Ch2

2.1 1^{st} order linear DE with variable coefficients (Method of Integrating factors)

- Recall that in ch1 we have solved any 1^{st} order linear DE with constant coefficients of the form

$$y' = ay - b \quad , \quad a \neq 0 \, , \quad y(0) = y_0 \quad \cdots \text{(A)}$$

whose sol. is $y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}$ --- A^*

Exp How to solve 1^{st} order linear DE with variable coefficients of the form:

$$y' + p(t)y = g(t) \quad \cdots \text{(B)}$$

- Note that the DE (B) is more general than (A).
- This means that (A) is special case of (B).
- If $p(t)$ and $g(t)$ are constants, then (B) becomes (A).
- Hence, the sol. of (B) will solve (A).

- Here the method of calculus does not work, so we look for new method called Integrating factor.

- The idea of this method is to multiply the DE (B) by a positive function $\mu(t)$ so that the resulting equation is easy to integrate:

$$\underbrace{\mu(t)}_{\text{الأول}} \underbrace{\frac{dy}{dt}}_{\text{مشتقة الثاني}} + \underbrace{y}_{\text{الثاني}} \boxed{\underbrace{\mu(t)p(t)}_{\text{مشتقة الأول}}} = \mu(t)g(t)$$

$$\mu'(t) = \mu(t)p(t)$$

$$\left(\mu(t)\, y(t)\right)' = \mu(t)\, g(t)$$

$$\mu(t)\, y(t) = \int \mu(t)\, g(t)\, dt + c$$

Hence, the general sol. of the DE Ⓑ is

$$\boxed{y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\, g(t)\, dt + c\right]} \rightarrow \overset{*}{B}$$

$$\frac{\mu'(t)}{\mu(t)} = p(t)$$

$$\ln|\mu(t)| = \int p(t)\, dt$$

$$\ln(\mu(t)) = \int p(t)\, dt$$

$$\boxed{\mu(t) = e^{\int p(t)\, dt}}$$

integrating factor

Exp Solve the IVP: $y' + 2y - 4 = 0$, $y(0) = 1$

t_0 y_0

Sol. 1 :• This DE has the form of Ⓐ $\Rightarrow$

$$y' = -2y + 4 \quad \text{with} \quad \left.\begin{matrix} a = -2 \\ b = -4 \end{matrix}\right\} \Rightarrow \frac{b}{a} = 2$$

• Apply $\overset{*}{A}$ $\Rightarrow$

$$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}$$

$$y(t) = 2 + (1 - 2)\, e^{-2t}$$

✓ $y(t) = 2 - e^{-2t}$ Note $\lim_{t\to\infty} y(t) = 2$ = Eq. Sol.

Sol. 2 • We can write this DE in the form of Ⓑ $\Rightarrow$

$$y' + 2y = 4 \quad \text{with} \quad p(t) = 2 \quad \text{and} \quad g(t) = 4$$

• $\mu(t) = e^{\int p(t) dt} = e^{\int 2\, dt} = e^{2t}$ is the integrating factor

• The gen. sol. $\overset{*}{B}$ is $y(t) = \frac{1}{\mu}\left[\int \mu g\, dt + c\right]$

$$y(t) = \frac{1}{e^{2t}}\left[\int 4 e^{2t} dt + c\right] = e^{-2t}\left[2 e^{2t} + c\right]$$

$y(t) = 2 + c\, e^{-2t}$ To find c we use the IC $\Rightarrow$

$y(0) = 2 + c\, e^{0}$

$1 = 2 + c$

$\boxed{c = -1}$

The sol. becomes

$y(t) = 2 - e^{-2t}$ ✓

Exp Solve the IVP:

$$ty' - 2y = 5t^2, \quad t>0, \quad y(1) = 2$$

- Since the DE is 1^{st} order linear with variable coefficients
  $\Rightarrow$ we can only use B^* to solve it

- But first we arrange the DE of the form B to write $p(t)$ and $g(t)$ correctly:

$$y' - \frac{2}{t}y = 5t, \quad p(t) = \frac{-2}{t}, \quad g(t) = 5t$$

- Integrating factor $\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{-2}{t}\,dt} = e^{-2\ln|t|}$

$$= e^{-2\ln t} = e^{\ln t^{-2}} = t^{-2} = \frac{1}{t^2}$$

- Apply $B^* \Rightarrow$

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\,g(t)\,dt + c\right]$$

$$y(t) = \frac{1}{\frac{1}{t^2}}\left[\int \frac{1}{t^2}(5t)\,dt + c\right] = t^2\left[5\ln t + c\right]$$

- To find c we use the IC

$$y(1) = 1^2[5\ln 1 + c]$$
$$2 = [0 + c]$$
$$2 = c$$

- Hence, the gen. sol. is $y(t) = t^2(5\ln t + 2)$

Exp Given the IVP:

$$y' - 2xy - x = 0 \quad , y(0) = \alpha$$

Find α so that the sol. approaches $\frac{-1}{2}$ as $x \longrightarrow \infty$.

- We need to find α s.t $\lim\limits_{x \to \infty} y(x) = \frac{-1}{2}$
- So first we find $y(x) \Rightarrow$ Apply Ⓑ and Ⓑ*

$$y' - 2xy = x \quad , p(x) = -2x \quad , g(x) = x$$

- $\mu(x) = e^{\int p(x)\,dx} = e^{\int -2x\,dx} = e^{-x^2}$ is the integrating factor
- Apply $B^* \Rightarrow y(x) = \frac{1}{\mu(x)}\left[\int \mu(x) g(x)\,dx + c\right]$

$$= \frac{1}{e^{-x^2}}\left[\int e^{-x^2}(x)\,dx + c\right]$$

$$y(x) = e^{x^2}\left[\frac{-1}{2} e^{-x^2} + c\right]$$

- To find c we use the IC $\Rightarrow y(0) = e^{0}\left[\frac{-1}{2} e^{0} + c\right] = \alpha$

$$\frac{-1}{2} + c = \alpha \Rightarrow \boxed{c = \alpha + \frac{1}{2}}$$

- The sol. becomes $y(x) = e^{x^2}\left[-\frac{1}{2} e^{-x^2} + \alpha + \frac{1}{2}\right]$

$$\boxed{y(x) = \frac{-1}{2} + \left(\alpha + \frac{1}{2}\right) e^{x^2}}$$

- But $\lim\limits_{x \to \infty} y(x) = \cancel{\frac{-1}{2}} = \lim\limits_{x \to \infty}\left(\cancel{\frac{-1}{2}} + \left(\alpha + \frac{1}{2}\right) e^{x^2}\right)$

$$0 = \lim_{x \to \infty}\left(\alpha + \frac{1}{2}\right) e^{x^2} \Leftrightarrow \boxed{\alpha = \frac{-1}{2}}$$

2.2 Separable DE's

<u>Remark</u>. Any 1^{st} order DE can be written as

$$y' = \frac{dy}{dt} = f(t, y)$$

- If we can separate the variables one on each side, then the DE is called separable
- Separable DE's can be solved by integrating each side with respect to its variable.

<u>Exp</u> Solve the DE: $\frac{dy}{dx} = \frac{x^2}{1-y^2}$ Sep. DE

$$\int (1-y^2)\, dy = \int x^2\, dx$$

nonlinear ✓

$$\boxed{y - \frac{y^3}{3} = \frac{x^3}{3} + C} \Rightarrow \text{Implicit solution}$$

<u>Exp</u> Solve the IVP: $\frac{dy}{dx} = \frac{y\cos x}{1+3y^3}$, $y(0) = 1$

$$(1+3y^3)\, dy = y\cos x\, dx$$

nonlinear ✓

$$\int \left(\frac{1}{y} + 3y^2\right) dy = \int \cos x\, dx$$

Sep. DE

$$\ln|y| + y^3 = \sin x + c$$

To find c we use IC:

$$\ln 1 + 1^3 = \sin 0 + c$$

$$\boxed{1 = c}$$

$$\boxed{\ln|y| + y^3 = \sin x + 1} \Rightarrow \text{Implicit solution}$$

Exp Consider the IVP: $\frac{dy}{dx} = \frac{3x^2+4x+2}{2y-2}$, $y(0)=-1$

[1] Solve this IVP for implicit sol.

$\boxed{\text{sep. DE}}$
nonlinear ✓

$$\int (2y-2)\,dy = \int (3x^2+4x+2)\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

⇒ To find c
⇒ $x=0$, $y=-1$

$$(-1)^2 - 2(-1) = 0 + 0 + 0 + c$$

$$1 + 2 = c \implies \boxed{c = 3}$$

$$\boxed{y^2 - 2y = x^3 + 2x^2 + 2x + 3}$$ ⇒ Implicit Solution

[2] Find Explicit Solution

$$y^2 - 2y + 1 = x^3 + 2x^2 + 2x + 4$$

$$(y-1)^2 = x^3 + 2x^2 + 2x + 4$$

$$|y-1| = \sqrt{x^2(x+2) + 2(x+2)}$$

$$y - 1 = \pm\sqrt{(x^2+2)(x+2)}$$

$$y(x) = 1 \pm \sqrt{(x^2+2)(x+2)}$$

$$\boxed{y_1(x) = 1 - \sqrt{(x^2+2)(x+2)}}$$ ✓ only sol. since it satisfies the IC: $y(0)=-1$

$$y_2(x) = 1 + \sqrt{(x^2+2)(x+2)}$$ X not solution since it does not satisfy the IC: $y(0)=-1$

[3] Find interval where the sol. is defined

The sol. is defined on

$$I = (-2, \infty)$$

Exp Solve the IVP: $y' = y^{\frac{1}{3}}$, $y(0) = 0$

$$\frac{dy}{dt} = y^{\frac{1}{3}} \Rightarrow \int y^{-\frac{1}{3}}\, dy = \int dt$$

$\boxed{\text{sep. DE}}$
nonlinear ✓

$$\frac{3}{2} y^{\frac{2}{3}} = t + c$$

To find c we use IC:

$$\frac{3}{2}(0) = 0 + c \Rightarrow \boxed{c = 0}$$

$$\frac{3}{2} y^{\frac{2}{3}} = t$$

$$\left(\frac{3}{2}\right)^3 y^2 = t^3 \Rightarrow y^2 = \frac{8}{27} t^3$$

$$|y| = \sqrt{\frac{8}{27} t^3} \Rightarrow y(t) = \pm\sqrt{\frac{8}{27} t^3}$$

$y_1(t) = \sqrt{\frac{8}{27} t^3}$ is sol. since it also satisfies the IC

$y_2(t) = -\sqrt{\frac{8}{27} t^3}$ is sol. since it also satisfies the IC

$y_3(t) = 0$ is also sol.

Note that if $y(0) = 1$ in the above Exp, then there is a unique sol. whic is

$$y(t) = \sqrt{\left(\frac{2}{3} t + 1\right)^3}$$

Exp solve the DE $\frac{dy}{dx} = -\frac{4x+3y}{2x+y}$

- $(2x+y)\,dy = -(4x+3y)\,dx$ — not sep. DE, nonlinear
- This DE is not separable

Question Can we change it to sep. DE?

Answer: Yes if the DE is homogenuous. That is

$$y' = \frac{dy}{dx} = F(x,y)$$

If we can rewrite F as function of $V = \frac{y}{x}$ then the DE is homogenuous and so the DE can be changed to separable DE

$V = \frac{y}{x}$

$y = xV$

$y' = x\frac{dV}{dx} + V$

$$y' = \frac{dy}{dx} = -\frac{4+3\frac{y}{x}}{2+\frac{y}{x}}$$

$$xV' + V = -\frac{4+3V}{2+V} \quad \text{Homo.}$$

$$x\frac{dV}{dx} = -V - \frac{4+3V}{2+V}$$

$$-x\frac{dV}{dx} = V + \frac{4+3V}{2+V}$$

$$= \frac{2V+V^2+4+3V}{2+V}$$

$$= \frac{V^2+5V+4}{2+V}$$

$$-x\frac{dV}{dx} = \frac{(V+1)(V+4)}{2+V}$$

$$\int \frac{2+v}{(v+1)(v+4)}\,dv = \int -\frac{dx}{x}$$ (sep. DE)

$$\int \left(\frac{A}{v+1} + \frac{B}{v+4}\right) dv = -\ln|x| + c$$

$$A = \frac{2+\boxed{-1}}{\boxed{-1}+4} = \frac{1}{3}$$

$$B = \frac{2+\boxed{-4}}{\boxed{-4}+1} = \frac{2}{3}$$

$$\int \left(\frac{\frac{1}{3}}{v+1} + \frac{\frac{2}{3}}{v+4}\right) dv = -\ln|x| + c$$

$$\frac{1}{3}\ln|v+1| + \frac{2}{3}\ln|v+4| = -\ln|x| + c$$

$$\frac{1}{3}\ln\left|\frac{y}{x}+1\right| + \frac{2}{3}\ln\left|\frac{y}{x}+4\right| = -\ln|x| + c$$

$$\frac{1}{3}\ln\left|\frac{y+x}{x}\right| + \frac{2}{3}\ln\left|\frac{y+4x}{x}\right| = -\ln|x| + c$$

$$\frac{1}{3}\ln|y+x| - \cancel{\frac{1}{3}\ln|x|} + \frac{2}{3}\ln|y+4x| - \cancel{\frac{2}{3}\ln|x|} = \cancel{-\ln|x|} + c$$

$$\frac{1}{3}\ln|y+x| + \frac{2}{3}\ln|y+4x| = c$$

Implicit solution

Exp ① $y' = \dfrac{x^3+y^3}{x^3-y^3}$ is homogenous DE

② $\dfrac{dy}{dx} = \dfrac{x^2-y}{x}$ is not homogenous DE

2.3 Modeling with 1^{st} Order DE's

The idea here is to construct MM's that characterize problems in physical, biological and social sciences using DE's.

Newton's law of Cooling: The rate of change of the Temperature $T(t)$ "heat loss" of an object is proportional to the difference between its own Temperature T and the ambient Temperature T_m (the temperature of its surroundings).

That is, the MM (IVP) that describes this phenomena is

$$\frac{dT}{dt} = \alpha (T - T_m) \quad , \; T(0) = T_0 \quad , \; \alpha < 0$$

This DE has the form of (A) $\Rightarrow$

$$T' = \alpha T - \alpha T_m \quad \text{with } a = \alpha, \; b = \alpha T_m$$

Hence, to find its solution $T(t)$ we apply (A*)

$$T(t) = \frac{b}{a} + \left(T_0 - \frac{b}{a}\right) e^{at}$$

$$\frac{b}{a} = \frac{\alpha T_m}{\alpha} = T_m$$

Eq. Sol.

$$\boxed{T(t) = T_m + (T_0 - T_m)\, e^{\alpha t}} \; - *$$

- So any problem obeys to Newton's law of Cooling according to the DE above can be solved directly using $*$.

- If $\alpha > 0 \Rightarrow$ the problem becomes heating instead of cooling.

Exp Suppose that the temperature of a cup of coffee obeys to Newton's law of cooling. If the coffee has temperature of 200 F when freshly poured, and one minute later has cooled to 190 F in a room at 70 F. How long will it take the coffee to reach temperature of 150 F.

• Let $T(t)$ be the temperature of the cup at time t

$$T(t) = T_m + (T_0 - T_m)\, e^{\alpha t}$$

$$= 70 + (200-70)\, e^{\alpha t}$$

$T_0 = 200$
$T_m = 70$
$T(1) = 190$

$$\boxed{T(t) = 70 + 130\, e^{\alpha t}}$$

$$T(1) = 70 + 130\, e^{\alpha}$$
$$190 = 70 + 130\, e^{\alpha}$$
$$120 = 130\, e^{\alpha}$$
$$\frac{12}{13} = e^{\alpha}$$
$$\boxed{\alpha = \ln \frac{12}{13}}$$

We need to find the time t^* such that

$$T(t^*) = 150$$
$$70 + 130\, e^{\alpha t^*} = 150$$
$$130\, e^{\alpha t^*} = 80$$
$$e^{\alpha t^*} = \frac{8}{13}$$
$$\alpha t^* = \ln \frac{8}{13}$$

$$t^* = \frac{\ln \frac{8}{13}}{\alpha} = \frac{\ln \frac{8}{13}}{\ln \frac{12}{13}}$$

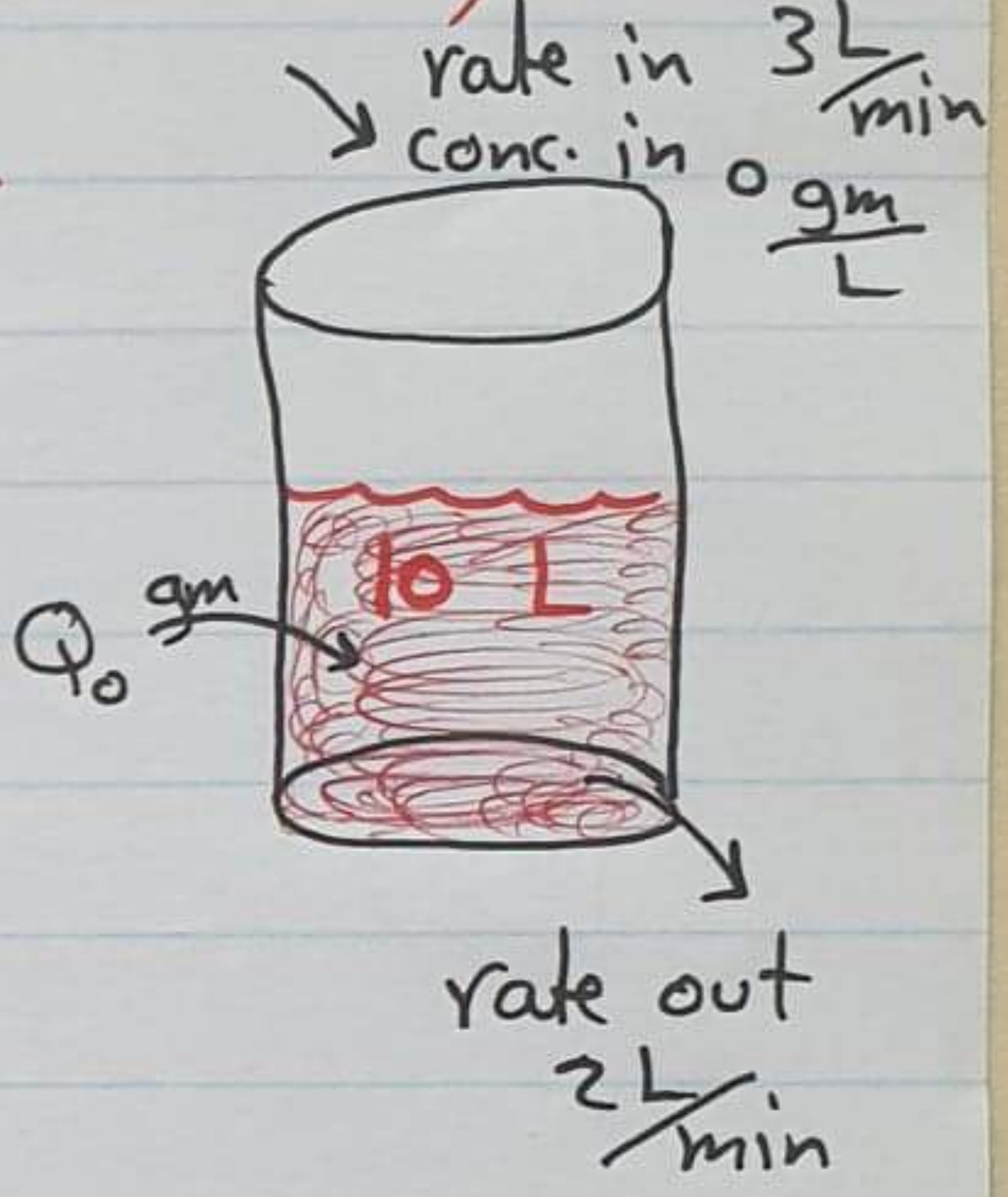
Exp At $t=0$, Q_0 gm of salt are dissolved in 10 L of water. Fresh water flows into the tank at rate 3L/min and the well-stirred mixture flows out at rate 2L/min. Denote the quantity of salt in the tank at time t by $Q(t)$.

1. Set up an IVP that describes this process

$$\frac{dQ}{dt} = (\text{rate in})(\text{Conc. in}) - (\text{rate out})(\text{conc. out})$$

$$= (3)(0) - (2)\frac{Q(t)}{10+t}$$

$$Q' = -\frac{2Q}{10+t}, \quad Q(0) = Q_0$$



2. Solve the IVP in 1
" Find the quantity of salt in the tank at any time t "

$$Q' + \frac{2}{10+t} Q = 0 \quad \dots (B) \quad \text{with } p(t) = \frac{2}{10+t}$$

$g(t) = 0$

$$\mu(t) = e^{\int p(t)\,dt} = e^{2\int \frac{dt}{10+t}} = e^{2\ln|10+t|} = (10+t)^2$$

Apply (B^*) $\Rightarrow Q(t) = \frac{1}{\mu(t)}\left[\int \mu(t) g(t)\,dt + c\right]$

$$Q(t) = \frac{c}{\mu(t)} = \frac{c}{(10+t)^2}$$

$$Q(0) = \frac{c}{100} = Q_0 \Rightarrow c = 100\,Q_0$$

$$Q(t) = \frac{100\,Q_0}{(10+t)^2}$$

(3) Assume at $t = 10$ min the quantity of salt in the tank is 20 gm. Find Q_0.

Since $Q(10) = 20 \Rightarrow$

$$\frac{100\, Q_0}{(10+10)^2} = 20$$

$$100\, Q_0 = (20)(400)$$

$$Q_0 = 80 \text{ gm}$$

[4] Find the time where the concentration of salt in the tank is $\frac{1}{8}$ g/L

We need to find time t^* s.t

$$\text{Concentration} = \frac{1}{8}$$

$$\frac{\text{Quantity}}{\text{Volume}} = \frac{1}{8}$$

$$\frac{\frac{100\, Q_0}{(10+t)^2}}{10+t} = \frac{1}{8}$$

$$\frac{100\, Q_0}{(10+t)^3} = \frac{1}{8}$$

$$800\, Q_0 = (10+t)^3$$

$$800\,(80) = (10+t)^3$$

$$(8)(8)(1000) = (10+t)^3$$

$$(2)(2)(10) = (10+t)$$

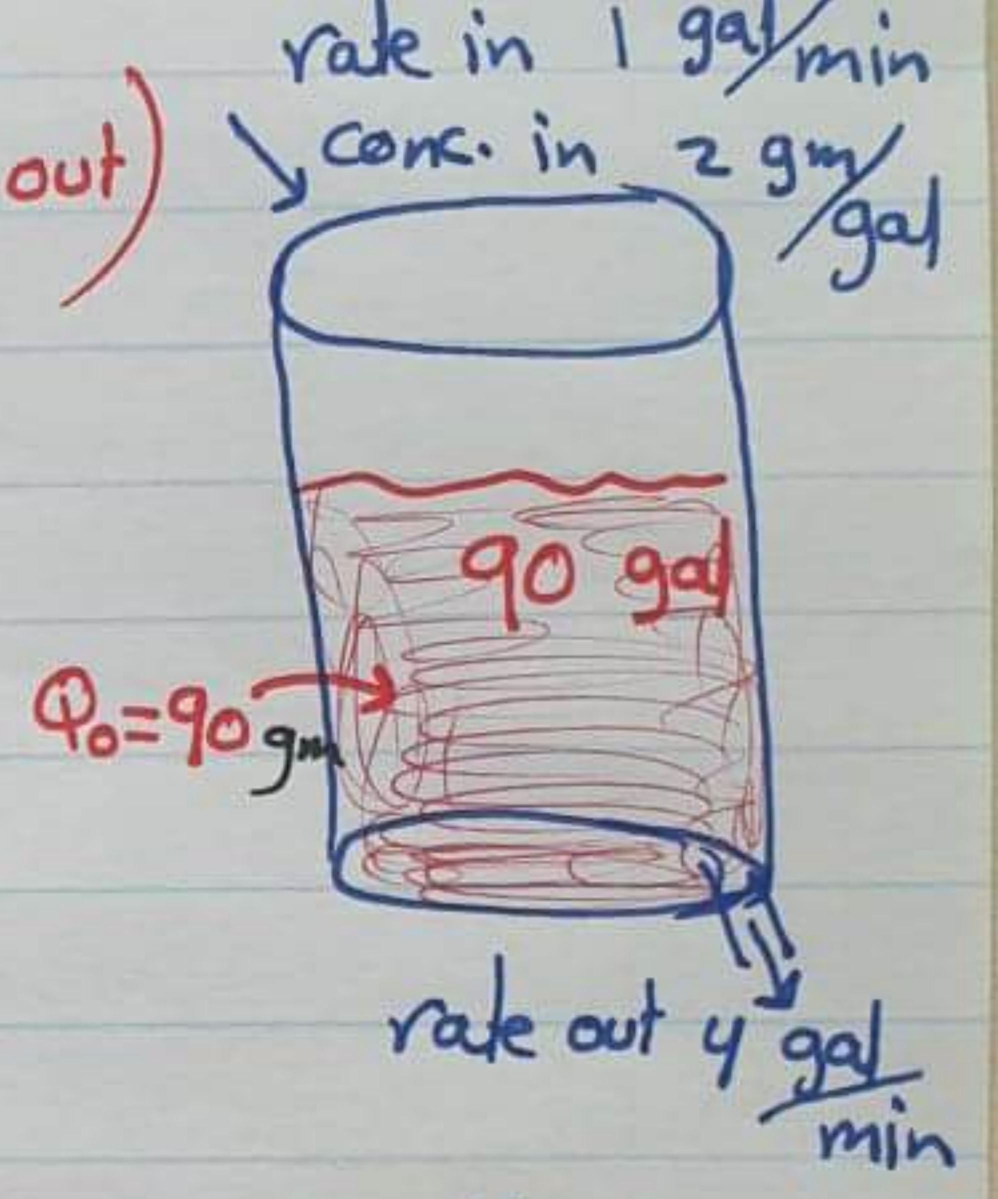
$$(2)(20) = 10+t$$

$$40 = 10+t$$

$$t = 30 \text{ min}$$

Exp A 120 gallon tank initially contains 90 gm of salt dissolved in 90 gallon of water. Water containing 2 gm/gal of salt enters the tank at rate of 1 gal/min. The well-stirred mixture flows out at rate of 4 gal/min.

(1) Set up the IVP that models the change on $Q(t)$, where $Q(t)$ is the amount of salt in the tank at time t



$$\frac{dQ}{dt} = (\text{rate in})(\text{conc. in}) - (\text{rate out})(\text{conc. out})$$

$$= (1)(2) - (4)\frac{Q(t)}{90-3t}$$

$$Q' = 2 - \frac{4Q}{90-3t} \quad , \quad Q(0) = 90$$

(2) Solve the IVP

$$Q' + \frac{4}{90-3t} Q = 2 \quad \cdots \text{(B)} \quad \text{with } p(t) = \frac{4}{90-3t}, \quad g(t) = 2$$

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{4\,dt}{90-3t}}$$

$$= e^{-\frac{4}{3}\ln|90-3t|} = (90-3t)^{-4/3}$$

$$Q(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\, g(t)\, dt + c\right]$$

$$= \frac{1}{(90-3t)^{-4/3}}\left[\int 2(90-3t)^{-4/3} dt + c\right]$$

$$= \frac{1}{(90-3t)^{-4/3}}\left[2\,\frac{(90-3t)^{\frac{-1}{3}}}{-\left(\frac{1}{3}\right)}\left(\frac{-1}{3}\right) + c\right]$$

$$Q(t) = 2\,(90-3t) + \frac{c}{(90-3t)^{-4/3}}$$

To find c we use the IC : $Q(0) = 90$

$$Q(0) = (2)\,(90) + \frac{c}{(90)^{-4/3}}$$

$$90 = (180) + \frac{c}{(90)^{-4/3}}$$

$$1 = 2 + \frac{c}{(90)^{-1/3}} \Rightarrow \frac{c}{(90)^{-1/3}} = -1 \Rightarrow c = -(90)^{\frac{-1}{3}}$$

$$Q(t) = 2\,(90-3t) - \frac{(90)^{-1/3}}{(90-3t)^{-4/3}}$$

$\boxed{3}$ When the tank becomes empty.

rate in 1 gal/min and rate out 4 $\frac{gal}{min}$ $\Rightarrow$

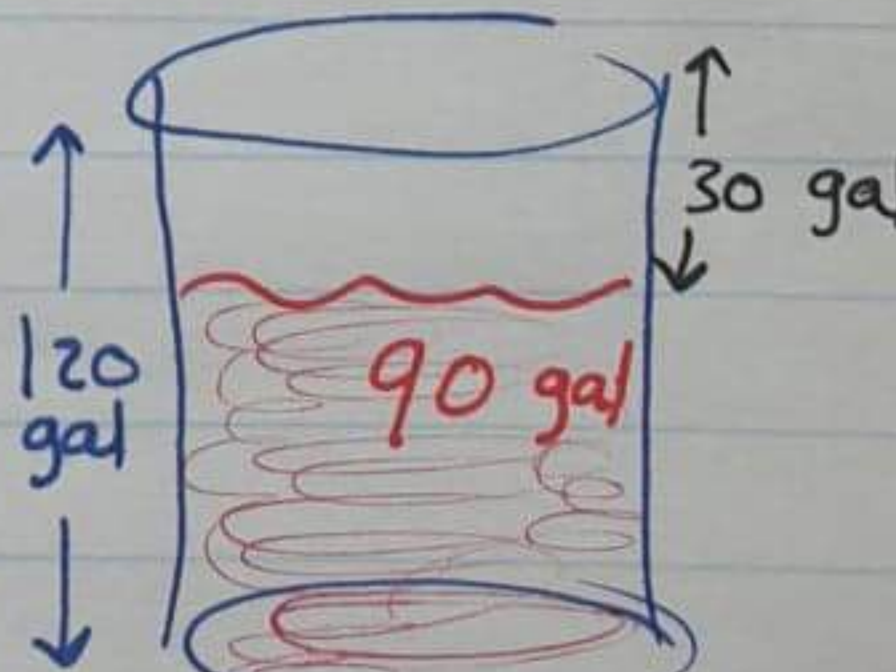
The tank looses 3 $\frac{gal}{min}$ $\Rightarrow \frac{90}{3} = 30$ min

Hence, after 30 min the tank becomes empty

Exp Consider the same Example above but with rate in 4 gal/min and rate out 3 gal/min and answer $\boxed{1}$ and $\boxed{2}$

$\boxed{1}$ $Q' = (2)(4) - (3)\dfrac{Q(t)}{90+t}$, $Q(0) = 90$

$\boxed{2}$ $Q(t) = 2(90+t) - 90\left(\dfrac{90}{90+t}\right)^3$



$\boxed{3}$ When the tank overflows?

after $t = 30$ min since the tank increases 1 gal/min

$\boxed{4}$ what is the quantity of salt in the tank when it becomes to overflow?

$$Q(30) = 2(90+30) - 90\left(\frac{90}{90+30}\right)^3$$

2.4 Difference Between linear and nonlinear DE's

- Recall that any 1^{st} order ODE has the general form

$$y' = \frac{dy}{dt} = f(t,y) \quad \cdots \ *$$

- The DE $*$ is linear if f is linear in y.
  Otherwise, the DE $*$ is nonlinear

Question: When the DE $*$ has a unique solution?
How can we find the interval in which the solution is defined?

Th 2.4.1 (linear)
Consider the 1^{st} order linear DE

$$\boxed{y' + p(t)y = g(t) \text{ with } y(t_0) = y_0} \quad \cdots \ \text{B}$$

If $p(t)$ and $g(t)$ are cont. on an open interval $I = (\alpha, \beta)$ containing t_0, then $\exists$ a unique solution $y(t) = \phi(t)$ satisfies the IVP B on I.

Proof Existence is done in section 2.1 pages 18 + 19

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t) g(t)\, dt + c\right], \quad \mu(t) = e^{\int p(t)\,dt}$$

Uniqueness $\mu(t) = e^{\int_{t_0}^{t} p(t)\,dt}$, $\quad y(t_0) = y_0$

$$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(t) g(t)\, dt + y_0\right]$$

Remark (2.4.1) If the DE is 1^{st} order linear (satisfies (B)), then we can find the interval $I = (\alpha, \beta)$ that contains t_0 without solving the DE

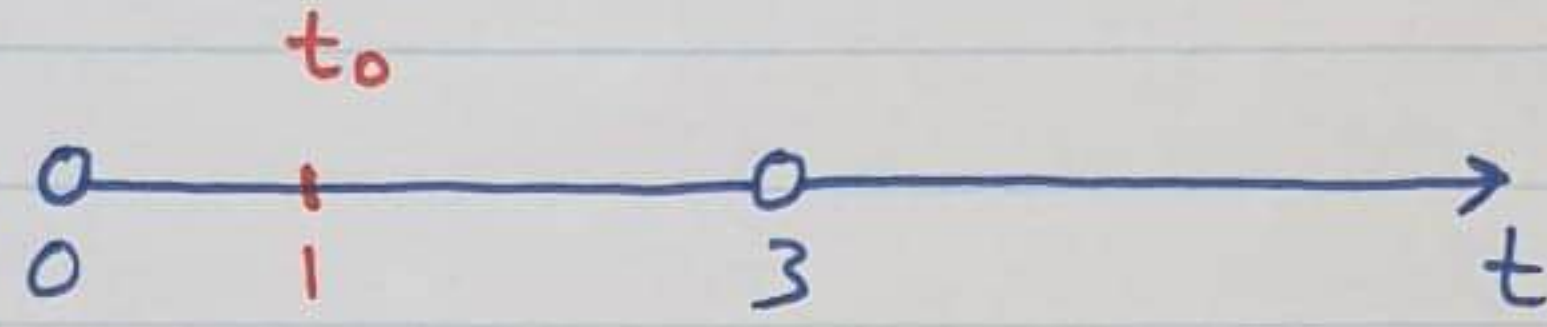
Exp Find the largest interval in which the solution of the following IVP's is valid (defined).

[1] $(t-3)y' + (\ln t)y = 2t \quad , \quad y(1) = 2$ linear

$$y' + \left(\frac{\ln t}{t-3}\right)y = \frac{2t}{t-3} \quad \cdots \text{(B)}$$

$p(t)$ cont. on $\mathbb{R}^+ \setminus \{3\}$ $\qquad$ $g(t)$ cont. on $\mathbb{R}\setminus\{3\}$

$p(t)$ and $g(t)$ are cont. on $(0,3)$ containing $t_0 = 1$



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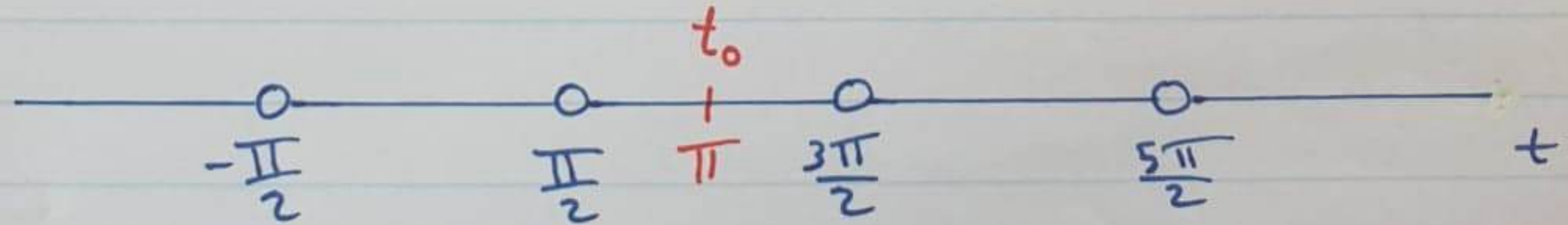
[2] $(\cos t)\, y' = \sin t\,(\cos t - y) \quad , \quad y(\pi) = 0$

$$y' = \tan t\,(\cos t - y) \qquad \text{linear}$$

$$y' + (\tan t)\, y = \sin t \quad \cdots \text{(B)}$$

$p(t)$ cont. on $\mathbb{R}\setminus\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots\}$ $\qquad$ $g(t)$ cont. on $\mathbb{R}$

$p(t)$ and $g(t)$ are cont. on $I = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$



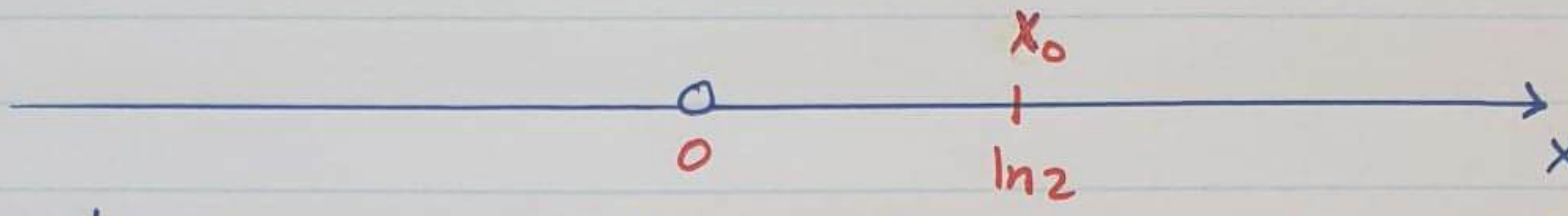
Exp Consider the IVP:

$$xy' = y + x^2 e^{-x}, \quad y(\ln 2) = \ln 2$$

1. Find an interval in which this IVP has a unique sol.

$$y' - \frac{1}{x} y = x e^{-x} \quad \ldots \; (B) \qquad \text{linear}$$

$p(x)$ cont. on $\mathbb{R}\setminus\{0\}$ $\qquad$ $g(x)$ cont. on $\mathbb{R}$



$p(x)$ and $g(x)$ are cont. on $I = (0, \infty)$ containing x_0

2. Find the unique solution on this interval.

Apply B^* $\Rightarrow$ $\mu(x) = e^{\int p(x)dx} = e^{\int -\frac{1}{x} dx} = e^{-\ln|x|} = e^{\ln x^{-1}} = \frac{1}{x}$

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x) g(x)\, dx + c\right]$$

$$= \frac{1}{\frac{1}{x}}\left[\int \left(\frac{1}{x}\right)(x e^{-x})\, dx + c\right]$$

$$= x\left[-e^{-x} + c\right]$$

To find c we use IC: $y(\ln 2) = \ln 2\left[-e^{-\ln 2} + c\right]$

$$\ln 2 = \ln 2\left[-\left(\tfrac{1}{2}\right) + c\right]$$

$$1 = -\tfrac{1}{2} + c$$

$$c = \frac{3}{2}$$

$$y(x) = x\left(-e^{-x} + \frac{3}{2}\right)$$

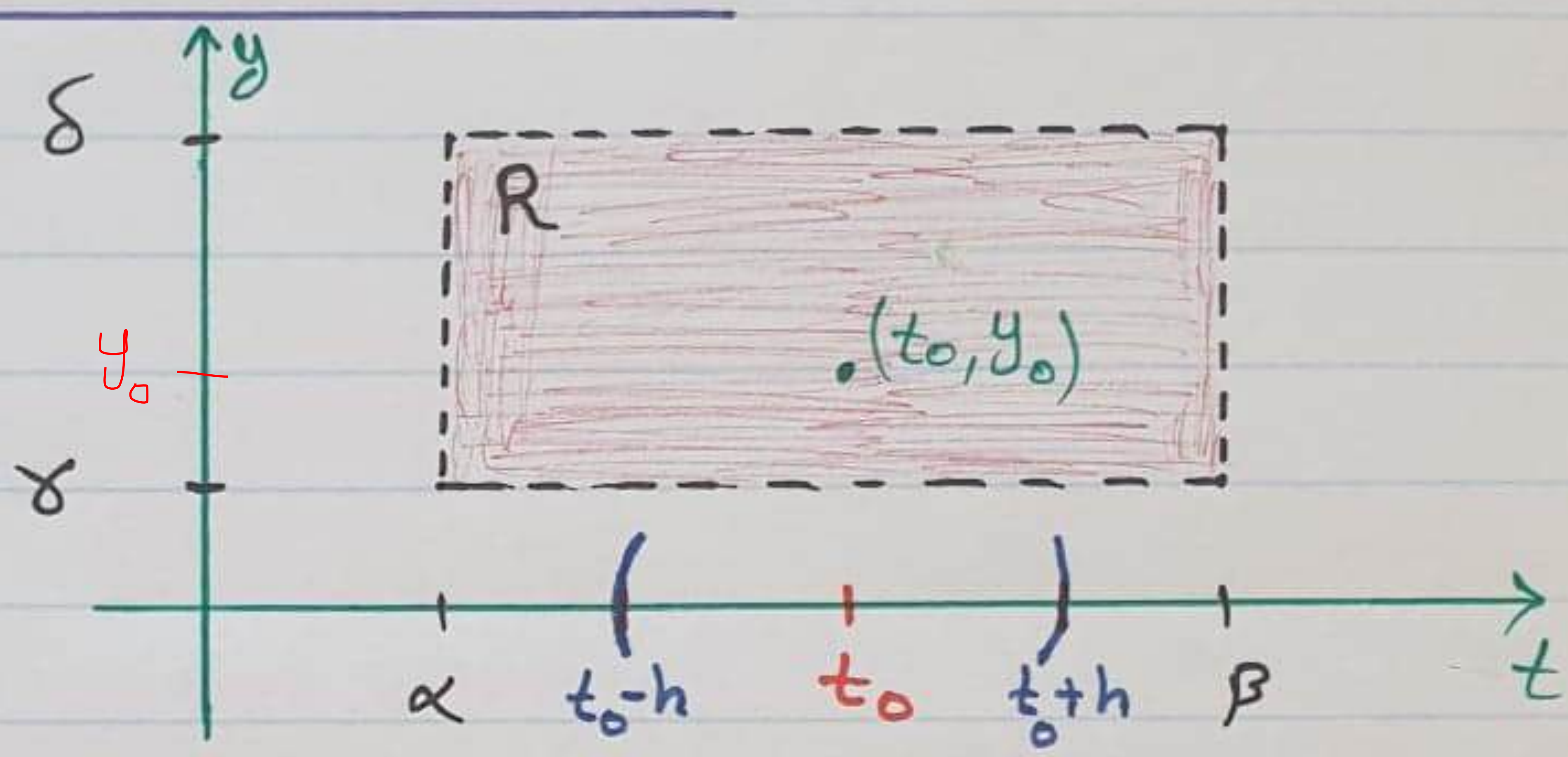
$$= \frac{3}{2}x - x e^{-x}$$

Th 2.4.2 (linear or nonlinear)
Consider the IVP $\boxed{\dfrac{dy}{dt} = f(t,y)\ ,\ y(t_0) = y_0}$

If f and f_y are cont. on an open rectangle $R = \{ (t,y) : \alpha < t < \beta \text{ and } \gamma < y < \delta \}$ containing the initial point (t_0, y_0), then $\exists$ a unique solution $y(t) = \phi(t)$ on an open sub-interval $I = (t_0 - h, t_0 + h) \subset (\alpha, \beta)$ s.t $y = \phi(t)$ satisfies the IVP on I, $h >> 0$.

---

we can draw an open rectangle R containing the initial point (t_0, y_0)



Remark (2.4.2): If the DE is 1^{st} order nonlinear, then to find the interval in which the solution is defined, we need to solve the DE first and find the domain of the solution which contains t_0.

Exp Find the largest interval in which the solution of the following IVP's is valid (defined):

[I] $y' = y^2 \quad , \quad y(0) = 1$

nonlinear
$\Downarrow$
1^{st} solve to find the interval

$$\frac{dy}{dt} = y^2$$

separable DE

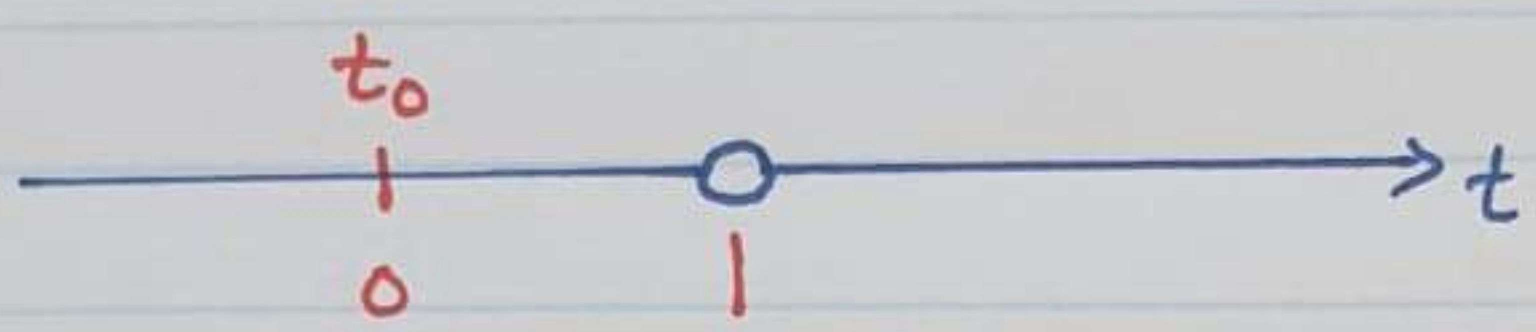
$$\int y^{-2}\, dy = \int dt$$

$-\frac{1}{y} = t + c$ To find c we use IC:

$-\frac{1}{\boxed{1}} = \boxed{0} + c \Rightarrow c = -1$

$\frac{-1}{y} = t - 1 \Rightarrow \frac{1}{y} = 1 - t$

$y(t) = \frac{1}{1-t}$



$I = (-\infty, 1)$

$\boxed{2}$ $\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$, $y(0) = -1$

we have solved this Exp in section 2.2 page 23 $\Rightarrow$

nonlinear $\Downarrow$ 1st solve to find the interval

$y(x) = 1 - \sqrt{(x^2+2)(x+2)}$

with interval $I = (-2, \infty)$

Exp show that the IVP: $y' = y^2$, $y(0) = 1$ has a unique solution

nonlinear Apply Th 2.4.2

Compare with $y' = f(t, y) \Rightarrow$

$f = y^2$ cont. on $\mathbb{R}$
$f_y = 2y$ cont. on $\mathbb{R}$

By Th 2.4.2 $\exists$ a unique sol. since we can draw an open rectangle containing $(t_0, y_0) = (0, 1)$

Exp show that the IVP:
$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \quad y(0)=-1$$
has a unique sol.

nonlinear
Apply Th2.4.2

Compare with $y' = f(x,y) \Rightarrow$

$f = \dfrac{3x^2+4x+2}{2(y-1)}$ is cont. on $\mathbb{R} \setminus \{y=1\}$

$f_y = -\dfrac{3x^2+4x+2}{2(y-1)^2}$ is cont. on $\mathbb{R} \setminus \{y=1\}$

Hence, we can draw an open rectangle R containing the initial point $(x_0, y_0) = (0,-1)$, so $\exists$ a unique sol. by Th 2.4.2

Exp Consider the IVP: $\dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)}, \quad y(0)=1$

[I] Does this IVP have unique sol.?

problem

nonlinear
Apply Th2.4.2

Compare with $y' = f(x,y) \Rightarrow$

$f = \dfrac{3x^2+4x+2}{2(y-1)}$ is cont. on $\mathbb{R} \setminus \{\underline{y=1}\}$

$f_y = -\dfrac{3x^2+4x+2}{2(y-1)^2}$ is cont. on $\mathbb{R} \setminus \{\underline{y=1}\}$

We can not draw an open rectangle R containing $(x_0, y_0) = (0,1)$ where f and f_y are cont. Hence, the conditions of Th 2.4.2 do not hold $\Rightarrow$ we can not guarantee the uniqueness.

$\boxed{2}$ Find the interval in which the sol. is defined.

nonlinear

1st solve to find the interval

$$\int (2y-2)\,dy = \int (3x^2+4x+2)\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

To find c $\Rightarrow$ we use IC : $y(0)=1$

$$1^2 - 2(1) = 0 + 0 + 0 + c \quad \Rightarrow \boxed{c=-1}$$

$$y^2 - 2y = x^3 + 2x^2 + 2x - 1 \quad \text{Implicit sol.}$$

$$y^2 - 2y + 1 = x^3 + 2x^2 + 2x - 1 + 1$$

$$(y-1)^2 = x(x^2+2x+2)$$

$$|y-1| = \sqrt{x(x^2+2x+2)}$$

The sol. is not unique

$$y(x) = 1 \pm \sqrt{x(x^2+2x+2)}$$

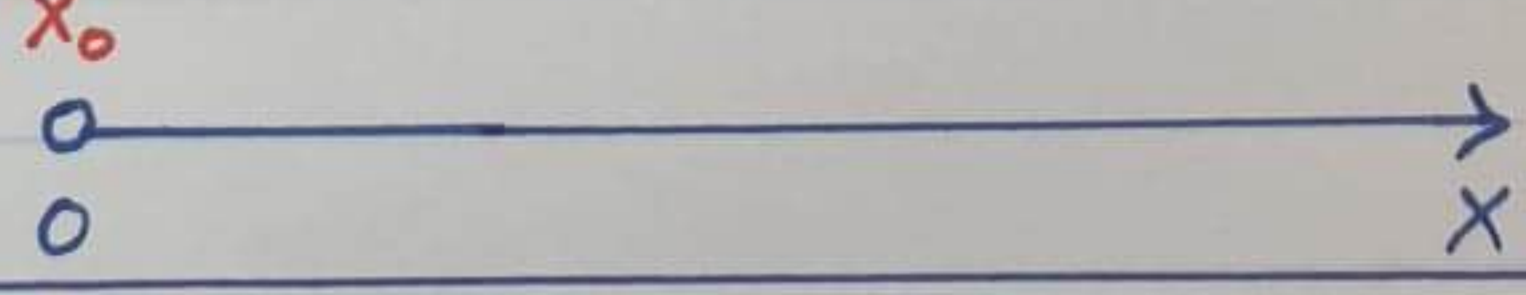
$y_1(x) = 1 + \sqrt{x(x^2+2x+2)}$ ✓ sol. satisfies $y(0)=1$

$y_2(x) = 1 - \sqrt{x(x^2+2x+2)}$ ✓ sol. satisfies $y(0)=1$

$x^2+2x+2 = (x^2+2x+1)+1 = (x+1)^2+1$ is positive

The interval in which the sol. is defined is $I=(0,\infty)$

$y_1(x_0) = y_2(x_0) = 1 = y_0$



$y_0 = 1$ is V. Asy.

Exp Given the IVP: $y' = y^{\frac{1}{3}}$, $y(0) = 0$

nonlinear
Apply Th 2.4.2

problem

1 Does this IVP have unique sol. ?

compare with $y' = f(t, y) \Rightarrow$

$f = y^{\frac{1}{3}}$ cont. on $\mathbb{R}$

$f_y = \frac{1}{3} y^{-\frac{2}{3}} = \frac{1}{3} \frac{1}{\sqrt[3]{y^2}}$ cont. on $\mathbb{R} \setminus \{y = 0\}$

We can not draw an open rectangle containing $(t_0, y_0) = (0, 0)$ in which f and f_y are cont.
Hence, conditions of Th 2.4.2 do not hold $\Rightarrow$
we can not guarantee the uniqueness.

2 Solve this IVP

We solved this IVP in section 2.2 page 24 $\Rightarrow$

$$y_1(t) = \sqrt{\frac{8}{27} t^3}$$

$$y_2(t) = -\sqrt{\frac{8}{27} t^3}$$

$$y_3(t) = 0$$

So there is no unique sol.

Bernoulli DE's (Problem 27 page 73 in book)

Bernoulli DE has the form

$$\boxed{y' + p(t)y = q(t)\,y^n \quad , n \in \mathbb{R}} - *$$

- Special case of Bernoulli DE when $n=0 \Rightarrow *$ becomes

$$y' + p(t)y = q(t) \quad \cdots \text{(B)}$$ linear

whose solution is $y(t) = \frac{1}{\mu}\left[\int \mu g\,dt + c\right]$, $\mu = e^{\int p\,dt}$

- Special case of Bernoulli DE when $n=1 \Rightarrow *$ becomes

$$y' + p(t)y = q(t)\,y$$ linear

$$y' + \big(p(t) - q(t)\big)y = 0 \quad \cdots \text{(B)}$$

whose solution is $y(t) = \frac{1}{\mu}\left[\int \mu g\,dt + c\right]$, $\mu = e^{\int (p-q)dt}$

$\Rightarrow y(t) = \frac{c}{\mu}$

- Special case of Bernoulli DE when $n \neq 0$ and $n \neq 1$
- we use change of variables to solve $*$: nonlinear

(2) $V = y^{1-n} \quad \Rightarrow V' = (1-n)\,y^{-n}\,y'$

- Multiply $*$ by $(1-n)\,y^{-n} \Rightarrow$

$$(1-n)\,y^{-n}\,y' + (1-n)\,p(t)\,y^{-n}\,y = (1-n)\,q(t)\,y^{-n}\,y^{n}$$

(1) $$\boxed{V' + (1-n)p(t)\,V = (1-n)\,q(t)}$$

we solve (1) for V then we solve (2) for y

using B*

Exp Solve this DE: $t^2 y' + 2ty - y^3 = 0, \ t > 0$

- This DE is nonlinear $\Rightarrow$ think of Bernoulli or Separable
  $\Rightarrow$ This DE is not separable

Bernoulli $\Rightarrow$ write the DE in the form of $*$

$$y' + \frac{2}{t} y = \frac{1}{t^2} y^3 , \quad t > 0$$

its Bernoulli with $n = 3$, $p(t) = \frac{2}{t}$, $q(t) = \frac{1}{t^2}$

- First solve ① $\Rightarrow$ $v' + (1-n) p(t) v = (1-n) q(t)$

$$v' + (1-3)\left(\frac{2}{t}\right) v = (1-3) \frac{1}{t^2}$$

$$v' - \frac{4}{t} v = \frac{-2}{t^2}$$

$$v(t) = \frac{1}{\mu}\left[\int \mu g \, dt + c\right], \quad \mu(t) = e^{\int \frac{-4}{t} dt}$$

$$= e^{-4 \ln t} = \frac{1}{t^4}$$

$$= \frac{1}{\frac{1}{t^4}}\left[\int \frac{1}{t^4}\left(\frac{-2}{t^2}\right) dt + c\right]$$

$$= t^4 \left[-2 \int t^{-6} dt + c\right]$$

$$= t^4 \left(-2 \frac{t^{-5}}{-5} + c\right)$$

$$v(t) = \frac{2}{5} \frac{1}{t} + c t^4$$

- Now solve ② $\Rightarrow$ $v = y^{1-n}$

$$\frac{2}{5} \frac{1}{t} + c t^4 = y^{1-3}$$

$$\frac{2 + 5ct^4 t}{5t} = y^{-2}$$

$$y^2 = \frac{5t}{2 + 5ct^5}$$

$$y(t) = \pm \sqrt{\frac{5t}{2 + 5ct^5}}$$

Exp Consider this IVP:

$$xy' + y = \frac{1}{y^2} \quad , \quad x>0 \quad , \quad y(1) = (2)^{\frac{1}{3}}$$

1 Solve this IVP using Bernoulli

$$y' + \frac{1}{x} y = \frac{1}{x} y^{-2} \qquad p(x) = q(x) = \frac{1}{x} \; , \; n = -2$$

First solve $v' + (1-n)p(x)v = (1-n)q(x)$

$$v' + \frac{3}{x} v = \frac{3}{x} \qquad \Rightarrow \mu(x) = e^{\int p(x)dx} = e^{\int \frac{3}{x} dx}$$

$$= e^{3\ln x} = x^3$$

$$v(x) = \frac{1}{\mu}\left[\int \mu q \, dx + c\right]$$

$$= \frac{1}{x^3}\left(\int x^3 \left(\frac{3}{x}\right) dx + c\right)$$

$$= \frac{1}{x^3}\left(3\frac{x^3}{3} + c\right)$$

$$v(x) = 1 + \frac{c}{x^3}$$

Second solve $v = y^{1-n} \iff v = y^3$

$$y^3 = 1 + \frac{c}{x^3} \qquad \text{To find } c \text{ we use IC} \Rightarrow$$

$$2 = 1 + \frac{c}{1} \Rightarrow c = 2 - 1 \Rightarrow c = 1$$

$$y(x) = \sqrt[3]{1 + \frac{1}{x^3}}$$

2) Solve this IVP using separable

$$x\frac{dy}{dx} + y = \frac{1}{y^2} \Rightarrow x\frac{dy}{dx} = \frac{1}{y^2} - y = \frac{1-y^3}{y^2}$$

$$\frac{1}{-3}\int \frac{-3y^2}{1-y^3}\,dy = \int \frac{dx}{x}$$

$$\frac{-1}{3}\ln|1-y^3| = \ln x + c$$ To find c we use IC

$$\frac{-1}{3}\ln|1-2| = \ln 1 + c \Rightarrow 0 = 0 + c \Rightarrow c = 0$$

$$\frac{-1}{3}\ln|1-y^3| = \ln x \Rightarrow \ln|1-y^3| = -3\ln x$$

$$|1-y^3| = x^{-3} \Rightarrow 1-y^3 = \pm\frac{1}{x^3}$$

we consider only $1-y^3 = -\frac{1}{x^3}$ since $y(1) = 2^{\frac{1}{3}}$

$$y^3 = 1 + \frac{1}{x^3}$$

$$y(x) = \sqrt[3]{1+\frac{1}{x^3}}$$

3) Show that this IVP has a unique solution

$$y' = \frac{dy}{dx} = \frac{1-y^3}{xy^2}$$ $y(1) = (2)^{\frac{1}{3}}$

nonlinear

Apply Th2.4.2

$f = \frac{1-y^3}{xy^2}$ cont. on $\mathbb{R}\setminus\{y=0\}$

$f_y = \frac{(xy^2)(-3y^2) - (1-y^3)(2xy)}{(xy^2)^2}$ cont. on $\mathbb{R}\setminus\{y=0\}$

$\Rightarrow$ by Th 2.4.2 $\exists$ unique sol. since we can draw an open rectangle R contains $(1, \sqrt[3]{2})$ in which f, f_y are cont.

2.6 Exact DE's and Integrating Factors

Question: How to solve this DE :

$$y' = \frac{x - y^2}{2xy + 1}$$

This DE is nonlinear, not separable, not Bernoulli ??

Th. Given the DE : $\boxed{M(x,y) + N(x,y)\, y' = 0}$ — ①

where M, N, M_y, N_x are cont. on an open rectangle $R = \{(x,y) : x \in (\alpha, \beta) \text{ and } y \in (\gamma, \delta)\}$.

- The DE ① is called Exact iff $\boxed{M_y = N_x}$ — ②
- Exact mean $\exists$ a function $\psi(x,y)$ s.t

$\psi_x = M$ and $\psi_y = N$ iff M and N satisfy ②.

Exp Solve the DE : $y' = \dfrac{x - y^2}{2xy + 1}$

$$(2xy+1)y' = x - y^2 \Rightarrow \underbrace{(y^2 - x)}_{M} + \underbrace{(2xy+1)}_{N} y' = 0$$

$M = y^2 - x \Rightarrow M_y = 2y$
$N = 2xy + 1 \Rightarrow N_x = 2y$ } Exact $\Rightarrow \exists\ \psi(x,y)$ s.t

$\psi_x = M$ and $\psi_y = N$

$\psi_x = M$

$$\psi = \int \psi_x\, dx = \int M\, dx = \int (y^2 - x)\, dx = y^2 x - \frac{x^2}{2} + h(y)$$

To find $h(y)$ we use $\psi_y = N$

$$\psi_y = \cancel{2xy} - 0 + h'(y) = N = \cancel{2xy} + 1 \iff h'(y) = 1$$

$$\iff h(y) = y$$

$$\psi(x,y) = y^2x - \frac{x^2}{2} + y$$

Since $\psi(x,y) = c \Rightarrow \boxed{y^2x - \frac{x^2}{2} + y = c}$ Implicit Solution

---

Exp Show that $\psi(x,y) = c$ where c is constant.

$$M(x,y) + N(x,y)\, y' = 0$$

$$\psi_x(x,y) + \psi_y(x,y)\frac{dy}{dx} = 0$$

$$\frac{d}{dx}\left(\psi(x,y)\right) = 0$$

$$\psi(x,y) = c$$

Exp Find explicit solution to the IVP:

$$2x + y^2 + 2xy\, y' = 0, \quad y(1) = 1, \quad x > 0$$

$\left.\begin{array}{ll} M = 2x + y^2 & \Rightarrow M_y = 2y \\ N = 2xy & \Rightarrow N_x = 2y \end{array}\right\}$ The DE is Exact $\Rightarrow$

$\exists\ \psi(x,y)$ s.t $\psi_x = M$ and $\psi_y = N$

$\psi_y = N \Rightarrow \psi = \int \psi_y\, dy = \int N\, dy = \int 2xy\, dy = xy^2 + g(x)$

To find $g(x)$: $\psi_x = \cancel{y^2} + g'(x) = M = 2x + \cancel{y^2}$

$$g'(x) = 2x$$

$$g(x) = x^2$$

$\psi(x,y) = xy^2 + x^2 = c \Rightarrow xy^2 + x^2 = c$

To find c we use $y(1) = 1 \Rightarrow (1)(1)^2 + (1)^2 = c \Rightarrow c = 2$

$xy^2 + x^2 = 2$ is the implicit solution

$xy^2 = 2 - x^2 \Rightarrow y^2 = \frac{2-x^2}{x} \Rightarrow y = \pm\sqrt{\frac{2-x^2}{x}} \Rightarrow y = \sqrt{\frac{2-x^2}{x}}$

Exp solve this IVP: $\frac{dy}{dx} = -\frac{x+4y}{4x-y}$, $y(0)=1$

- This IVP is similar to problem solved page 25 in section 2.2
This is homogenuous DE but we can use the method of today

$(4x-y)y' = -(x+4y) \Rightarrow \underbrace{(x+4y)}_{M} + \underbrace{(4x-y)}_{N}y' = 0$

$M = x+4y \Rightarrow M_y = 4$
$N = 4x-y \Rightarrow N_x = 4$ } The DE is Exact $\Rightarrow$

$\exists\ \psi(x,y)$ s.t $\psi_x = M$ and $\psi_y = N$

$\psi = \int \psi_y\, dy = \int N\, dy = \int (4x-y)\, dy = 4xy - \frac{y^2}{2} + h(x)$

To find $h(x) \Rightarrow$

$\psi_x = \cancel{4y} - 0 + h'(x) = M = x + \cancel{4y}$

$h'(x) = x$

$h(x) = \frac{x^2}{2}$

$\psi = 4xy - \frac{y^2}{2} + \frac{x^2}{2} = C$

To find c we use the IC: $y(0)=1$

$4(0)(1) - \frac{(1)^2}{2} + \frac{(0)^2}{2} = c \Rightarrow c = \frac{-1}{2}$

$4xy - \frac{y^2}{2} + \frac{x^2}{2} = \frac{-1}{2}$

$8xy - y^2 + x^2 = -1$ Implicit Solution

Exp Solve the DE : $\underbrace{(3xy+y^2)}_{M}+\underbrace{(xy+x^2)}_{N}y'=0 \quad , x>0$

$$\left.\begin{array}{l} M=3xy+y^2 \Rightarrow M_y = 3x+2y \\ N=xy+x^2 \Rightarrow N_x = y+2x \end{array}\right\} \text{This DE is not Exact}$$

- In this case, the results of our Theorem do not hold.
- So we need to find a positive function I called integrating factor s.t when we multiply the non exact DE by I, it becomes exact so that we can apply this theorem.

Question: How to find the positive integrating factor I? There are three cases to check:

$\boxed{1}$ If $\dfrac{M_y - N_x}{N} = f(x)$ then $I(x) = e^{\int f(x)\,dx}$

$\boxed{2}$ If $\dfrac{M_y - N_x}{M} = g(y)$ then $I(y) = e^{-\int g(y)\,dy}$

~~$\boxed{3}$ If $\dfrac{M_y - N_x}{yN - Mx} = h(v)$ then $I(v) = e^{\int h(v)\,dv}$ where $v = xy$~~

Exp Solve the DE: $(3xy+y^2)+(xy+x^2)y'=0 \quad , x>0$

$$\left.\begin{array}{l} M=(3xy+y^2) \Rightarrow M_y = 3x+2y \\ N=(xy+x^2) \Rightarrow N_x = y+2x \end{array}\right\} \text{Not Exact}$$

- Check $\cancel{\boxed{2}}$ $\dfrac{M_y - N_x}{M} = \dfrac{3x+2y-(y+2x)}{3xy+y^2} = \dfrac{x+y}{3xy+y^2} \neq g(y)$

• check (II) $\frac{M_y - N_x}{N} = \frac{x+y}{xy+x^2} = \frac{x+y}{x(x+y)} = \frac{1}{x} = f(x)$

Hence, the integrating factor is

$$I(x) = e^{\int f(x)\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

Now multiply the nonexact DE by x to become exact:

$$(3x^2y + xy^2) + (x^2y + x^3)y' = 0 \quad , x > 0$$

$$\left.\begin{array}{l} M = 3x^2y + xy^2 \Rightarrow M_y = 3x^2 + 2xy \\ N = x^2y + x^3 \Rightarrow N_x = 3x^2 + 2xy \end{array}\right\} \text{Exact} \Rightarrow$$

$\exists\ \psi(x,y)$ s.t $\psi_x = M$ and $\psi_y = N$

$$\psi = \int \psi_x\,dx = \int M\,dx = \int (3x^2y + xy^2)\,dx$$

$$= x^3y + \frac{x^2}{2}y^2 + h(y)$$

To find $h(y) \Rightarrow \psi_y = \cancel{x^3} + \cancel{x^2y} + h'(y) = N = \cancel{x^2y} + \cancel{x^3}$

$h'(y) = 0$

$h(y) = c$

$$\psi(x,y) = x^3y + \frac{x^2}{2}y^2 + c$$

$$\boxed{x^3y + \frac{x^2}{2}y^2 = c}$$ Implicit Solution

Exp (Q31) Solve the DE $\left(3x+\frac{6}{y}\right)+\left(\frac{x^2}{y}+\frac{3y}{x}\right)y'=0$

$M=3x+\frac{6}{y} \Rightarrow M_y=-\frac{6}{y^2}$

$N=\frac{x^2}{y}+\frac{3y}{x} \Rightarrow N_x=\frac{2x}{y}-\frac{3y}{x^2}$

→ not Exact

- Now we need to find integrating factor $I \Rightarrow$

Case 1

$$\frac{M_y-N_x}{N}=\frac{-\frac{6}{y^2}-\left(\frac{2x}{y}-\frac{3y}{x^2}\right)}{\frac{x^2}{y}+\frac{3y}{x}} \neq f(x)$$

Case 2

$$\frac{M_y-N_x}{M}=\frac{-\frac{6}{y^2}-\left(\frac{2x}{y}-\frac{3y}{x^2}\right)}{3x+\frac{6}{y}} \neq g(y)$$

Case 3

$$\frac{M_y-N_x}{yN-xM}=\frac{-\frac{6}{y^2}-\frac{2x}{y}+\frac{3y}{x^2}}{x^2+\frac{3y^2}{x}-3x^2-\frac{6x}{y}}$$

$$=\frac{\frac{3y}{x^2}-\left(\frac{6+2xy}{y^2}\right)}{\frac{3y^3-6x^2}{xy}-2x^2}=\frac{\frac{3y^3-6x^2-2x^3y}{x^2y^2}}{\frac{3y^3-6x^2-2x^3y}{\cancel{xy}}}$$

$$=\frac{1}{xy}=\frac{1}{v}=h(v) \checkmark$$

Hence, the integrating factor is $I(v)=e^{\int h(v)\,dv}=e^{\int\frac{dv}{v}}$

$$=e^{\ln v}=v=xy$$

- Now multiply our DE by xy to become exact

$$\left(3x^2y+6x\right)+\left(x^3+3y^2\right)y'=0$$

$M = 3x^2y + 6x \Rightarrow M_y = 3x^2$

$N = x^3 + 3y^2 \Rightarrow N_x = 3x^2$

$\rightarrow$ Exact $\Rightarrow$

$\exists$ a function $\psi(x,y)$ s.t $\psi_x = M$ and $\psi_y = N$

• To find $\psi \Rightarrow$ use

$$\psi_x = M \Rightarrow \psi(x,y) = \int \psi_x \, dx = \int M \, dx = \int (3x^2y + 6x)dx$$

$$= x^3y + 3x^2 + g(y)$$

• To find $g(y) \Rightarrow$ use $\psi_y = N$

$$\psi_y = x^3 + 0 + g'(y)$$

$$N = x^3 + g'(y)$$

$$\cancel{x^3} + 3y^2 = \cancel{x^3} + g'(y)$$

$$g'(y) = 3y^2 \Rightarrow g(y) = y^3$$

• Hence, ψ becomes

$$\psi = x^3y + 3x^2 + y^3 \Rightarrow \psi = c$$

$$x^3y + 3x^2 + y^3 = c$$ Implicit solution

not Bernoulli

not separable

nonlinear

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Exp Solve the DE: $\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$, $x>0$, $y>0$

$(x^3+y^3)y' = x^2 y \Rightarrow x^2 y - (x^3+y^3)y' = 0$ — $*$

$M = x^2 y \Rightarrow M_y = x^2$

$N = -(x^3+y^3) \Rightarrow N_x = -3x^2$

→ not exact DE

case 1: $\frac{M_y - N_x}{N} = \frac{x^2 - -3x^2}{-(x^3+y^3)} = \frac{4x^2}{-(x^3+y^2)} \neq f(x)$

case 2: $\frac{M_y - N_x}{M} = \frac{4x^2}{x^2 y} = \frac{4}{y} = g(y)$ ✓

Hence, the integrating factor is

$I = e^{-\int g(y)dy} = e^{-\int \frac{4}{y}} = e^{-4\ln y} = e^{\ln y^{-4}} = \frac{1}{y^4}$

now multiply the non exact DE $*$ by $\frac{1}{y^4}$ to become exact $\Rightarrow$

$\frac{x^2}{y^3} - \left(\frac{x^3+y^3}{y^4}\right)y' = 0 \Rightarrow x^2 y^{-3} - (x^3 y^{-4} + y^{-1})y' = 0$

$M = x^2 y^{-3} \Rightarrow M_y = -3x^2 y^{-4}$

$N = -(x^3 y^{-4} + y^{-1}) \Rightarrow N_x = -3x^2 y^{-4}$

→ exact DE

Hence, $\exists$ a function $\psi(x,y)$ s.t $\psi_x = M$ and $\psi_y = N$

$\psi = \int \psi_x \, dx = \int M \, dx = \int x^2 y^{-3} dx = \frac{x^3}{3} y^{-3} + h(y)$

To find $h(y)$ we use $\psi_y = N$

$$\psi_y = -\cancel{x^3 y^{-4}} + h'(y) = N = -\cancel{x^3 y^{-4}} - y^{-1}$$

$$h'(y) = -\frac{1}{y}$$

$$h(y) = -\ln y$$

Hence,

$$\psi = \frac{x^3}{3} y^{-3} - \ln y = c$$

$$\frac{x^3}{3y^3} - \ln y = c$$ Implicit solution

2.8 The Existance and Uniqueness Theorem

Th 2.8.1 Consider the IVP:

$$\frac{dy}{dt} = f(t, y) \quad , \quad y(0) = 0 \qquad \text{---- } *$$

If f and f_y are cont. on a rectangle
$R = \{(t, y) : -a \le t \le a \text{ and } -b \le y \le b\}$,
then $\exists$ a unique solution $y(t) = \phi(t)$ defined on a sub-interval $|t| \le h \le a$ that satisfies the IVP $*$.

Note that Th2.8.1 differes from Th2.4.2 only in the initial condition. That is, Th2.8.1 has IC starts at origin.

Remark: In general we can transform any IVP starts at (t_0, y_0) to an equivalent one starts at origin.

Exp Transform the following IVP's to an equivalent ones starting at origin:

① $y' - y^3 = t^2 \quad , \quad y(2) = -6$

Let $s = t - 2 \Rightarrow t = 2 + s$
$z = y + 6 \Rightarrow y = z - 6$
$y' = z'$

The equivalent IVP is

$$z' - (z-6)^3 = (2+s)^2 \quad , \quad z(0) = 0$$

- Now we will learn a method used to prove the existance of solution for Th2.8.1

- This method is called Picard's Iteration or it is also called The Method of Successive Approximation

$$\frac{dy}{dt} = f(t,y) \quad , \quad y(\underset{t_0}{0}) = \underset{y_0}{0} \quad \text{---} \; *$$

$$\int_{t_0}^{t} dy = \int_{t_0}^{t} f(t,y)\, dt$$

بخلي ال dy على جهة

والباقي على جهة ونكامل الطرفين

$$y(t)\Big|_0^t = \int_0^t f(t,y)\, dt$$

$$y(t) - y_0 = \int_0^t f(t,y)\, dt$$

$$\boxed{y(t) = \phi(t) = \int_0^t f(t,\phi(t))\, dt \quad \text{---}(T)}$$

where $\phi(t)$ is the solution of the IVP $*$

- (T) is called integral equation

- The solution of $*$ is the solution of (T).

- Now we will construct a sequence of functions

$$\phi_1, \phi_2, \phi_3, \dots, \phi_n, \dots$$

all satisfy the IC $y(0)=0$ but in general non of them satisfies the DE in $*$

- If the sequence $\phi_n(t)$ converges to $y = \phi(t)$, then $y = \phi(t)$ will be the solution for the IVP *

Here how to construct the sequence (iteration) ϕ_n:

Determine $f(t,y)$ from *

$\phi_0 = y_0 = 0$

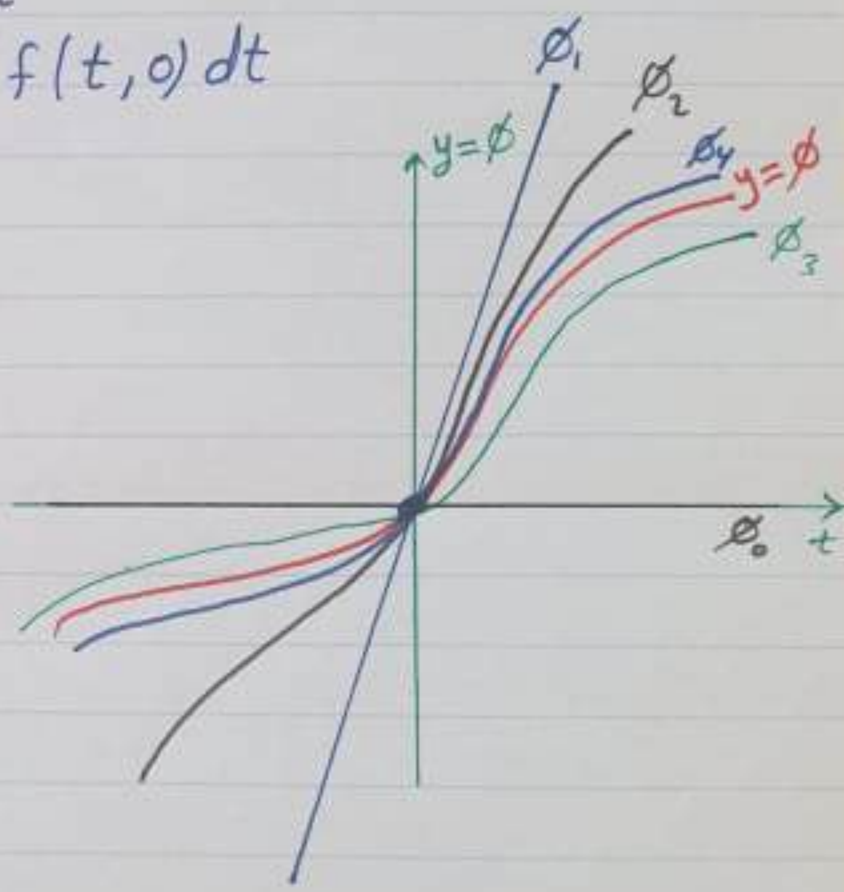
$$\phi_1 = \int_0^t f(t, \phi_0)\,dt = \int_0^t f(t, 0)\,dt$$

$$\phi_2 = \int_0^t f(t, \phi_1)\,dt$$

$$\phi_3 = \int_0^t f(t, \phi_2)\,dt$$

$\vdots$

$$\phi_n(t) = \int_0^t f(t, \phi_{n-1}(t))\,dt$$



- If $\lim_{n \to \infty} \phi_n(t) = \phi(t)$, then $\phi(t)$ is the solution of the IVP *.

Remark
- If the iteration diverges, then this method will not be able to find the solution
- We may apply Ratio Test to prove an infinite series converges

Exp Use Picard's iteration to solve the IVP

$$y' = 2t\,(1+y)\ ,\ y(0)=0$$

Compare with $y' = f(t,y) \Rightarrow \boxed{f(t,y) = 2t\,(1+y)}$

$\phi_0 = y_0 = 0$

$$\phi_1 = \int_0^t f(t,\phi_0)\,dt = \int_0^t f(t,0)\,dt = \int_0^t 2t\,dt = t^2 \Big|_0^t = t^2$$

$$\phi_2 = \int_0^t f(t,\phi_1)\,dt = \int_0^t f(t,t^2)\,dt = \int_0^t 2t\,(1+t^2)\,dt$$

$$= \int_0^t (2t + 2t^3)\,dt = t^2 + \frac{t^4}{2}$$

$$\phi_3 = \int_0^t f(t,\phi_2)\,dt = \int_0^t f\left(t, t^2 + \frac{t^4}{2}\right)dt = \int_0^t 2t\left(1 + t^2 + \frac{t^4}{2}\right)dt$$

$$= \int_0^t (2t + 2t^3 + t^5)\,dt = t^2 + \frac{t^4}{2} + \frac{t^6}{6}$$

$$\phi_4 = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \frac{t^8}{24}$$

$\vdots$

$$\phi_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \frac{t^8}{4!} + \cdots + \frac{t^{2n}}{n!}$$ ✓

$$\lim_{n\to\infty} \phi_n(t) = \sum_{n=1}^{\infty} \frac{t^{2n}}{n!} = e^{t^2} - 1 = \phi(t) \quad \text{since} \Downarrow$$

The Maclurine Series of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{t^2} = 1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} = 1 + \sum_{n=1}^{\infty} \frac{t^{2n}}{n!}$$

$$e^{t^2} - 1 = \sum_{n=1}^{\infty} \frac{t^{2n}}{n!} \checkmark$$

Exp Consider the IVP : $y' - 2y - 2 = 0$, $y(0) = 0$

I Use the Method of Successive Approximation to find ϕ_3

compare with $y' = f(t,y) \Rightarrow y' = 2y + 2$
$= 2(y+1)$

$$\boxed{f(t,y) = f(y) = 2(y+1)}$$

$\phi_0 = y_0 = 0$

$$\phi_1 = \int_0^t f(t,\phi_0)\,dt = \int_0^t f(t,0)\,dt = \int_0^t 2\,dt = 2t \Big|_0^t = 2t$$

$$\phi_2 = \int_0^t f(t,\phi_1)\,dt = \int_0^t f(t,2t)\,dt = \int_0^t 2(2t+1)\,dt = \int_0^t (4t+2)\,dt$$

$$= 2t + \frac{4t^2}{2}$$
$$= 2t + 2t^2$$

$$\phi_3 = \int_0^t f(t,\phi_2)\,dt = \int_0^t f(t, 2t+2t^2)\,dt = \int_0^t 2(2t+2t^2+1)\,dt$$

$$= \int_0^t (4t + 4t^2 + 2)\,dt = 2t + 2t^2 + \frac{4t^3}{3}$$

(2) Find the solution of this IVP using this method

$$\phi_4(t) = 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4$$

$$\vdots$$

$$\boxed{\phi_n(t) = 2t + \frac{2^2 t^2}{2!} + \frac{2^3 t^3}{3!} + \frac{2^4 t^4}{4!} + \cdots + \frac{2^n t^n}{n!}}$$

$$\lim_{n\to\infty} \phi_n(t) = \sum_{n=1}^{\infty} \frac{2^n t^n}{n!} = e^{2t} - 1 = \phi(t) \quad \text{since} \Downarrow$$

The Maclurine series of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{2t} = 1 + 2t + \frac{2^2 t^2}{2!} + \frac{2^3 t^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{2^n t^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{2^n t^n}{n!}$$

$$e^{2t} - 1 = \sum_{n=1}^{\infty} \frac{2^n t^n}{n!}$$

$y' - 2y - 2 = 0 \qquad y(0) = 0$

① Find ϕ_3

$y' = f(t,y) \Rightarrow y' = 2y - 2$

$\phi_n = \int_0^t f(t,\phi)\,dt$

$\phi_0 = 0$

$\phi_1 = \int_0^t f(t,0) \Rightarrow \int_0^t -2\,dt = -2t$

$\phi_2 = \int_0^t f(t,-2t) = \int_0^t -4t - 2 = -2t^2 - 2t$

$\phi_3 = \int_0^t f(t, -2t^2 - 2t) = \int_0^t -4t^2 - 4t - 2$

$\phi_3 = \frac{-4}{3}t^3 - 2t^2 - 2t$

$\phi_n = -\left(\frac{2^{n-1}t^n}{n}\right)$

Miscellaneous Problems
End of chapter 2

How to solve some 2^{nd} order DE's ?

- If $y(t)$ is a solution for a given DE then t is called independent variable (Indep. Var.) and y is called dependent variable (Dep. Var.)

Now if the 2^{nd} order DE misses the

missing y

[A] Dep. Var. y then let $\boxed{V = y'}$ and $\boxed{V' = y''}$

solve first for V then solve for y

missing t

[B] Indep. Var. t then let $\boxed{V = y'} = \frac{dy}{dt}$

and $V' = y'' = \frac{dV}{dt}$

solve first for V then solve for y

$= \frac{dV}{dy}\frac{dy}{dt}$

$= \frac{dV}{dy} y'$

الهدف :- اوجد y' عشان اعملها
تكامل واوجد y

$$\boxed{y'' = \frac{dV}{dy} V}$$

$$\frac{t^2 y''}{t^2} + \frac{2ty'}{t^2} - \frac{1}{t^2} = 0$$

$$y'' + \frac{2}{t} y' = \frac{1}{t^2}$$ missing y

$V = y'$

$$V' + \frac{2}{t} V = \frac{1}{t^2}$$ $V' = y''$

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{2}{t}} = e^{2\ln t} = t^2$$

$$V(t) = \frac{1}{\mu} \left[\int \mu(t)\, g(t)\, dt \right]$$

$$V(t) = \frac{1}{t^2} * \int t^2 * \frac{1}{t^2}\, dt) = \frac{1}{t^2} * t = \frac{1}{t} + \frac{C}{t^2}$$

$$y' = \frac{1}{t} + \frac{C}{t^2}$$

$$y = \ln t - \frac{C}{t}$$

Exp Solve the DE : Q36 / p.134

$$t^2 y'' + 2t y' - 1 = 0 \quad , t > 0$$

Missing $y \Rightarrow$ Apply $\boxed{A} \Rightarrow y' = V$, $y'' = V'$

$$t^2 V' + 2t V - 1 = 0$$

$$V' + \underbrace{\frac{2}{t}}_{p(t)} V = \underbrace{\frac{1}{t^2}}_{g(t)} \quad \text{--- B}$$

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t} = t^2$$

$$V = \frac{1}{\mu}\left[\int \mu g \, dt + c_1\right]$$

$$= \frac{1}{t^2}\left[\int t^2 \frac{1}{t^2} dt + c_1\right]$$

$$= \frac{1}{t^2}\left[t + c_1\right]$$

$$y' = \frac{1}{t} + \frac{c_1}{t^2}$$

$$\boxed{y(t) = \ln t - \frac{c_1}{t} + c_2}$$

Exp Solve the IVP: Q49/p.135

$t_0 \quad y_0 \quad t_0 \quad y'_0 = v_0$

$$y'' - 3y^2 = 0 \quad , y(0) = 2 \, , \, y'(0) = 4$$

Missing $t \Rightarrow$ Apply $\boxed{B} \Rightarrow \quad y' = v$

$$y'' = \frac{dv}{dy} v$$

$$\frac{dv}{dy} v - 3y^2 = 0 \quad \text{"Seperable"}$$

$$\frac{dv}{dy} v = 3y^2$$

$$v \, dv = 3y^2 \, dy$$

$$\frac{v^2}{2} = y^3 + c_1$$

$$v^2 = 2y^3 + 2c_1$$

$$(y')^2 = 2y^3 + 2c_1 \quad \text{To find } c_1 \Rightarrow$$

$$(4)^2 = 2(2)^3 + 2c_1$$

$$16 = 16 + 2c_1 \Leftrightarrow \boxed{c_1 = 0}$$

$$(y')^2 = 2y^3$$

$$y' = \pm\sqrt{2y^3}$$

$$4 = \boxed{+}\sqrt{2(2)^3}$$

$$y' = \sqrt{2y^3}$$

$$\frac{dy}{dt} = \sqrt{2}\, y^{\frac{3}{2}}$$

$$\int y^{-\frac{3}{2}} dy = \int \sqrt{2} \, dt$$

$$-2y^{-\frac{1}{2}} = \sqrt{2}t + c_2$$

$$\frac{-2}{\sqrt{y}} = \sqrt{2}t + c_2$$

To find $c_2 \Rightarrow$

$$\frac{-2}{\sqrt{2}} = \sqrt{2}(0) + c_2$$

$$-\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}} = c_2$$

$$\boxed{c_2 = -\sqrt{2}}$$

$$\frac{-2}{\sqrt{y}} = \sqrt{2}t - \sqrt{2}$$

$$\frac{\sqrt{y}}{-2} = \frac{1}{\sqrt{2}t - \sqrt{2}}$$

$$\sqrt{y} = \frac{-2}{\sqrt{2}(t-1)}$$

$$\sqrt{y} = \frac{\sqrt{2}}{1-t}$$

$$\boxed{y(t) = \frac{2}{(1-t)^2}}$$

$yy'' + y'^2 = 0 \quad , \; y(0) = 1 \quad , \; y'(0) = 1 \quad , y > 0$

$$y v \frac{dv}{dy} + v^2 = 0$$

Missing t

$y' = v$

$y'' = \frac{dv}{dy} v$

$$v \frac{dv}{dy} + \frac{v^2}{y} = 0 \Rightarrow \cancel{v} \frac{dv}{dy} * \frac{1}{v} = -\frac{\cancel{v^2}}{y} * \frac{1}{\cancel{v^2}}$$

$$\frac{1}{v} \frac{dv}{dy} = \frac{-1}{y} \Rightarrow \frac{dv}{v} = -\frac{dy}{y}$$

$$\int \frac{1}{v} dv = -\int \frac{1}{y} dy$$

$$\ln v = -\ln y + c = \ln y' = -\ln y + C \Rightarrow C = 0$$

$$y' = \frac{1}{y} \Rightarrow y = \ln y$$

Exp (Q42) Solve the IVP

$$y\,y'' + (y')^2 = 0 \quad , y(0) = 1 \, , \, y'(0) = 1 \quad , \, y > 0$$

Missing $t \Rightarrow y' = v$ and $y'' = v\dfrac{dv}{dy}$

$$y\left(v\frac{dv}{dy}\right) + v^2 = 0$$

$$v\left[y\,\frac{dv}{dy} + v\right] = 0$$

either $v = 0 \Rightarrow y' = 0$ not possible since $y'_0 = 1$

or $y\,\dfrac{dv}{dy} + v = 0 \quad \Rightarrow \displaystyle\int \frac{dv}{v} = -\int \frac{dy}{y}$

$\ln|v| = -\ln|y| + c_1 \quad \Rightarrow$ To find c_1 we use IC's

$\ln|y'| = -\ln y + c_1$

$\ln 1 = -\ln 1 + c_1 \quad \Rightarrow 0 = 0 + c_1 \Rightarrow c_1 = 0$

$\ln|y'| = \ln y^{-1} \quad \Rightarrow |y'| = \dfrac{1}{y}$

either $y' = -\dfrac{1}{y}$ not possible since $y_0 = 1 = y'_0$

or $y' = \dfrac{1}{y} \quad \Rightarrow \dfrac{dy}{dt} = \dfrac{1}{y} \quad \Rightarrow \displaystyle\int y\,dy = \int dt$

$\dfrac{y^2}{2} = t + c_2 \quad \Rightarrow$ To find c_2 we use $y(0) = 1$

$\dfrac{1}{2} = 0 + c_2 \quad \Rightarrow c_2 = \dfrac{1}{2} \quad \Rightarrow \dfrac{y^2}{2} = t + \dfrac{1}{2}$

$y^2 = 2t + 1 \quad \Rightarrow y(t) = \pm\sqrt{2t+1} \quad \Rightarrow \boxed{y(t) = \sqrt{2t+1}}$ since $y_0 = 1$