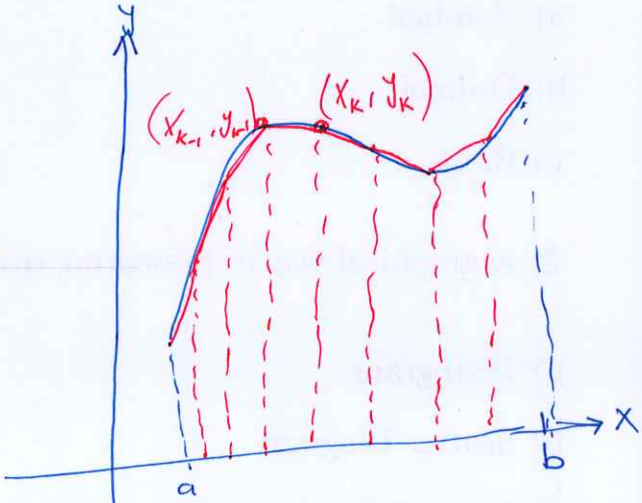
Further Application of Integration

6.3 Arc length



Length of a curve $y = f(x)$:-

To find the length of $y = f(x)$ from

$x = a$ to $x = b$:-

* We partition the interval $[a,b]$ into n subintervals

$$a = x_0 < x_1 < x_2 < x_3 < x_4 \cdots\cdots < x_n = b$$

$$L_k = \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$$

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

Arc length $L = \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$

$$L = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_a^b dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \longrightarrow L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Def If f' is Cont. on $[a,b]$, then the length (Arc Length) of the curve $y = f(x)$ from $x=a$, to $x=b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

- If $x = g(y)$ from $y=c$ to $y=d$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Exp① Find arc-Length of the curve $y = \frac{3}{2} x^{2/3}$ over the interval $[0,8]$

Sol.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$\frac{dy}{dx} = \frac{3}{2} \cdot \frac{2}{3} x^{-1/3}$

$\left(\frac{dy}{dx}\right)^2 = x^{-2/3}$

$$L = \int_0^8 \sqrt{x^{-2/3} + 1}\, dx$$

$$L = \int_0^8 \sqrt{\frac{1 + x^{2/3}}{x^{2/3}}}\, dx$$

$$= \int_0^8 \frac{1}{x^{1/3}} \sqrt{1 + x^{2/3}}\, dx$$

Assume

$u = 1 + x^{2/3}$

$du = \frac{2}{3} x^{-1/3}\, dx$

$x=0 \longrightarrow u=1$

$x=0 \longrightarrow u=5$

$$\int_1^5 \frac{3}{2} \sqrt{u}\, du = u^{3/2}\Big]_1^5 = 5\sqrt{5} - 1$$

Exp2: Find the arc length for the curve

$$x - \frac{y^4}{8} - \frac{1}{4y^2} = 0 \quad \text{for} \quad 1 \leq y \leq 2.$$

$x = y^4/8 + 1/4y^2$

$\frac{dx}{dy} = \frac{4y^3}{8} + \frac{-2y^{-3}}{4}$

$\frac{dx}{dy} = \frac{y^3}{2} - \frac{y^{-3}}{2}$

$\left(\frac{dx}{dy}\right)^2 = \frac{y^6}{4} - 2\left(\frac{y^3}{2}\right)\left(\frac{y^{-3}}{2}\right) + \frac{y^{-6}}{4}$

$= \frac{y^6}{4} - \frac{1}{2} + \frac{y^{-6}}{4}$

Sol:-

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^2 \sqrt{\frac{y^6}{4} + \frac{1}{2} + \frac{y^{-6}}{4}}\, dy$$

$$= \int_1^2 \sqrt{\left(\frac{y^3}{2} + \frac{y^{-3}}{2}\right)^2}\, dy$$

$$= \int_1^2 \frac{y^3}{2} + \frac{y^{-3}}{2}\, dy = \left.\frac{y^4}{8} + \frac{y^{-2}}{-4}\right]_1^2$$

$$= \frac{33}{16}$$

Exp3. Find the arc length [Don't evaluate the integral]

$x = \frac{1}{2}(y^3+1)^2$ from $(0,-1)$ to $(2,1)$.

$x = \frac{1}{2}(y^3+1)^2$

$\frac{dx}{dy} = 3y^2(y^3+1)$

$\left(\frac{dx}{dy}\right)^2 = 9y^4(y^3+1)^2$

$= 9y^4\left[y^6 + 2y^3 + 1\right]$

Sol.

$$L = \int_{-1}^{1} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_{-1}^{1} \sqrt{9y^{10} + 18y^7 + 9y^4 + 1}\, dy$$

Q_4

Page 330

Find ~~arc~~ arc length $X = \frac{y^{3/2}}{3} - y^{1/2}$ $\boxed{91}$ $y=1 \to 9$

$$\frac{dX}{dy} = \frac{1}{2} y^{1/2} - \frac{1}{2} y^{-1/2}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{4} y - 2\left(\frac{1}{2} y^{1/2}\right)\left(\frac{1}{2} y^{-1/2}\right) + \frac{1}{4} y^{-1}$$

$$= \frac{1}{4} y - \frac{1}{2} + \frac{1}{4} y^{-1}$$

$$L = \int_1^9 \sqrt{\frac{1}{4} y + \frac{1}{2} + \frac{1}{4} y^{-1}} \, dy$$

$$= \int_1^9 \sqrt{\left(\frac{1}{2} y^{1/2} + \frac{1}{2} y^{-1/2}\right)^2} \, dy$$

$$= \int_1^9 \frac{1}{2} y^{1/2} + \frac{1}{2} y^{-1/2} \, dy$$

$$= \frac{1}{3} y^{3/2} + y^{1/2} \Big]_1^9$$

$$= \frac{1}{3}(27) + 3 - \left[\frac{1}{3} + 1\right] = 12 - \frac{4}{3} = \frac{32}{3}.$$

Q19
Page 331

Find the Curve $y=f(x)$ through the point $(1,1)$ whose length integral is

$$L = \int_1^4 \sqrt{1 + \left(\frac{1}{4x}\right)}\, dx$$

$$\left(\frac{dy}{dx}\right)^2$$

$$\longrightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4x}$$

$$\longrightarrow \frac{dy}{dx} = \pm\frac{1}{2\sqrt{x}}$$ "Just the positive answer because the curve is to have a positive derivative"

$$\longrightarrow y = \int \frac{1}{2\sqrt{x}}\, dx = \sqrt{x} + c$$

$$y = \sqrt{x} + c$$

To find the value of c $f(1)=1$ its positive function

at $x=1 \rightarrow y=1$ $1 = 1 + c \rightarrow c = 0$

$y = \sqrt{x}$ from $x=1 \longrightarrow x=4$

* An equation of the curve $y=\sqrt{x}$ as x increase from lower limit $x=1$ to the upper limit $x=4$

The curve ~~run~~ run from $y=1$ to $y=4$

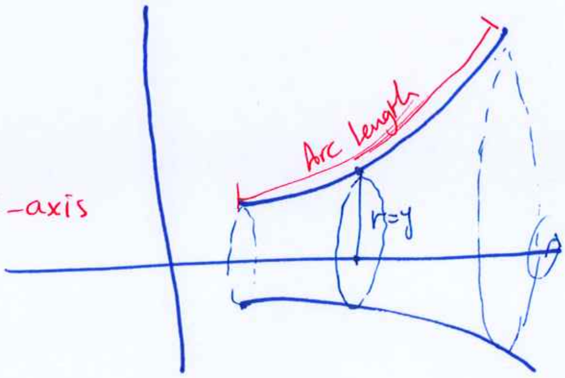
$\longrightarrow$ Its only one curve corresponding to the given derivative and the value of $f(x)$

6.4 Surface Area

If the fun. $f(x) \geq 0$ is continuously differentiable on $[a,b]$ the area to the surface generated by revolving the graph about x-axis

$$S = \int_a^b 2\pi r \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$r = y$, since the cross section $\perp$ x-axis



. If $x = g(y)$, $c \leq y \leq d$

the area of the surface generated by revolving the graph about y-axis

$$S = \int_c^d 2\pi r \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$r = x$, since the cross section $\perp$ y axis

Exp① Find Area of the surface obtained by rotation $y = \sqrt{16-x^2}$ about x-axis from $x=-1$ to $x=1$

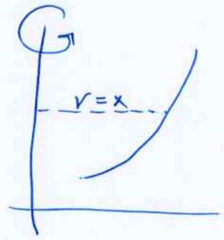
$$S = \int_{-1}^{1} 2\pi\, y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \frac{dy}{dx} = \frac{-x}{\sqrt{16-x^2}}$$

$$= \int_{-1}^{1} 2\pi \sqrt{16-x^2} \sqrt{1+\frac{x^2}{16-x^2}}$$

$$= \int_{-1}^{1} 2\pi \sqrt{16-x^2} \frac{\sqrt{16}}{\sqrt{16-x^2}}\, dx = 8\pi[1 - -1] = 16\pi$$

Exp. The area of the surface obtained by rotating the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $1 \leq x \leq 2$ about y-axis

$$S = \int 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$$S = \int_1^2 2\pi x \sqrt{\frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}}\, dx$$

$$\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}$$

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}$$

$$S = \int_1^2 2\pi x \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\, dx$$

$$= \int_1^2 2\pi x \left(\frac{x}{2} + \frac{1}{2x}\right) dx$$

$$= 2\pi \int_1^2 \frac{x^2}{2} + \frac{1}{2}\, dx = 2\pi \left[\frac{x^3}{6} + \frac{1}{2}x\right]_1^2$$

$$= 2\pi \left[\left(\frac{8}{6} + 1\right) - \left(\frac{1}{6} + \frac{1}{2}\right)\right] = 2\pi \left(\frac{10}{6}\right)$$

$$= \frac{10\pi}{3}$$

Find the surface area generated by revolving the curve

$\boxed{95}$

$x = \sqrt{2y-1} \qquad \frac{5}{8} \leq y \leq 1 \quad , \quad y\text{-axis}$

$$S = \int_{5/8}^{1} 2\pi \, x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$\frac{dx}{dy} = \frac{1}{2}(2y-1)^{-1/2} \, 2$$

$$\left(\frac{dx}{dy}\right)^2 = (2y-1)^{-1}$$

$$\left(\frac{dx}{dy}\right)^2 + 1 = \frac{1}{2y-1} + \frac{2y-1}{2y-1}$$

$$\left(\frac{dx}{dy}\right)^2 + 1 = \frac{2y}{2y-1}$$

$$= 2\pi \int_{5/8}^{1} \sqrt{2y-1} \sqrt{\frac{2y}{2y-1}} \, dy$$

$$= 2\pi \int_{5/8}^{1} \sqrt{2} \, y^{1/2} \, dy$$

$$= 2\sqrt{2}\,\pi \cdot \frac{2}{3} \, y^{3/2} \Big]_{5/8}^{1}$$

$$= \frac{4\sqrt{2}\,\pi}{3} \left[1 - \left(\frac{5}{8}\right)^{3/2} \right]$$

Find the surface area obtained by revolving $y = e^x$, $0 \leq x \leq 1$ about x-axis [Don't evaluate the integral]

$$S = \int 2\pi \, y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$\int_{0}^{1} 2\pi \, e^x \sqrt{1 + e^{2x}} \, dx$$