exercises Chapter 14:

Q4: Find a subring of $\mathbb{Z} \oplus \mathbb{Z}$ that is Not an ideal of $\mathbb{Z} \oplus \mathbb{Z}$.

$A = \{(a,a) : a \in \mathbb{Z}\}$ is a subring of $\mathbb{Z} + \mathbb{Z}$ since

→ $A \neq \phi$ since $(0,0) \in A$

→ let $(a,a) \in A$, $(b,b) \in A$ then

$(a,a) - (b-b) = (a-b , a-b) . \in A$

→ let $(a,a) \in A$, $(b,b) \in A$ then

$(a,a) . (b,b) = (ab , ab) \in A$.

But A is not ideal ~~since~~ of $\mathbb{Z} \oplus \mathbb{Z}$

let $(2,5) \in \mathbb{Z} \oplus \mathbb{Z}$, $(6,6) \in A$ then

$(2,5) . (6,6) = (12,30) \notin A$.

Q5: let $S = \{a+bi : a,b \in \mathbb{Z} , b \text{ is even}\}$. show that S is a subring of $\mathbb{Z}[i]$ But Not an ideal of $\mathbb{Z}[i]$.

→ $S \neq \phi$ since $0 \in S$.

→ let $a+bi$ and $c+di \in S \Rightarrow a,b,c,d \in \mathbb{Z}$ with $b = 2m$, $d = 2n$

$\Rightarrow (a+bi) - (c+di) = a - c + (b-d)i$

$= a - c + 2(m-n)i \in S$

→ let $(a+bi) \in S$ and $(c+di) \in S$ then

$(a+bi)(c+di) = ac - bd + (ad - bc)i$

$= (ac - bd) + 2(an + bm)i \in S$

So S is subring of $\mathbb{Z}[i]$.

continue →

S in Not an ideal :

$1 = 1 + 0i \in S$ and $i \in \mathbb{Z}[i]$

yet $1 \cdot i = i \notin S$

So its Not ideal .

Q11: In the Ring of integers, find a Positive integer a s.t

a. $\langle a \rangle = \langle 2 \rangle + \langle 3 \rangle$.

$a = 1$

b. $\langle a \rangle = \langle 6 \rangle + \langle 8 \rangle$.

$a = 2$

c. $\langle a \rangle = \langle m \rangle + \langle n \rangle$.

$a =$ g.c.d (m, n) .

Q13: Find a positive integer a such that

a. $\langle a \rangle = \langle 3 \rangle \langle 4 \rangle$

$a = 12$

b. $\langle a \rangle = \langle 6 \rangle \langle 8 \rangle$

$a = 48$ →

c. $\langle a \rangle = \langle m \rangle \langle n \rangle$.

$a = mn$.

proof b :

every element of $\langle 6 \rangle \langle 8 \rangle$ has the form $6t_1 8k_1 + 6t_2 8k_2 + \cdots + 6t_n k_n$

$= 48s \in \langle 48 \rangle$

so $\langle 6 \rangle \langle 8 \rangle \subseteq \langle 48 \rangle$. ①

also since $48 \in \langle 6 \rangle \langle 8 \rangle$

we have $\langle 48 \rangle \subseteq \langle 6 \rangle \langle 8 \rangle$. ②

By 1 and 2 $\Rightarrow$ $\langle 48 \rangle = \langle 6 \rangle \langle 8$.

Q14: let A and B be ideals of a Ring. Prove that $AB \subseteq A \cap B$.

let $x \in AB \Rightarrow x = \underbrace{a_1 b_1}_{} + \cdots + a_n b_n \in A \cap B$

$\in A$ $\qquad\qquad$ $\in B$

since $a_1 \in A, b_1 \in R$ $\qquad$ since $a_1 \in R, b_1 \in B$

$\Rightarrow x \in A \cap B$

Q18: suppose that in the ring $\mathbb{Z}$, the ideal $\langle 35 \rangle$ is a proper ideal of J and J is a proper ideal of I. What are the posibilities for J? What are the posibilities for I?

all ideals of $\mathbb{Z}$ are of the form $m\mathbb{Z}$.

we have $m\mathbb{Z} \subseteq n\mathbb{Z}$ iff $n | m$

therefor the ideals that contain $\langle 35 \rangle = 35\mathbb{Z}$ are $\mathbb{Z}, 5\mathbb{Z}, 7\mathbb{Z}$ and $35\mathbb{Z}$

$\mathbb{Z}, 5\mathbb{Z}, 7\mathbb{Z} \Rightarrow$ proper ideal.

and we must have $I = \mathbb{Z}$

So the possibilities for J are $5\mathbb{Z}$ and $7\mathbb{Z}$.

Q19: give an example of a Ring that has exactly two maximal ideals.

$\mathbb{Z}_6$ has two maximal ideal $\langle 2 \rangle$ and $\langle 3 \rangle$.

By " maximal ideal of $\mathbb{Z}_n$ are only $\langle p \rangle$ where p is a prime divisor of n ".

Q20: suppose that R is commutative Ring and $|R| = 30$. If I is an ideal of and $|R| = 10$. prove that I is maximal ideal.

since if B is a proper ideal of R s.t $I \subseteq B \subseteq R$ then

$(R,+)$, $(I,+)$, $(B,+)$ are groups and $I \subseteq B \subseteq R$.

so if $|B| = X$ then $|I| \mid |B| = X$

$|B| = X \mid |R|$ $\rightarrow$ $X = 10$ or 30

so B can't be proper subset of R and I proper in B.

i.e either $B = I$ or $B = R$

so R is maximal ideal

Exercises of chapter 14 :

Q6 : Find all maximal ideals in

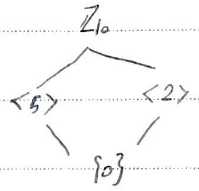
a. $\mathbb{Z}_8$:

only $\langle 2 \rangle = \{0, 2, 4, 6\}$

b. $\mathbb{Z}_{10}$:

$\langle 2 \rangle = \{0, 2, 4, 6, 8\}$

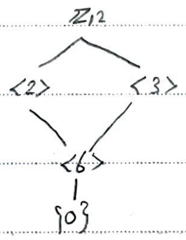
$\langle 5 \rangle = \{0, 5\}$



c. $\mathbb{Z}_{12}$:

$\langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$

$\langle 3 \rangle = \{0, 3, 6, 9\}$



d. $\mathbb{Z}_n$:

are only $\langle p \rangle$ where p is a prime divisor of n.

Q7: let a belong to a commutative Ring R. show that $aR = \{ar : r \in R\}$ is an ideal of R. If R is the rings of even integers list the element of $4R$.

since $ar_1 - ar_2 = a(r_1 - r_2)$

and $(ar_1)r = a(r_1 r)$

$\rightarrow 4R = \{ \dots, -16, -8, 0, 8, 16, \dots \}$.

Q8: Prove that the intersection of any set of ideals of a Ring is an ideal.

let A_i, $i \in I$ be an indexed family of ideals of R, then $\bigcap_{i \in I} A_i$ is an ideal of R since.

$\rightarrow \bigcap_{i \in I} A_i \neq \phi$ since $0 \in \bigcap_{i \in I} A_i$

$\rightarrow$ let $a, b \in \bigcap_{i \in I} A_i \Rightarrow a, b \in A_i, \forall i \in I$

$\Rightarrow a - b \in A_i, \forall i \in I$

$\Rightarrow a - b \in \bigcap_{i \in I} A_i$

$\rightarrow$ let $a \in \bigcap_{i \in I} A_i$ and suppose $x \in R$, then

$\forall i \in I$, $a \in A_i$, $x \in R \Rightarrow ax$ and $xa \in A_i$ since A_i is an ideal.

$\Rightarrow ax$ and $xa \in \bigcap_{i \in I} A_i$

Q9: If n is an integer greater than 1, show that $\langle n \rangle = n\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$ iff n is prime.

" By → I is a prime iff R/I is an integral domain "

So if n is a prime ideal iff $\mathbb{Z}/n\mathbb{Z}$ is an integral domain iff n is prime.

OR ⇒

($\Rightarrow$) suppose that n is a prime ideal.

If $a, b \in n$ then $n | ab$

$\Rightarrow$ $a \in n$ or $b \in n$

$\Rightarrow$ $n | a$ or $n | b$

$\Rightarrow$ n is prime.

($\Leftarrow$) suppose that n is prime number

If $ab \in n$, then $n | ab$

$\Rightarrow$ $n | a$ or $n | b$

$\Rightarrow$ $a \in n$ or $b \in n$

$\Rightarrow$ n is prime ideal.

Q10: If A and B are ideals of a Ring, show that the sum of A and B $A+B=\{a+b : a\in A, b\in B\}$ is an ideal.

$\rightsquigarrow$ $A+B \neq \phi$ since $0 = \underbrace{0}_{\in A} + \underbrace{0}_{\in B} \in A+B$

$\rightsquigarrow$ let $x = a_1 + b_1$, $y = a_2 + b_2$ $\Big\} \rightarrow \in A+B$ since $a_1, a_2 \in A$, $b_1, b_2 \in B$

$$\Rightarrow x-y = (a_1+b_1)-(a_2+b_2)$$
$$= \underbrace{(a_1-a_2)}_{\in A} + \underbrace{(b_1-b_2)}_{\in B}$$

$\Rightarrow$ $x-y \in A+B$.

$\rightarrow$ let $x = a_1 + b_1$, $r\in R$ then $xr = r(a_1+b_1)$

$$= \underbrace{ra_1}_{\in A} + \underbrace{rb_1}_{\in B}$$

since $r\in R$, $a\in A$, $b\in B$, and A, B are ideals.

$\Rightarrow$ $xr \in A+B$.

similarly, $rx \in A+B$ since $rx = r(a_1+b_1)$

$$= \underbrace{ra_1}_{\in A} + \underbrace{rb_1}_{\in B} \Rightarrow \in A+B.$$

Q12: If A and B are ideals of a Ring, show that the product of A and B $AB = \{a_1b_1 + a_2b_2 + \cdots + a_nb_n : a_i \in A_i, b_i \in B_i, n \text{ positive integer}\}$ is an ideal.

$\rightsquigarrow$ $A.B \neq \phi$ since $0 = \underset{\in A}{0} . \underset{\in B}{0} \in AB$.

$\rightsquigarrow$ let $x = a_1b_1 + a_2b_2 + \cdots + a_nb_n \in A.B$

$y = \bar{a}_1\bar{b}_1 + \bar{a}_2\bar{b}_2 + \cdots + \bar{a}_m\bar{b}_m \in A.B$

$\Rightarrow x - y = (a_1b_1 + \cdots + a_nb_n) - (\bar{a}_1\bar{b}_1 + \cdots + \bar{a}_m\bar{b}_m)$

$= a_1b_1 + \cdots + a_nb_n - \bar{a}_1\bar{b}_1 - \cdots - \bar{a}_m\bar{b}_m$

$\in A.B$.

$\rightsquigarrow$ suppose $r \in R$, $x = a_1b_1 + \cdots + a_nb_n \in AB$ then

$rx = (ra_1)b_1 + \cdots + (ra_n)b_n$

$\in AB$.

Q15: If A is an ideal of a ring R and 1 belongs to A, prove that $A = R$.

let $1 \in A$

let $x \in R \Rightarrow x = \underset{\in R}{x} . \underset{\in A}{1} \in A$ since A is ideal.

$\Rightarrow R = A$.

Q16: If A and B are ideals of commutative Ring R with unity and $A + B = R$ show that $AB = A \cap B$.

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$AB \subseteq A \cap B$ --- ① ("prove it Q14")

$1 \in R \rightarrow R = A + B \Rightarrow 1 = a + b$ for some $a \in A$, $b \in B$.

let $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

$x = x.1 = x(a+b) = \underset{\in B}{x}\underset{\in A}{a} + \underset{\in A}{x}\underset{\in B}{b} \Rightarrow \in AB \Rightarrow A \cap B \subseteq AB$ --- ②

By 1 and 2

$\Rightarrow A \cap B = AB$

Q17: If an ideal I of a ring R contains a unit, show that $I = R$.

since I is an ideal of R then $I \subseteq R$ --- ①

let a is a unit in I, let $r \in R$, Then

$$r = r(a a^{-1})$$

$$= \underbrace{(r a^{-1})}_{\in R} \underbrace{a}_{\in I} \in I \quad \text{since } I \text{ ideal}$$

$\Rightarrow R \subseteq I$ --- ②

By ① and ② $I = R$

Q33: How many elements are in $\mathbb{Z}[i] / \langle 3+i \rangle$?

has 2 element, A maximal $\to$ Thm 14.4

First, we notice that $(3+i)(3-i) = 10 \in \langle 3+i \rangle$.

Hence, $10 + \langle 3+i \rangle = \langle 3+i \rangle$

Now, $i + \langle 3+i \rangle = i + (-3-i) + \langle 3+i \rangle$

$$= -3 + \langle 3+i \rangle$$

$$= 7 + \langle 3+i \rangle$$

Hence, $a + bi + \langle 3+bi \rangle$ can be expressed as just $a < 3+bi$, $0 \le a \le 9$

it is clear, $1 + \langle 3+i \rangle$ has (additive) order 10

Thus, the quotient Ring simply $\{ k + \langle 3+i \rangle : k \in \{0, \ldots, 9\} \}$

so there are 10 element in the quotient Ring.

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Q34: In $Z[x]$, the Ring of polynomials with integer coefficients, let $I = \{ f(x) \in Z[x] : f(0) = 0 \}$ prove that I is Not maximal ideal.

let $B = \{ f(x) \in \mathbb{Z}[x] : f(0) \text{ is even} \}$

and $I \subset B \subset Z[x]$.

OR :

Q35: In $Z \oplus Z$, let $I = \{ (a,0) : a \in Z \}$. show that I is a prime ideal but not a maximal ideal.

$(a,b), (c,d) \in \mathbb{Z} \oplus \mathbb{Z}$

$\Rightarrow (a,b) \cdot (c,d) = (ac, bd) \in I$

But $bd = 0 \Rightarrow b = 0$ or $d = 0$

$\Rightarrow (a,b) \in I$ or $(c,d) \in I$

So I is prime ideal.

- $I = (a,0) \subsetneq J = (a, 2b) \subset \mathbb{Z} \oplus \mathbb{Z}$

So it is Not maximal ideal.

Q36: let R be a Ring and let I be an ideal of R. prove that the factor Ring R/I is commutative iff $rs - sr \in I$ for all r and s in R.

let $R|I$ commutative, then

$(r+I)(s+I) = (s+I)(r+I)$

$\Rightarrow \quad rs+I = sr+I$

$\Rightarrow \quad rs - sr \in I$

conversly,

let for any $r, s \in R$, $rs - sr \in I$

So $R|I$ is comm.

Q45: let R be a commutative Ring and let A be any subset of R. Show that the annihilator of A, $Ann(A) = \{ r \in R : ra = 0 \ \forall a \in A \}$ is an ideal.

→ $Ann(A) \neq \phi$ since $0 \in Ann(A)$ since $0 \cdot a = 0 \ \forall a \in A$.

→ let $x, y \in Ann(A)$ then $xa = 0$ and $ya = 0$ $\forall a \in A$

$\Rightarrow (x-y)a = x \cdot a - y \cdot a$

$= 0 - 0 \quad \forall a \in A$

$\Rightarrow (x-y) \in Ann(A)$.

→ let $x \in Ann(A)$ and $r \in R$ then

$r \cdot x \cdot a = r(xa)$

$= r \cdot 0$

$= 0 \quad \forall r \in R$.

So $Ann(A)$ is ideal.

Q46: let R be a commutative Ring and let A be any ideal of R. Show that the nil radical of A, $N(A) = \{ r \in R : r^n \in A$ for some integer $\underset{\uparrow}{n} \}$ is an ideal of R.

depends on r

$\rightsquigarrow$ $N(A) \neq \phi$ since $0 \in N(A)$ since $0 = 0^n \ \forall n$.

$\rightsquigarrow$ let $a, b \in N(A) \Rightarrow \exists n, m \in \mathbb{Z}$ s.t $a^m, b^n \in A$

$\Rightarrow (a-b)^{m+n} = a^{m+n} + a^{m+n-1} b + \cdots + a b^{m+n-1} + b^{m+n} \in A$ ← أكمليه

$\Rightarrow a - b \in N(A)$.

$\rightsquigarrow$ let $a \in N(A)$ and $r \in R$,

$(ra)^m = r^m a^m \in A$ since A ideal

$\Rightarrow ra \in N(A)$

and $ar \in N(A)$.

Q48: let $R = \mathbb{Z}_{36}$, Find

a. $N(\langle 0 \rangle) = \{ m \in \mathbb{Z}_{36} : m^k \in \langle 0 \rangle \} = \langle 6 \rangle$

b. $N(\langle 4 \rangle) = \{ m \in \mathbb{Z}_{36} : m^k \in \langle 4 \rangle \} = \{0, 2, 4, 6, 8, \ldots, 34\} = \langle 2 \rangle$.

c. $N(\langle 6 \rangle) = N(N\langle 0 \rangle) = \{0, 6, 12, \ldots, 30\} = \langle 6 \rangle$.

