2h) = (x - x_0) - 2h = ht - 2h = h(t-2)$

- Hence, $\int_{x_0}^{x_2} P_2(x) = \int_0^2 \frac{f_0}{2\cancel{h^2}} \cancel{h}(t-1)\cancel{(h)}(t-2)\,h\,dt - \int_0^2 \frac{f_1}{\cancel{h^2}} \cancel{(h}t)\cancel{(h)}(t-2)\,h\,dt$

$$+ \int_0^2 \frac{f_2}{2\cancel{h^2}} (\cancel{h}t)\cancel{(h)}(t-1)\,h\,dt$$

$$= \frac{hf_0}{2}\int_0^2 (t^2 - 3t + 2)\,dt - hf_1 \int_0^2 (t^2 - 2t)\,dt + \frac{hf_2}{2}\int_0^2 (t^2 - t)\,dt$$

$$= \frac{hf_0}{2}\left(\frac{t^3}{3} - \frac{3}{2}t^2 + 2t\right)\Bigg|_0^2 - hf_1\left(\frac{t^3}{3} - t^2\right)\Bigg|_0^2 + \frac{hf_2}{2}\left(\frac{t^3}{3} - \frac{t^2}{2}\right)\Bigg|_0^2$$

$$= \frac{hf_0}{\cancel{2}}\left(\frac{\cancel{2}}{3}\right) - hf_1\left(\frac{-4}{3}\right) + \frac{hf_2}{\cancel{2}}\left(\frac{\cancel{2}}{3}\right)$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2\right]$$

The Degree of Precision (DP)

- Recall that the quadrature formula:

$$\int_a^b f(x)\,dx = Q[f] + E[f]$$

- To derive the truncation error $E[f]$ for any quadrature formula $Q[f]$, we first study the Degree of Precision (DP) for this quadrature formula $Q[f]$.

Def. The DP of a quadrature formula $Q[f]$ is a positive integer n s.t $Q[f]$ is exact "$E[f]=0$" for $f_k = x^k$ where $k = 0, 1, 2, \dots, n$

- That is: $E[f_0] = E[x^0] = E[1] = \int_a^b dx - Q[1] = 0$

$$E[f_1] = E[x] = \int_a^b x\,dx - Q[x] = 0$$

$$E[f_2] = E[x^2] = \int_a^b x^2\,dx - Q[x^2] = 0$$

$$\vdots$$

$$E[f_n] = E[x^n] = \int_a^b x^n\,dx - Q[x^n] = 0$$

But $E[f_{n+1}] = E[x^{n+1}] = \int_a^b x^{n+1}\,dx - Q[x^{n+1}] \neq 0$

- In this case we have $n =$ DP of $Q[f]$

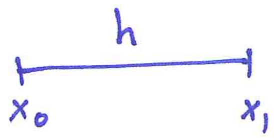
- We use n to find the truncation error $E[f]$ which has the general form:

* ... $E[f] = K f^{(n+1)}(c)$ where $c \in [a,b]$ and K is a constant that depends on the DP $= n$ and h.

Exp Determine the DP of the Trapezoidal Rule and use it to find the truncation error.

- Recall that the Trapezoidal Rule is:

$$\int_a^b f(x)\,dx = Q[f] + E[f]$$
$$= \frac{h}{2}\left[f_0 + f_1\right] + \frac{-h^3 f''(c)}{12}$$



- It will be enough to apply Trapezoidal Rule over the interval $[0,1] \Rightarrow$

$$\int_0^1 f(x)\,dx = \frac{1}{2}\left[f(0)+f(1)\right] + \frac{-h^3 f''(c)}{12}$$

- $\int_0^1 dx = 1 = \frac{1}{2}[1+1]$ with $E[1]=0$ since $f=1$

$\int_0^1 x\,dx = \frac{1}{2} = \frac{1}{2}[0+1]$ with $E[x]=0$ since $f=x$

$\int_0^1 x^2\,dx = \frac{1}{3} \neq \frac{1}{2}[0+1]$ with $E[x^2] = \frac{1}{3} - \frac{1}{2} = \frac{-1}{6} \neq 0$ since $f = x^2$

- Hence the DP $= n = 1$ for the Trapezoidal Rule.
- And so, by $*$ we have $E = K f^{(n+1)}(c) = K f''(c)$
- Now to find K, we consider $f(x) = (x-x_0)^{n+1} = (x-x_0)^2$

$\Rightarrow f'(x) = 2(x-x_0) \Rightarrow f''(x) = 2 \Rightarrow \boxed{E = 2K}$ ①

- But $E = \text{True} - \text{Estimate} = \int_{x_0}^{x_1} (x-x_0)^2\,dx - \frac{h}{2}\left(f(x_0) + f(x_1)\right)$

$$= \frac{(x-x_0)^3}{3}\Bigg|_{x_0}^{x_1} - \frac{h}{2}\left(0 + (x_1-x_0)^2\right)$$
$$= \frac{h^3}{3} - \frac{h^3}{2}$$
$$\boxed{E = \frac{-h^3}{6}} \text{ ②}$$

From ① and ② we have $2K = \frac{-h^3}{6}$

$\Leftrightarrow \boxed{K = \frac{-h^3}{12}}$ and hence, $E = K f''(c)$

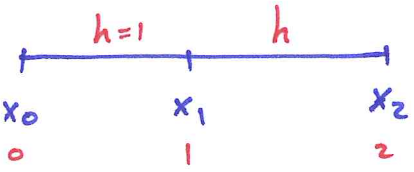
$$= \frac{-h^3}{12} f''(c)$$

Exp Determine the DP of the Simpson's Rule and use it to find the truncation error.

- Recall that the Simpson's Rule is:

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f_0 + 4f_1 + f_2\right] + \frac{-h^5 f^{(4)}(c)}{90}$$

- It will be enough to apply Simpson's Rule over the interval $[0,2]$ $\Rightarrow$



$$\int_0^2 f(x)\,dx = \frac{1}{3}\left[f(0) + 4f(1) + f(2)\right] - \frac{h^5 f^{(4)}(c)}{90}$$

- $\int_0^2 dx = 2 = \frac{1}{3}[1+4+1]$ with $E[1] = 0$ since $f = 1$

$\int_0^2 x\,dx = 2 = \frac{1}{3}[0+4+2]$ with $E[x] = 0$ since $f = x$

$\int_0^2 x^2\,dx = \frac{8}{3} = \frac{1}{3}[0+4+4]$ with $E[x^2] = 0$ since $f = x^2$

$\int_0^2 x^3\,dx = 4 = \frac{1}{3}[0+4+8]$ with $E[x^3] = 0$ since $f = x^3$

$\int_0^2 x^4\,dx = \frac{32}{5} \neq \frac{1}{3}[0+4+16] = \frac{20}{3}$ with $E[x^4] = \frac{32}{5} - \frac{20}{3} \neq 0$ since $f = x^4$

- Hence, the $DP = n = 3$ for the Simpson's Rule.

- And so, by $*$ the truncation error is

$$E = K f^{(n+1)}(c) = K f^{(4)}(c)$$

• Now to find K, we consider $f(x) = (x-x_0)^{n+1}$

$= (x-x_0)^4$

$$\Rightarrow f'(x) = 4(x-x_0)^3$$

$$f''(x) = 12(x-x_0)^2$$

$$f'''(x) = 24(x-x_0) \quad \Rightarrow \quad f^{(4)}(x) = 4!$$

$$\Rightarrow f^{(4)}(c) = 4!$$

• Hence, $E = K f^{(4)}(c)$

$$\boxed{E = 24K} \text{ — } ①$$

• But $E = \int_{x_0}^{x_2} (x-x_0)^4 \, dx - \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$

$$= \left.\frac{(x-x_0)^5}{5}\right|_{x_0}^{x_2} - \frac{h}{3}\left[0 + 4(x_1-x_0)^4 + (x_2-x_0)^4\right]$$

$$= \frac{(x_2-x_0)^5}{5} - \frac{h}{3}\left[0 + 4h^4 + (2h)^4\right]$$

$$= \frac{(2h)^5}{5} - \frac{h}{3}\left(4h^4 + 16h^4\right)$$

$$= \frac{32h^5}{5} - \frac{20h^5}{3}$$

$$\boxed{E = \frac{-4h^5}{15}} \text{ — } ②$$

• From ① and ② we get $24K = \frac{-4h^5}{15} \Rightarrow K = \frac{-h^5}{90}$

• Hence, $E = K f^{(4)}(c)$

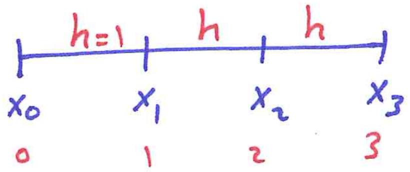
$$= \frac{-h^5 f^{(4)}(c)}{90}$$

Exp Determine the DP of the Simpson's $\frac{3}{8}$ Rule and use it to find the truncation error.

- Recall that the Simpson's $\frac{3}{8}$ Rule is :

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[f_0 + 3f_1 + 3f_2 + f_3\right] + \frac{-3h^5 f^{(4)}(c)}{80}$$

- It will be enough to apply Simpson's $\frac{3}{8}$ Rule over the interval $[0, 3]$ $\Rightarrow$



$$\int_0^3 f(x)\,dx = \frac{3}{8}\left[f(0) + 3f(1) + 3f(2) + f(3)\right] - \frac{3h^5 f^{(4)}(c)}{80}$$

- $\int_0^3 dx = 3 = \frac{3}{8}[1+3+3+1]$ with $E[1] = 0$ since $f = 1$

$\int_0^3 x\,dx = \frac{9}{2} = \frac{3}{8}[0+3+6+3]$ with $E[x] = 0$ since $f = x$

$\int_0^3 x^2\,dx = 9 = \frac{3}{8}[0+3+12+9]$ with $E[x^2] = 0$ since $f = x^2$

$\int_0^3 x^3\,dx = \frac{81}{4} = \frac{3}{8}[0+3+24+27]$ with $E[x^3] = 0$ since $f = x^3$

$\int_0^3 x^4\,dx = \frac{243}{5} \neq \frac{3}{8}[0+3+48+81] = \frac{99}{2}$ with $E[x^4] = \frac{243}{5} - \frac{99}{2} \neq 0$ since $f = x^4$

- Hence, the DP $= n = 3$ for the Simpson's $\frac{3}{8}$ Rule.
- And so, by $*$ the truncation error is

$$E = K f^{(n+1)}(c) = K f^{(4)}(c)$$

- Now to find K, we consider $f(x) = (x-x_0)^{n+1}$

$= (x-x_0)^4$

$\Rightarrow f^{(4)}(c) = 4!$

- Hence, $E = K f^{(4)}(c)$

$$\boxed{E = 24K} - (1)$$

- But $E = $ True Value $-$ Estimate Value

$$= \int_{x_0}^{x_3} (x-x_0)^4 dx - \frac{3}{8} h\left[f_0 + 3f_1 + 3f_2 + f_3\right]$$

$$= \frac{(x-x_0)^5}{5}\Bigg|_{x_0}^{x_3} - \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

$$= \frac{(x_3-x_0)^5}{5} - \frac{3h}{8}\left[0 + 3(x_1-x_0)^4 + 3(x_2-x_0)^4 + (x_3-x_0)^4\right]$$

$$= \frac{(3h)^5}{5} - \frac{3h}{8}\left[3h^4 + 3(2h)^4 + (3h)^4\right]$$

$$= \frac{243h^5}{5} - \frac{99h^5}{2}$$

$$\boxed{E = \frac{-9h^5}{10}} - (2)$$

- From (1) and (2) we get $24K = \frac{-9h^5}{10} \Rightarrow K = \frac{-3h^5}{80}$

- Hence, $E = K f^{(4)}(c)$

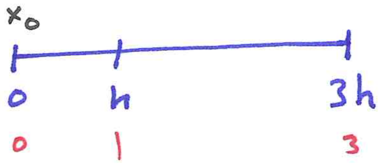
$$= \frac{-3h^5 f^{(4)}(c)}{80}$$

Exp Given the following quadrature formula: 152

$$\int_0^{3h} f(x)\,dx \approx Q[f] = \frac{3h}{4}\left[3f(h) + f(3h)\right].$$

Find its DP and its truncation error $E[f]$.

$$\int_0^{3} f(x)\,dx \approx \frac{3}{4}\left[3f(1) + f(3)\right]$$



- $\int_0^3 dx = 3 = \frac{3}{4}[3+1]$ with $E[1]=0$ since $f=1$

$\int_0^3 x\,dx = \frac{9}{2} = \frac{3}{4}[3+3]$ with $E[x]=0$ since $f=x$

$\int_0^3 x^2\,dx = 9 = \frac{3}{4}[3+9]$ with $E[x^2]=0$ since $f=x^2$

$\int_0^3 x^3\,dx = \frac{81}{4} \neq \frac{3}{4}[3+27] = \frac{90}{4}$ with $E[x^3] = \frac{81}{4} - \frac{90}{4} = \frac{-9}{4} \neq 0$ since $f=x^3$

- Hence, $DP = n = 2$ and therefore, $E = K f^{(n+1)}(c) = K f'''(c)$

- Take $f(x) = x^3 \xrightarrow{\text{since } x_0=0} \Rightarrow f'''(c) = 3! = 6 \Rightarrow$ so $\boxed{E = 6K}$ — ①

- But $E = \text{True} - \text{Estimate} = \int_0^{3h} x^3\,dx - \frac{3h}{4}\left[3f(h) + f(3h)\right]$

$$\left.\frac{x^4}{4}\right|_0^{3h} - \frac{3h}{4}\left[3h^3 + 27h^3\right]$$

$$= \frac{81h^4}{4} - \frac{90h^4}{4}$$

$$\boxed{E = \frac{-9h^4}{4}} \text{ ②}$$

- From ① and ② we get

$6K = \frac{-9h^4}{4} \Rightarrow K = \frac{-3h^4}{8}$

Hence, $E = K f'''(c)$

$$= \frac{-3h^4 f'''(c)}{8} \checkmark$$

Exp Given the following quadrature formula:

$$\int_{-1}^{1} f(x)\,dx \approx Q[f] = \frac{1}{2}\left[f(-1) + 3f\left(\tfrac{1}{3}\right)\right]$$

Find its DP and its truncation error $E[f]$.

- $\int_{-1}^{1} dx = 2 = \frac{1}{2}[1+3]$ with $E[1]=0$ since $f=1$

  $\int_{-1}^{1} x\,dx = 0 = \frac{1}{2}[-1+1]$ with $E[x]=0$ since $f=x$

  $\int_{-1}^{1} x^2\,dx = \frac{2}{3} = \frac{1}{2}\left[1+\frac{1}{3}\right]$ with $E[x^2]=0$ since $f=x^2$

  $\int_{-1}^{1} x^3\,dx = 0 \neq \frac{1}{2}\left[-1+\frac{1}{9}\right] = \frac{-4}{9}$ with $E[x^3] = 0 - \frac{-4}{9} = \frac{4}{9} \neq 0$ since $f=x^3$

- Hence, $DP = n = 2$ and therefore $E = K f^{(n+1)}(c) = K f'''(c)$

- Now Take $f(x) = (x-x_0)^{(n+1)} = (x+1)^3$ $\Rightarrow f'''(c) = 3! = 6$

  $\Rightarrow \boxed{E = 6K}$ (1)

- But $E = \int_{-1}^{1}(x+1)^3\,dx - \frac{1}{2}\left[f(-1) + 3f\left(\tfrac{1}{3}\right)\right]$

  $= \left.\frac{(x+1)^4}{4}\right|_{-1}^{1} - \frac{1}{2}\left[0 + 3\left(\tfrac{1}{3}+1\right)^3\right]$

  $= 4 - \frac{32}{9}$

  $\boxed{E = \frac{4}{9}}$ (2)

- From (1) and (2) we get

  $6K = \frac{4}{9}$

  $\boxed{K = \frac{2}{27}}$

  Hence, $E = K f'''(c)$

  $= \frac{2 f'''(c)}{27}$

Composite Trapezoidal Rule (CTR)

This method approximates the area under the curve $y = f(x)$ over $[a,b]$ using a series of trapezoids that lie above the intervals $\{[x_k, x_{k+1}]\}$.

Th (CTR)

- Assume that the interval $[a,b]$ is subdivided into M subintervals $[x_k, x_{k+1}]$ each of width $h = \frac{b-a}{M}$ using equally spaced nodes $x_k = a + kh$ for $k = 0, 1, 2, \dots, M$:

- Then, the composit trapezoidal rule is

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_M} f(x)\,dx \approx T(f,h) = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{M-1}}^{x_M} f(x)\,dx$$

$$= \frac{h}{2}(f_0 + f_1) + \frac{h}{2}(f_1 + f_2) + \cdots + \frac{h}{2}(f_{M-1} + f_M)$$

$$= \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{M-1} + f_M\right)$$

- Furthermore, the total error of $T(f,h)$ is

$$E_T(f,h) = \frac{-h^3 f''(c)}{12} \cdot M = \frac{-h^3 f''(c)}{12} \cdot \frac{b-a}{h} = \frac{-h^2 f''(c)}{12}(b-a)$$

Note that
- Number of points is $M+1$
- M is the number of subintervals = number of subintegrals = number of composition

Exp Use CTR to estimate $\int_2^7 e^x dx$ with 10 composition. (use 4 chopping digits). $\boxed{155}$

- $h = \frac{b-a}{M} = \frac{7-2}{10} = \frac{5}{10} = 0.5$ and $f(x) = e^x = (2.718)^x$

a	h	h	h	h	h	h	h	h	h	h	b
2		2.5	3	3.5	4	4.5	5	5.5	6	6.5	7
x_0		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}

- $\int_2^7 e^x dx \approx T(e^x, 0.5)$

$$= \frac{h}{2}\left[f_0 + 2f_1 + 2f_2 + 2f_3 + 2f_4 + 2f_5 + 2f_6 + 2f_7 + 2f_8 + 2f_9 + f_{10}\right]$$

$$= \frac{0.5}{2}\left[(2.718)^2 + 2(2.718)^{2.5} + 2(2.718)^3 + \cdots + 2(2.718)^{6.5} + (2.718)^7\right]$$

$$= 0.25\left[7.387 + 2(12.17) + 2(20.07) + \cdots + 2(664.6) + 1095\right]$$

$$= 0.25\left[7.387 + 24.34 + 40.14 + 66.2 + 109.1 + 179.9 + 296.6 + 489 + 806.2 + 1329 + 1095\right]$$

$$= 0.25(4333)$$

$$= 1083$$

True Vaule is $\int_2^7 e^x dx = e^x \Big|_2^7 = e^7 - e^2$

$= 1089.2441023295$

Exp Given

x	0	2	4	6
f(x)	10	15	-10	8

Use CTR to estimate $\int_0^6 f(x)dx$.

$$\int_0^6 f(x)dx \approx \frac{h}{2}\left[f_0 + 2f_1 + 2f_2 + f_3\right] = \frac{2}{2}\left[10 + 2(15) + 2(-10) + 8\right] = 28$$

Exp Given

x	0	2	3	6
f(x)	10	15	-10	8

Use CTR to estimate $\int_0^6 f(x)dx$.

$$\int_0^6 f(x)dx \approx \int_0^2 f(x)dx + \int_2^3 f(x)dx + \int_3^6 f(x)dx = \frac{2}{2}\left[f_0 + f_1\right] + \frac{1}{2}\left[f_1 + f_2\right] + \frac{3}{2}\left[f_2 + f_3\right]$$

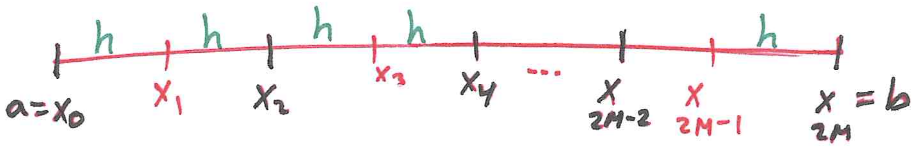
$$= \left[10 + 15\right] + \frac{1}{2}\left[15 - 10\right] + \frac{3}{2}\left[-10 + 8\right] = 24.5$$

Composit Simpson Rule (CSR)

This method approximates the area under the curve $y = f(x)$ over $[a,b]$.

Th (CSR)

- Assume that the interval $[a,b]$ is subdivided into $2M$ subintervals $[x_k, x_{k+1}]$ each of width $h = \frac{b-a}{2M}$ using equally spaced nodes $x_k = a + kh$ for $k = 0,1,2,\dots,2M$:



- Then, the Composite Simpson Rule is

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_{2M}} f(x)\,dx \approx S(f,h) = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \dots + \int_{x_{2M-2}}^{x_{2M}} f(x)\,dx$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2\right] + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right] + \dots + \frac{h}{3}\left[f_{2M-2} + 4f_{2M-1} + f_{2M}\right]$$

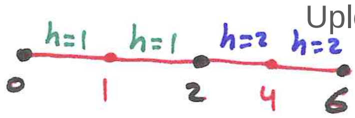
- Furthermore, the total error of $S(f,h)$ is

$$E_S(f,h) = \frac{-h^5 f^{(4)}(c)}{90}\cdot M = \frac{-h^4 f^{(4)}(c)}{180}(b-a)$$

Exp Given

x	0	1	2	4	6
$f(x)$	2	-1	3	0	10

Estimate $\int_0^6 f(x)\,dx$ using CSR.



$$\int_0^6 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx$$

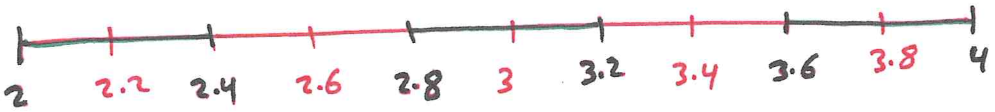
$$= \frac{1}{3}\left[f(0) + 4f(1) + f(2)\right] + \frac{2}{3}\left[f(2) + 4f(4) + f(6)\right]$$

$$= \frac{1}{3}\left[2 - 4 + 3\right] + \frac{2}{3}\left[3 + 0 + 10\right]$$

$$= \frac{1}{3} + \frac{26}{3} = \frac{27}{3} = 9$$

Exp Use CSR to estimate $\int_2^4 e^x\,dx$ with 5 compositions. $\boxed{157}$
Use 4 chopping digits.

- $h = \dfrac{b-a}{2M} = \dfrac{4-2}{2(5)} = \dfrac{2}{10} = 0.2$ and $f(x) = e^x = (2.718)^x$



- $\int_2^4 e^x\,dx = \int_2^{2.4} e^x\,dx + \int_{2.4}^{2.8} e^x\,dx + \int_{2.8}^{3.2} e^x\,dx + \int_{3.2}^{3.6} e^x\,dx + \int_{3.6}^{4} e^x\,dx$

$$= \frac{0.2}{3}\left[f(2)+4f(2.2)+f(2.4)\right] + \frac{0.2}{3}\left[f(2.4)+4f(2.6)+f(2.8)\right]$$

$$+ \frac{0.2}{3}\left[f(2.8)+4f(3)+f(3.2)\right] + \frac{0.2}{3}\left[f(3.2)+4f(3.4)+f(3.6)\right]$$

$$+ \frac{0.2}{3}\left[f(3.6)+4f(3.8)+f(4)\right]$$

$$= 0.06666\left[(2.718)^2 + 4(2.718)^{2.2} + 2(2.718)^{2.4} + 4(2.718)^{2.6} + 2(2.718)^{2.8} + 4(2.718)^3 + 2(2.718)^{3.2} + 4(2.718)^{3.4} + 2(2.718)^{3.6} + 4(2.718)^{3.8} + (2.718)^4\right]$$

$$= 0.06666\left[7.387 + 36.08 + 22.04 + 53.84 + 32.86 + 80.28 + 49.04 + 119.8 + 73.16 + 178.7 + 54.57\right]$$

$$= 0.06666(707.4)$$

$$= 47.15$$

Note that the True Value is $\int_2^4 e^x\,dx = e^x\Big|_2^4 = e^4 - e^2 = 47.2090939342$

Exp Find the number of compositions and the step size needed to estimate $\int_2^7 \frac{dx}{x}$ with accuracy 5×10^{-9} using

$\boxed{1}$ CTR

$\boxed{2}$ CSR

$\boxed{1}$ $|E| = \left| \frac{-h^2 f''(c)}{12}(b-a) \right| \leq 5\times10^{-9}$

$a=2$, $b=7$

$f(x) = \frac{1}{x}$

$f' = -\frac{1}{x^2}$

$f'' = \frac{2}{x^3} \leq \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4}$

$$\left| \frac{h^2 (\frac{1}{4})}{12} (\cancel{7-2}) \right| \leq \cancel{5}\times10^{-9}$$

$$h \leq \sqrt{12\times10^{-9}\times4} = 0.000219089$$

$$M = \frac{b-a}{h} \geq \frac{5}{0.000219089} = 22821.77562543$$

so the number of compositions is $M \geq 22822$ and # of points $= M+1$

$\boxed{2}$ $|E| = \left| \frac{h^4 f^{(4)}(c)}{180}(b-a) \right| \leq 5\times10^{-9}$

$f'''(x) = -\frac{6}{x^4}$

$f^{(4)} = \frac{24}{x^5} \leq \frac{24}{2^5} = \frac{24}{32} = \frac{3}{4}$

$$\frac{h^4 (\frac{3}{4})(\cancel{5})}{\cancel{180}_{60}} \leq \cancel{5}\times10^{-9}$$

$$h \leq (240\times10^{-9})^{\frac{1}{4}} = 0.0221336384$$

$$M = \frac{b-a}{2h} \geq \frac{5}{0.0442672768} = 112.9502504206$$

Hence, $M \geq 113$

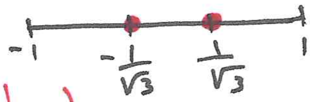
Gauss-Legendre Integration (optional)

To estimate the area under the curve $y=f(x)$, $-1 \le x \le 1 \Rightarrow$ one can use one of the following formulas:

1 Gauss-Legendre one-point Rule:

$$\int_{-1}^{1} f(x)\,dx \approx G_1(f) = 2 f_0$$

2 Gauss-Legendre two-points Rule:

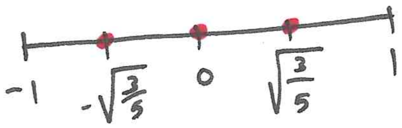


$$\int_{-1}^{1} f(x)\,dx \approx G_2(f) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Gauss-Legendre two-points Rule $G_2(f)$ has Degree of Precision $\boxed{DP = 2n-1 = 3}$ and error $\boxed{E_2[f] = \frac{f^{(4)}(c)}{135}}$

3 Gauss-Legendre three-points Rule

$$\int_{-1}^{1} f(x)\,dx \approx G_3(f) = \frac{5 f\left(-\sqrt{\frac{3}{5}}\right) + 8 f(0) + 5 f\left(\sqrt{\frac{3}{5}}\right)}{9}$$



$G_3(f)$ has Degree of Precision $\boxed{DP = 2n-1 = 5}$ and error $\boxed{E_3[f] = \frac{f^{(6)}(c)}{15750}}$

Exp. Given $\int_{-1}^{1} \frac{dx}{x+2} = \ln(x+2)\Big|_{-1}^{1} = \ln(3) - \ln(1) \approx 1.09861$

- Estimate this integral using [1] $G_2(f)$ [2] $G_3(f)$ [3] $T(f,h)$ with $h=2$ [4] $S(f,h)$ with $h=1$

Use 4 chopping

[1] $\int_{-1}^{1} \frac{dx}{x+2} \approx G_2(f) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$ $\qquad f(x) = \frac{1}{x+2}$

$$= f(-0.5773) + f(0.5773)$$

$$= \frac{1}{1.422} + \frac{1}{2.577} = 0.7032 + 0.3880$$

$$= 1.091$$

[2] $\int_{-1}^{1} \frac{dx}{x+2} \approx G_3(f) = \frac{5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)}{9}$

$$= \frac{5f(-0.7745) + 8f(0) + 5f(0.7745)}{9}$$

$$= \frac{4.081 + 4 + 1.802}{9} = \frac{9.883}{9} = 1.098$$

[3] $\int_{-1}^{1} \frac{dx}{x+2} \approx T(f,2) = \frac{h}{2}\left[f(-1) + f(1)\right] = f(-1) + f(1)$

$$= 1 + 0.3333 = 1.333$$

[4] $\int_{-1}^{1} \frac{dx}{x+2} \approx S(f,1) = \frac{h}{3}\left[f(-1) + 4f(0) + f(1)\right]$

$$= 0.3333\left[1 + 2 + 0.3333\right] = 0.3333(3.333) = 1.110$$

Exp Estimate $\int_{-1}^{1} \sin x \, dx$ using [1] $G_2(f)$ [2] $G_3(f)$

1) $\int_{-1}^{1} \sin x \, dx \approx G_2(f) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = -\sin\frac{1}{\sqrt{3}} + \sin\frac{1}{\sqrt{3}} = 0$

2) $\int_{-1}^{1} \sin x \, dx \approx G_3(f) = \frac{5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)}{9} = 0$

Gauss - Legendre Translation

- How to use $G_2(f)$ and $G_3(f)$ to estimate $\int_a^b f(x)\,dx$
- We use change of variables to transform the limits of integration from $[a,b]$ to $[-1,1]$:

$$x = \frac{a+b}{2} + \frac{b-a}{2}t \quad \Rightarrow \quad dx = \frac{b-a}{2}\,dt$$

- when $t=-1 \Rightarrow x = \frac{a+b}{2} - \frac{b-a}{2} = a$

  $t = 1 \Rightarrow x = \frac{a+b}{2} + \frac{b-a}{2} = b$

- Hence, $\displaystyle\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2}t\right)\frac{b-a}{2}\,dt$

$$= \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2}t\right)dt$$

$\Downarrow$

$$G_2(f) = \frac{b-a}{2}\left[f\left(\frac{a+b}{2} + \frac{b-a}{2}\left(\frac{-1}{\sqrt{3}}\right)\right) + f\left(\frac{a+b}{2} + \frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right)\right)\right]$$

and

$$G_3(f) = \frac{b-a}{2}\left[\frac{5f\left(\frac{a+b}{2} + \frac{b-a}{2}\left(-\sqrt{\frac{3}{5}}\right)\right) + 8f\left(\frac{a+b}{2}\right) + 5f\left(\frac{a+b}{2} + \frac{b-a}{2}\left(\sqrt{\frac{3}{5}}\right)\right)}{9}\right]$$

Exp Use two-points Gauss - Legendre rule to approximate $\boxed{162}$

$$\int_1^5 e^{-x}\,dx = -e^{-x}\Big|_1^5 = -e^{-5} + e^{-1} = 0.3746173882$$

- $x = \frac{a+b}{2} + \frac{b-a}{2}t = 3 + 2t$

$dx = 2\,dt$

- $\int_1^5 e^{-x}\,dx = \int_{-1}^{1} e^{-(3+2t)}\,2\,dt = 2\left[f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right]$

$$= 2\left[e^{-(3+\frac{-2}{\sqrt{3}})} + e^{-(3+\frac{2}{\sqrt{3}})}\right] = 0.3473369892$$

Exp Use three-points Gauss - Legendre rule to estimate

$$\int_1^5 \frac{dx}{x} = \ln x\Big|_1^5 = \ln 5 = 1.609437912 4$$. Use 4 chopping digits.

- $x = \frac{a+b}{2} + \frac{b-a}{2}t = 3 + 2t$ with $dx = 2\,dt$

- Now ➜ $\int_1^5 \frac{dx}{x} = \int_{-1}^{1} \frac{2\,dt}{3+2t}$ with $f(t) = \frac{2}{3+2t}$

$$= \frac{5f(-\sqrt{\frac{3}{5}}) + 8f(0) + 5f(\sqrt{\frac{3}{5}})}{9}$$

$$= \frac{1}{9}\left[5f(-0.7745) + 8f(0) + 5f(0.7745)\right]$$

$$= 0.1111\left[5(1.378) + 8(0.6666) + 5(0.4396)\right]$$

$$= 0.1111\left[6.890 + 5.332 + 2.198\right]$$

$$= 0.1111(14.41)$$

$$= 1.6$$