بعض الإثباتات لمادة اللينير

$\boxed{1}$ If x_1, x_2 are solutions of $Ax=b$, then $\alpha x_1+\beta x_2$ is a solution of $Ax=b$ iff $\alpha+\beta=1$.

Proof ⇒ Given ⇒ $Ax_1=b$ and $Ax_2=b$.

Then ⇒ $A(\alpha x_1+\beta x_2)$

$=\alpha(Ax_1)+\beta(Ax_2)$

$=\alpha b+\beta b = (\alpha+\beta)b = b$.

iff $\alpha+\beta=1$

$\boxed{2}$ x_1, x_2 are solutions of a homogeneous Linear System $Ax=0$, then $\alpha x_1+\beta x_2$ is a solution of $Ax=0$, $\forall \alpha, \beta \in \mathbb{R}$.

Proof ⇒ $A(\alpha x_1+\beta x_2) = \alpha \overset{0}{\boxed{Ax_1}} + \beta \overset{0}{\boxed{Ax_2}}$

$= 0, \ \forall \alpha$ and β.

$\boxed{5}$ If A is an $m\times n$ matrix. The A^TA and AA^T both symmetric.

Proof ⇒ $B = A^TA \Rightarrow B^T = (A^TA)^T = A^T(A^T)^T = A^TA$ is symmetric.

$B = (AA^T)^T = (A^T)^T A^T = AA^T$ is symmetric.

$\boxed{6}$ If A is symmetric and skew symmetric, then A must be zero matrix.

Proof ⇒ $(\alpha A)^T = \alpha A^T = \alpha A$

since A is symmetric

∴ αA is symmetric.

3 Consisting of the linear system:-

A linear system $Ax=b$ is consistent iff b is a linear combination of the columns of A.

Proof $\Rightarrow$ suppose that $Ax=b$ is consistent, so there exist real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$. such that $A\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix} = b$. so,

$\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n = b$.

and so b is a linear combination of the column of A.

conversely,

Suppose that b is a linear combination of the column of A, so there exist real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$b = \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n$.

$\Rightarrow b = A\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}$. So, $\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}$ is a solution of $Ax=b$. that is,

$Ax=b$ is consistent.

---

4 Let A, B be symmetric matrices. then $H = AB - BA$ is skew symmetric.

Proof $\Rightarrow$ $A^T = A$ and $B^T = B$.

$$H^T = (AB - BA)^T = (AB)^T - (BA)^T$$
$$= B^T A^T - A^T B^T$$
$$= BA - AB = \boxed{-H}$$

$\boxed{1}$ Show that $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has no inverse. (singular matrix).

Proof ⇒ if B is any 2x2 matrix, the

$$BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} b_{11} & 0 \\ b_{21} & 0 \end{bmatrix} \neq I$$

$\boxed{2}$ If A and B are nonsingular nxn matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$

Proof ⇒ $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$

$\therefore (AB)^{-1} = B^{-1}A^{-1}.$

$\boxed{3}$ If A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T.$

Proof ⇒ $A^T(A^T)^{-1} = A^T(A^{-1})^T$

$= (A^{-1}A)^T = I^T = I$

$(A^T)^{-1}A^T = (A^{-1})^T A^T$

$= (A^{-1}A)^T = I^T = I.$

$\boxed{4}$ If A and B are invertible matrices, then A+B is also invertible.

Proof ⇒ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\alpha = 1 \neq 0$ non singular. $\alpha = -1 \neq 0$ non singular.

$A+B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \alpha = 0$ sing.

$\boxed{5}$ The sum of singular matrices is also singular.

Proof ⇒ (False):-

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ but $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ non singular.

sing $\alpha = 0$ sing $\alpha = 0$. $\alpha = 1$.

$\boxed{6}$ $A^2 - B^2 = (A-B)\,(A+B)$, where A and B are matrices.

Proof ⇒ False

إذا كانت هيك الحل بتكون true.

if A and B are commute:- $AB = BA$, then:-

$$A^2 - B^2 = (A+B)\quad(A-B).$$

$\boxed{7}$ $(A+B)^2 = A^2 + 2AB + B^2$ ⇒ false.

$\boxed{8}$ If $AB = 0$, then $A = 0$ or $B = 0$. ⇒ False.

Proof ⇒ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ⇒ $AB = 0$ but $A \neq 0$ and $B \neq 0$.

$\boxed{9}$ If $A^2 = 0$, then $A = 0$. ⇒ False.

$\boxed{10}$ If $AB = AC$, then $B = C$ ⇒ false.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

To be true ⇒ if A is invertible and $AB = AC$ ⇒ $B = C$.

$\boxed{11}$ If $A^2 = A$, then $A = 0$ or $A = I$ ⇒ false.

$\boxed{12}$ If $A_{n \times n}$ matrix, such that $A^2 = A$, then $I + A$ is nonsingular and $(I+A)^{-1} = I - \frac{1}{2}A$. ⇒ True.

Proof ⇒ $(I+A)(I-\frac{1}{2}A) = I^2 - \frac{1}{2}IA + AI - \frac{1}{2}A^2$

$= I - \frac{1}{2}A + A - \frac{1}{2}A$ given.

$= I + 0 = I.$

$(I-\frac{1}{2}A)(I+A) = I.$

$\therefore (I+A)^{-1} = I - \frac{1}{2}A.$

$\boxed{13}$ Let A be nxn matrix, then if $A^2=0$, then $I-A$ is nonsingular and $(I-A)^{-1} = I+A$. ⇒ True.

$\boxed{14}$ If A and B are nxn matrices. Then if $AB = A$ and $B \neq I$, then A must be singular. ⇒ True.

Proof ⇒ If A were nonsingular, then A^{-1} exists. Thus,

$A^{-1}(AB) = A^{-1}A \Rightarrow IB = I \Rightarrow B = I$

which is a contradiction. Therefore, A must be singular.

$\boxed{1}$ If A is row equivalent to B, then B is row equivalent to A.

↳ True:-

Proof:) suppose that A is row equivalent to B then

$A = (E_k E_{k-1} \cdots E_1) B$, where $E_1, E_2, ---, E_k$ elementary matrices.

$$(E_k E_{k-1} \cdots E_1)^{-1} A = \underbrace{(E_k \cdots E_1)^{-1} (E_k \cdots E_1)}_{I} B.$$

$\therefore B = E_1^{-1} E_2^{-1} \cdots E_k^{-1} A.$

↳ B is row equivalent to A.

$\boxed{2}$ if A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

↳ True:-

Proof:- We have:

$A = (E_k E_{k-1} \cdots E_1) B$

and $B = F_k F_{k-1} \cdots F_1) C$, where $E_1, ---, E_k, F_1, ---, F_k$ are elementary matrice then,

$A = (E_k E_{k-1} \cdots E_1 F_k F_{k-1} \cdots F_1) C.$

↳ A is row equivalent to C.

$\boxed{3}$ Let A be an nxn matrix. Then the following statements are equivalent.

$\boxed{a}$ A is nonsingular.

$\boxed{b}$ $Ax=0$ has only the trivial solution ($X=0$ the zero solution).

$\boxed{c}$ A is row equivalent to I_n.

Proof $\boxed{a} \rightarrow \boxed{b}$:- suppose that A is nonsingular and y is a solution of $Ax=0$, i.e, $Ay=0$. Multiply both sides by A^{-1} from left, we get $A^{-1}(Ay) = A^{-1}0$. So $Iy = 0$, i.e, $y=0$.

$b \Rightarrow c$:- Suppose that $Ax=0$ has only zero solution and suppose that A is not row equivalent to I, so the RREF of A has a free variable and so $Ax=0$ has infinitely many solutions which is a contradiction.

$c \Rightarrow a$:- Suppose that A is row equivalent to I_n, so there exist a finite sequence of elemetary matrices $E_1, E_2, \dots, E_k$ such that
$(E_k E_{k-1} \dots E_1) I = A$. So,
$A = E_k E_{k-1} \dots E_1$ and so
$A^{-1} = E_1^{-1} E_2^{-1} \dots E_k^{-1}$
A is invertible (non singular).

$\boxed{4}$ If A is a 4×4 matrix and $a_1 + a_2 = a_3 + 2a_4$ then A must be singular. $\Rightarrow$ True.
↳ $Ax=0$ يعني
has infinite.

Proof $\Rightarrow a_1 + a_2 - a_3 - 2a_4 = 0 \Rightarrow (1, 1, -1, -2)$ is a solution of $Ax=0$.
$\Rightarrow Ax=0$ has infinitely many solution.
$\Rightarrow A$ is singular. (theorem).

$\boxed{5}$ If A has no LU-factorization, then A is non singular iff L is nonsingular
↳ False. $\Downarrow$

Elementary matrix ضرب لأنه non singular دائماً

$\boxed{6}$ If A has an LU-factorization, then A is non singular iff U is nonsingular
↳ True $\Rightarrow$ $Ax=0 \Rightarrow L(Ux)=0 \Rightarrow Ux = L^{-1}0 = 0$.

$\boxed{7}$ If A has an LU-factorization, then A is row equivalent to U.
↳ True.

$\boxed{1}$ If A is an $n \times n$ matrix, then $det(A^T) = det(A)$. $\Rightarrow |A^T| = |A|$.

$\hookrightarrow$ Proof:- by induction.

$\boxed{1}$ Lemma:- Let A be an nxn matrix. then $a_{i1}A_{j1}+a_{i2}A_{j2}+\dots+a_{in}A_{jn}$

$$= \begin{cases} 0 & , i \neq j \\ |A| & , i = j \end{cases}$$

↳ Proof:- If $i=j$ then.

$a_{i1}A_{j1}+a_{i2}A_{j2}+\dots+a_{in}A_{jn} = a_{i1}A_{i1}+\dots+a_{in}A_{in} = |A|$.

if $i \neq j$, Let A^1 be the matrix obtained from A by replacing the j^{th} row by the i^{th} row, i.e.,

$$A^1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{matrix} \\ j^{th} \text{ row.} \\ i^{th} \text{ row.} \\ \\ \end{matrix}$$

since two rows of A^1 are the same, so $|A^1|=0$.

So $0 = \det(A^1) = a_{i1}A_{j1}^1 + a_{i2}A_{j2}^1 + \dots + a_{in}A_{jn}^1$

$= a_{i1}A_{j1} + a_{i2}A_{j2} + \dots + a_{in}A_{jn}$.

$\boxed{2}$ Let $E_1, \dots, E_k$ be elementary matrices then $|E_1 \dots E_k| = |E_1| \dots |E_k|$.

↳ proof:- Use math. induction.

$\boxed{3}$ An nxn matrix A is singular iff $\det(A)=0$.

oR ⇒ An nxn matrix A is non singular iff $|A| \neq 0$.

↳ Proof ⇒ Let A be nonsingular. So, A is row equivalent to I_n. that is, there exist elementary matrices $E_1, E_2, \dots, E_k$ such that $A = E_1 \dots E_k I_n$.

So $|A| = |E_1|\,|E_2| \dots |E_k| \neq 0$.

⇒ conversely ⇒ suppose that $|A| \neq 0$ then the matrix A can be changed to RREF. With a finite number of row operations. That is, there exist elementary matrice $E_1 \dots E_k$, and a matrix U in RREF such that $E_k E_{k-1} \dots E_1 U = A$. since

$|A| \neq 0$, So, $|U| \neq 0$. since, all E_i's are invertible, and $|A| = |E_1|\,|E_2| \ldots\ldots |E_k|\,|U|$

So, $U = I_n$, and so A is invertible.

$\boxed{4}$ If A, B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

$\hookrightarrow$ proof:- If A is singular, then $|A| = 0$ and so AB is singular and therefore $|AB| = 0 = |A|\,|B|$

If A is nonsingular, then A is row equivalent to I_n. That is, $A = E_1 \cdots E_k I_n = E_1 \cdots E_k$.

Thus, $|AB| = |E_1 E_2 \cdots E_k B|$

$= |E_1|\,|E_2| \cdots |E_k|\,|B| = |A|\,|B|$.

$\boxed{5}$ $|A+B| = |A| + |B|$, where $+$ is defined $\Rightarrow$ False.

$\boxed{6}$ $|A^n| = |A|^n$, $n = 0, 1, \ldots\ldots$ $\Rightarrow$ True.

$\boxed{3}$ $|kA| = k^n |A|$ where $A_{n \times n}$, $k \in \mathbb{R}$. $\Rightarrow$ True.

$\boxed{4}$ If A is nonsingular, then $\det(A^{-1}) = \dfrac{1}{\det A}$ $\Rightarrow$ True.

$\hookrightarrow$ proof:- A nonsingular $|A^{-1}| = \dfrac{1}{|A|}$

$$A^{-1} A = I$$
$$|A^{-1} A| = |I|$$
$$|A^{-1}|\,|A| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}.$$

$\boxed{5}$ If $A^2 = A$, then $|A| = 0$ or $|A| = 1$ ⇒ False.

└ proof ⇒ $A^2 = A \Rightarrow |A| = 0$ or $|A| = 0$

$|A^2| = |A|$

$|A|^2 = |A|.$

$|A|(|A| - 1) = 0$

$|A| = 0$ or $|A| = 1.$

$\boxed{6}$ If $A^T A = I$, then $|A| = \mp 1$ ⇒ true.

└ proof:- if $|A^T A| = |I|$

$|A^T|\,|A| = 1$

$|A|\,|A| = 1 \Rightarrow |A|^2 = 1 \Rightarrow |A| = \mp 1.$

$\boxed{7}$ If $A_{n\times n}$ is skew symmetric and n is odd, then A must be singular.

└ True. ⇒ proof ⇒ $A^T = -A.$

$|A^T| = |-A|.$

$|A| = (-1)^n\,|A| = -|A|$, n odd.

$\Rightarrow 2|A| = 0.$

$|A| = 0 \Rightarrow A$ singular.

$\boxed{8}$ If $A_{n\times n}$ is skew symmetric and n is even, then A must be nonsingular.

└ False. ⇒ proof ⇒ same as $\boxed{7}$ but n is even.

$\boxed{9}$ Let $A_{n\times n}$, $B_{n\times n}$. then AB is nonsingular iff A and B are Both nonsingular.

└ True. ⇒ proof ⇒ AB is nonsingular $\Leftrightarrow |AB| \neq 0$

$|A|\,|B| \neq 0.$

$\Leftrightarrow |A| \neq 0$ and $|B| \neq 0$

So A and B both nonsingular.

$\boxed{10}$ If A,B, and C are 3x3 matrices, $|A|=9$, $|B|=2$, $|C|=3$. then,
$|4C^TBA^{-1}| = \frac{128}{3}$ ⇒ True.

→ proof ⇒ $|4C^TBA^{-1}| = 4^3 |C^T| |B| |A^{-1}|$

$= 64 |C| |B| \frac{1}{|A|} = 64 \times 3 \times 2 \times \frac{1}{9}$

$= \boxed{\frac{128}{3}}$.

$\boxed{11}$ Let A be an mxn matrix. Explain why the matrix multiplications A^TA and AA^T are possible.

→ A is mxn matrix and A^T is nxm matrix.
A^TA is nxm mxn matrix which is valid.
AA^T is mxn nxm matrix which is valid.

$\boxed{12}$ A matrix A said to be skew symmetric if $A^T=-A$. show that if a matrix is skew symmetric, then its diagonal entries must all be 0.

→ A is skew symmetric matrix, so $A^T=-A$ which is a square matrix a_{ij} are the matrices on A and b_{ij} the entries of A^T. So we have $b_{ij}=a_{ji}$ since $A^T=-A$ so $b_{ij}=-a_{ji}$. Thus $a_{ji}=-a_{ij}$, and if we take $i=j$ we have $a_{ij}=-a_{ij}$ which gives us that $a_{ij}=0$, which are the diagonal entries.

$\boxed{1}$ $A\,\mathrm{adj}(A) = |A|\, I_n$.

↳ proof ⇒ the ij^{th} entry of $A\,\mathrm{adj}(A)$ is:-

$$A\,\mathrm{adj}(A) = a_{i1}A_{j1} + \cdots + a_{in}A_{jn} = \begin{cases} |A|, & i=j \\ 0, & i \neq j \end{cases} = |A|\, I_n.$$

$\boxed{2}$ $A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A)$.

↳ Proof ⇒ Since A is nonsingular, then A^{-1} exists. From last theorem We know, $A\,\mathrm{adj}(A) = |A|\, I_n$. Multiply both sides by A^{-1} from left:-

$$A^{-1}A\,\mathrm{adj}(A) = |A|\, A^{-1} I_n = |A|\, A^{-1}.$$

$\mathrm{adj}(A) = |A|\, A^{-1}$ and so, since $|A| \neq 0$, $A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A)$.

$\boxed{3}$ $X_i = \frac{|A_i|}{|A|}$, $i = 1, 2, \ldots, n$.

↳ Proof ⇒ since $x = A^{-1}b = \frac{1}{|A|}\,\mathrm{adj}(A)\, b$.

it follows that ⇒ $X_i = \frac{b_1A_{1i} + b_2A_{2i} + \cdots + b_nA_{ni}}{|A|} = \frac{|A_i|}{|A|}$.

$\boxed{4}$ Let A be a nonsingular $n \times n$ matrix with $n \geq 1$. Show that $|\mathrm{adj}A| = |A|^{n-1}$.

↳ If A is nonsingular, then $|A| \neq 0$ and hence $\mathrm{adj}A = |A|A^{-1}$ is also nonsingular ⇒ $|\mathrm{adj}A| = |\underbrace{(|A|)}_{\text{constant}}A^{-1}| = |A|^n\, |A^{-1}| = |A|^n \frac{1}{|A|} = |A|^{n-1}$.

$\boxed{5}$ Show that if A is nonsingular, then $\mathrm{adj}\,A$ is nonsingular and $(\mathrm{adj}\,A)^{-1} = |A^{-1}|\,A = \mathrm{adj}\,A^{-1}$.

↳ If A nonsingular, then $|A| \neq 0$ and hence $\mathrm{adj}\,A = |A|A^{-1}$ is also nonsingular.

To prove the equation, we have ⇒ $(\mathrm{adj}\,A)^{-1} = \left(\overset{ar}{|A|}\, A^{-1}\right)^{-1} = \frac{1}{|A|}\,(A^{-1})^{-1} = \frac{1}{|A|}\,A = |A^{-1}|\,A \Rightarrow \boxed{1}$

and $\mathrm{adj}\,A^{-1} = |A^{-1}|\,(A^{-1})^{-1} = |A^{-1}|\,A$, hence $\boxed{1}$ and $\boxed{2}$ gives the equation.

$\swarrow \boxed{2}$

$\boxed{6}$ if $|\text{adj} A| = |A|$ and nonsingular, then A is 2×2 matrix $\Rightarrow$ False.

$\hookrightarrow$ $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\Rightarrow$ $|\text{adj}(I)| = |I|^2 = (1)^2 = 1$

$|I| = 1$

$\hookrightarrow$ $|\text{adj} I| = |I|$ but, I is 3×3.

$\boxed{7}$ Let A be $n\times n$ matrix (nonsingular), then $\text{adj}(\text{adj} A) = |A|^{n-2} A$

$\hookrightarrow$ Proof:- since A is nonsingular, then $|A| \neq 0$. and hence $\text{adj} A = |A| A^{-1}$.

$\Rightarrow \text{adj}(\text{adj} A) = \text{adj}(|A| A^{-1}) = \left| |A| A^{-1} \right| (|A| A^{-1})^{-1} = |A|^n |A^{-1}| \frac{1}{|A|} (A^{-1})^{-1}$

$= |A|^n \frac{1}{|A|} \cdot \frac{1}{|A|} \cdot A = |A|^{n-2} A.$

$\boxed{8}$ Show that if $|A| = 1$, then $\text{adj}(\text{adj} A) = A$.

$\hookrightarrow$ This question is special case of the above example. We proved that $\text{adj}(\text{adj} A) = |A|^{n-2} A$.

if $|A| = 1$, then $\text{adj}(\text{adj} A) = (1)^{n-2} A = A$.

$\boxed{9}$ Let A and B are $n\times n$ matrices. show that if $AB = A$ and $B \neq I$ then A must be singular.

$\hookrightarrow$ $AB = A$, $B \neq I$ suppose A is nonsingular and A^{-1} exists such that

$A^{-1}A = AA^{-1} = I$ multiply by A^{-1}.

$A^{-1}(AB) = (A^{-1}A)B = A^{-1}A \Rightarrow B = I$ (contradiction).

Therefore A is singular.

$\boxed{10}$ Prove that if A is nonsingular then A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$

$\hookrightarrow$ $A^{-1}A = I \Rightarrow (A^{-1}A)^T = I^T$

$= A^T (A^{-1})^T = I^T$

$(A^T)^{-1} A^T (A^{-1})^T = (A^T)^{-1} I^T$

$(A^{-1})^T = (A^T)^{-1}$

$\boxed{1}$ $Q = \{ x = \frac{a}{b}$ are integers $b \neq 0 \}$. is not a vector space.

↳ $x = \frac{2}{5} \in Q$, $\alpha = \sqrt{2}$

$\alpha x = \sqrt{2} \times \frac{2}{5} = \frac{2\sqrt{2}}{5} \notin Q$.

$\boxed{2}$ $Z = \{0, \mp 1, \mp 2, \ldots\}$ is not a vector space.

↳ $\alpha = \frac{1}{2}$, $x = 5 \in Z$ but.

$\alpha x = \frac{1}{2} \times 5 = \frac{5}{2} \notin Z$.

$\boxed{3}$ Irrationals, $(\sqrt{2}, \sqrt{3}, e, \pi, \ldots)$. is not a vector since:-

↳ $\alpha = \frac{1}{2}$, $x = \sqrt{2} \in Q^c$.

but $\alpha x = \sqrt{2} \cdot \sqrt{2} = 2 \notin Q^c$.

$\boxed{4}$ $0 \vec{v} = 0$

↳ proof:- $0 = 0 + 0$, so $(0+0) v = 0v$.

thus $0v + 0v = 0v$ add to both sides

$-0v$. So, $0v + 0v + -0v = 0v + -0v$.

$\Rightarrow 0v + \vec{0} = \vec{0}$. Hence $0v = \vec{0}$.

$\boxed{5}$ If $x + y = \vec{0}$, then $y = -x$.

↳ proof:- Add $-x$ to both sides of $x + y = \vec{0}$.

So, $-x + x + y = -x + 0$. Thus, $\vec{0} + y = -x + 0$.

$\Rightarrow y = -x$.

$\boxed{6}$ $-1.v = -v, \forall v \in V$.

↳ $0 = 1 + -1$, So $(1 + -1) v = 0v = \vec{0}$.

Thus, $1v + -1v = \vec{0}$, so $v + -1v = \vec{0}$.

$\Rightarrow -v + v + -1v = -v + 0 = -v$.

$\Rightarrow 0 + -1v = -v \Rightarrow$ Thus $-1v = -v$.

$\boxed{1}$ Let S be a subspace of a vector space V, then $\vec{0} \in S$.

↳ proof ⇒ since S is a subspace of V, then $S \neq \emptyset$. Let $x \in S$, so $0x = \vec{0} \in S$.

$\boxed{2}$ $S \cap T$.

↳ * since $0 \in S$, $0 \in T$, then $0 \in S \cap T \Rightarrow S \cap T \neq \emptyset$.

* Let $x, y \in S \cap T$, then $x, y \in S$ and $x, y \in T$

$\Rightarrow x+y \in S$ and $x+y \in S$.

$x+y \in S \cap T$.

* let $\alpha \in \mathbb{R}$ and $x \in S \cap T$.

since $x \in S \cap T$, then $x \in S$ and $x \in T$. Since S and T are subspases, it follows $\alpha x \in S$ and $\alpha x \in T$. $\Rightarrow \alpha x \in S \cap T$.

$\boxed{3}$ $S \cup T$ is not always a subspase of V.

↳ Let $S = \{(x,0) : x \in \mathbb{R}\}$.

$T = \{(0,y) : y \in \mathbb{R}\}$.

Notice that S and T are subspase of $\mathbb{R}^2$ but $S \cup T = \{(x,y) : x \text{ or } y \text{ is zero}\}$ is not subspase, for example, $(0,1), (1,0) \in S \cup T$, but $(0,1)+(1,0) = (1,1) \notin S \cup T$.

$\boxed{4}$ Let A be $m \times n$ matrix, then $N(A)$ is a subspace of $\mathbb{R}^n$.

↳ $\boxed{i}$ since $A0 = 0$, then $0 \in N(A)$

$\therefore N(A) \neq \emptyset$

$\boxed{ii}$ Let $x, y \in N(A)$. Then $Ax = 0$ and $Ay = 0$, so $A(x+y) = Ax + Ay$
$= 0 + 0 = 0$.

$\Rightarrow x+y \in N(A)$.

$\boxed{iii}$ Let $x \in N(A)$, $\alpha \in \mathbb{R}$. Then $Ax = 0$, so $A(\alpha x) = \alpha Ax$
$= \alpha 0 = 0$.

$\alpha x \in N(A)$.

$\boxed{5}$ Let V be avector space and let $v_1, v_2, \dots, v_k \in V$. then span $(v_1, \dots, v_k)$ is asubspase of V.

→ $\boxed{i}$ since $\vec{0} = 0.v_1 + 0 v_2 + 0.v_k$, then $\vec{0}$ span $(v_1, \dots, v_k)$. That is, span $(v_1, \dots, v_k) \neq \emptyset$.

$\boxed{ii}$ Let $x, y \in$ span $(v_1, \dots, v_k)$. Then $x = \alpha_1 v_1 + \dots + \alpha_k v_k$.

$y = \beta_1 v_1 + \dots + \beta_k v_k$.

So, $x + y = (\alpha_1 + \beta_1) v_1 + \dots + (\alpha_k + \beta_k) v_k$.

$= \gamma_1 v_1 + \dots + \gamma_k v_k$

$\Rightarrow x + y \in$ span $(v_1, \dots, v_k)$.

$\boxed{iii}$ Let $x \in$ span $(v_1, \dots, v_k)$, $\alpha \in \mathbb{R}$.

Then $\alpha x = \alpha (c_1 v_1 + c_2 v_2 + \dots + c_n v_n)$

$= (\alpha c_1) v_1 + (\alpha c_2) v_2 + \dots + (\alpha c_n) v_n$.

$\Rightarrow \alpha x \in$ span $(v_1, \dots, v_k)$.

therefore, span $(v_1, \dots, v_k)$ is subspase of V.

$\boxed{6}$ Let A be an nxn matrix. show that if $A^2 = 0$, then $I - A$ is nonsingular and $(I - A)^{-1} = I + A$.

$(I + A)(I - A) = I + A - A + A^2 = I$

$(I - A)(I + A) = I - A + A + A^2 = I$ therefore $I - A$ is non singular and $(I - A)^{-1} = I + A$.

Let A be an nxn matrix. Then the following statement are equivalent:

A- A is non singular.

B- Ax=0 has only the trivial solution (x=0 The zero solution).

C- A is row equivalent to In.

إشا تهم صيغة ال

(الدكتور علاء).

---

* Let A be nxn matrix and α is scalar. show that $|\alpha A| = \alpha^n |A|$.

→ αI is adiagonal matrix with all diagonal entries equal α. So $|I\alpha| = \alpha^n$

$\alpha A = \alpha (IA) \Rightarrow (\alpha I) A \Rightarrow |\alpha A| = | |\alpha I| A | = |\alpha I| |A| = \alpha^n |A|$.
