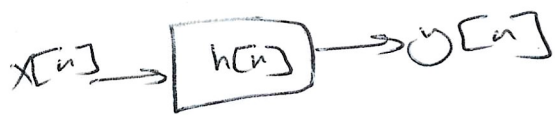


Properties of LTI ^{Discrete} system



$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k] h[k]$$

Causality of LTI system

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

$$y[n_0] = \sum_{k=-\infty}^{\infty} x[k] h[n_0-k]$$

the system is causal if $y[n]$ is independent of the future values of the input

for $k < 0$ then $y[n] \rightarrow x[n-k]$
 $x[n+k]$

for this case now the output depends on the future input, therefore to be causal $y[n]$ should be zero. So $h[n]$ should be zero

Stability of LTI system

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k] h[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |x[k]| |h[n-k]|$$

It is given that $|x[n]|$ is bounded

So to have bounded $y[n]$, $\sum_{k=-\infty}^{\infty} |h[n-k]| < \infty$
 $\downarrow$
bounded $\rightarrow$ finite

Finite impulse response (FIR)

infinite impulse response (IIR)

FIR

$$h[n] = \{0, 0, 0, \dots, 0, \underset{\uparrow}{a_1}, a_2, a_3, 0, 0, 0, 0\}$$

$$h[n] = 0 \text{ for all } n > N$$

IIR

$$h[n] = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{o.w} \end{cases} \quad h = \{0, \dots, 0, \underset{\uparrow}{1}, a, a^2, a^3, \dots\}$$

Linear Difference Equations

$$\sum_{k=0}^N a_k y(n-k) = \sum_{r=0}^m b_r x(n-r)$$

Order $\equiv$ No of delays in output sequence

e.g zero order

$$a_0 y[n] + 0 = \sum_{r=0}^m b_r x[n-r]$$

$$a_0 = 1 \quad h[n] = \begin{cases} b_n, & n=0, 1, 2, 3, \dots, m \\ 0, & \text{o.w} \end{cases}$$

1st order

$$a_0 y[n] + a_1 y[n-1] = \sum_{r=0}^m b_r x[n-r]$$

In general

$$\sum_{k=0}^N a_k y[n-k] = \sum_{r=0}^M b_r x[n-r]$$

$$a_0 y[n] + \sum_{k=1}^{\infty} a_k y[n-k] = \sum_{r=0}^M b_r x[n-r]$$

If we assume that $a_0 = 1$ and $a_k = 0, k=1, 2, 3, 4, \dots, N$

$$y[n] = \sum_{r=0}^M b_r x[n-r], \quad \text{where } b_r = \begin{cases} b_n, & n=0, 1, 2, 3, \dots, M \\ 0, & \text{o.w} \end{cases}$$

Example 8- Consider the following system described by the following difference equation-

$$y[n] = x[n] + a y[n-1]$$

Evaluate the response of the system if

$$x[n] = K \delta[n] \text{ and } y[-1] = C$$

Soln- ~~once~~ $x[n]$ is impulse then $y[n] = h[n]$
= impulse response

* for $n \geq 0$

$$\begin{aligned} y[0] &= K \delta[0] + a y[-1] \\ &= K(1) + a C \end{aligned}$$

$$\begin{aligned} y[1] &= K \delta[1] + a y[0] \\ &= 0 + a [K + a C] \\ &= a K + a^2 C \end{aligned}$$

$$\begin{aligned} y[2] &= K \delta[2] + a y[1] \\ &= 0 + a [a K + a^2 C] \\ &= a^2 K + a^3 C \end{aligned}$$

$$\therefore y[n] = a^n K + a^{n+1} C$$

for $n \leq -1$

$$y[n-1] = a^{-1} [y[n] - x[n]]$$

$n = -1$

$$y[-2] = a^{-1} [y[-1] - x[-1]]$$

$$= a^{-1} [c - k\delta[-1]]$$

$$= a^{-1} [c - 0]$$

$$= a^{-1} c$$

$$y[-3] = a^{-1} [y[-2] - k\delta[-2]]$$

$$= a^{-1} [a^{-1} c - 0]$$

$$= a^{-2} c$$

$$y[n] = a^{n+1} c$$

$$y[n] = a^{n+1} c + a^n k u[n]$$

Example 2 - Determine the impulse of the first order system given as follows:

$$y[n] = x[n] + a y[n-1], \text{ Assume } y[n] = 0, n < 0$$

for impulse response $x[n] = \delta[n] \Rightarrow y[n] = h[n]$

$$h[n] = \delta[n] + a h[n-1]$$

$y[n] = 0$ for $n < 0$ causal system

$$y[-1] = 0 \Rightarrow h[-1] = 0$$

$$y[-2] = 0 \quad h[-2] = 0$$

$$h[0] = \delta[0] + a h[-1]$$

$\uparrow$
0

$$= 1$$

$$h[1] = y[1] = a$$

$$h[2] = a^2$$

$$h[n] = a^n u[n]$$

$$y[n] = x[n] + \alpha y[n-1]$$

$$y[n-1] = \frac{y[n] - x[n]}{\alpha}$$

$$= \alpha^{-1} [y[n] - x[n]]$$

Let us assume that $y[n] = 0$ for $n > 0$

non causal system

and $x[n] = \delta[n]$

$$y[n-1] = \alpha^{-1} [y[n] - \delta[n]]$$

$$y[1] = 0$$

$$y[0] = 0$$

$$y[-1] = \alpha^{-1} [y[0] - \delta[0]]$$

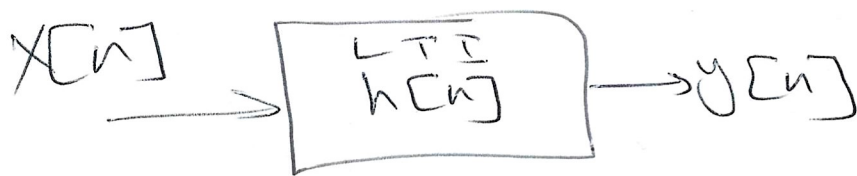
$$= -\alpha^{-1}$$

$$y[-2] = \alpha^{-1} [y[-1] - \delta[-1]]$$

$$= \alpha^{-1} [-\alpha^{-1} - 0]$$

$$= -\alpha^{-2}$$

* Frequency Response of the system



$$X[n] = e^{j\omega_0 n}$$

for LTI

$$Y[n] = \sum_{k=-\infty}^{\infty} h[k] X[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0(n-k)}$$

$$= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}$$

Eigen Function } Same as the input function
 $H(e^{j\omega_0})$ function of ω_0

$$Y[n] = H(e^{j\omega_0}) e^{j\omega_0 n}$$

where $H(e^{j\omega_0}) = |H(e^{j\omega_0})| \angle \angle H(e^{j\omega_0})$

when input to the system, produces the same input signal.

$$|H(e^{j\omega})| = |H(e^{-j\omega})|$$

Example 2: Consider the ideal delay system defined by

$$y[n] = x[n - nd]$$

find the frequency response of the system?

$$x[n] = e^{j\omega_0 n}$$

$$y[n] = x[n - nd] = e^{j\omega_0 (n - nd)} = e^{j\omega_0 n} \underbrace{e^{-j\omega_0 nd}}$$

$$H(e^{j\omega_0}) = \underbrace{1}_{\text{magnitude}} e^{\underbrace{-j\omega_0 nd}_{\text{phase}}}$$

Example 3: Consider $h[n] = \delta(n - nd)$ evaluate the frequency response

$$\begin{aligned} H(e^{j\omega_0}) &= \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \\ &= \sum_{k=-\infty}^{\infty} \delta(k - nd) e^{-j\omega_0 k} \end{aligned}$$

$$= \underbrace{1}_{\text{magnitude}} e^{\underbrace{-j\omega_0 nd}_{\text{phase}}}$$

$$H(e^{j\omega_0}) = \cos(\omega_0 nd) - j \sin(\omega_0 nd)$$

$$H_R(e^{j\omega}) + j H_I(e^{j\omega})$$

where $H_R(e^{j\omega}) = \cos(\omega nd)$ and $H_I(e^{j\omega}) = -\sin(\omega nd)$

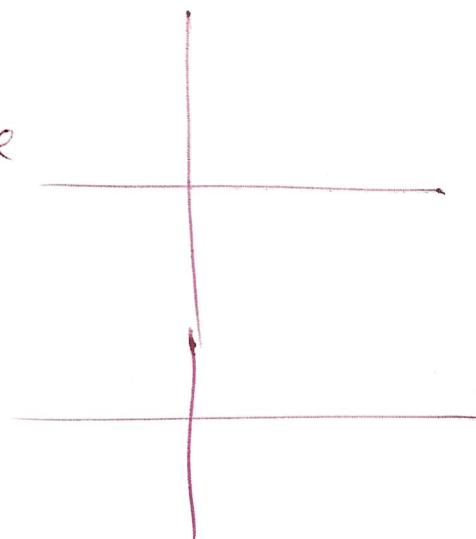
* Sinusoidal response

$$x[n] = A \cos(\omega_0 n + \phi)$$

$$H(e^{j\omega_0}) = |H(e^{j\omega_0})| e^{j\theta(\omega_0)}$$

$$y[n] = A |H(e^{j\omega_0})| \cos(\omega_0 n + \phi + \theta)$$

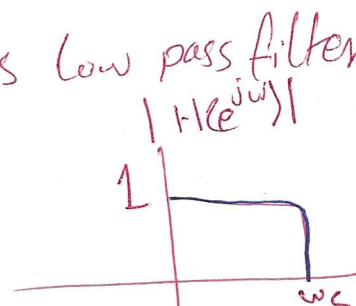
$$H(e^{j\omega_0}) \begin{matrix} \rightarrow \text{magnitude} \\ \rightarrow \text{phase} \end{matrix}$$



Example 2- consider the system as low pass filter

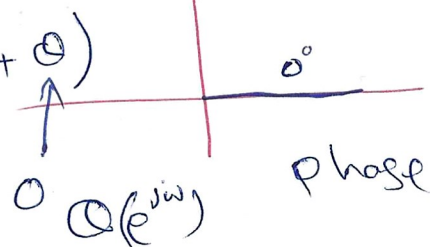
$$x[n] = A \cos(\omega_0 n + \phi)$$

based on the magnitude and the phase of the frequency response

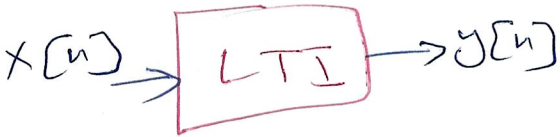


$$y[n] = A |H(e^{j\omega_0})| \cos(\omega_0 n + \phi + \theta)$$

$\uparrow$ if $\omega_0 < \omega_c$
 0 if $\omega_0 > \omega_c$



Example 2



$$x_1[n] = 2 \cos(0.2 \pi n)$$

$$y_1[n] = (0.5) 2 \cos(0.2 \pi n + \frac{\pi}{4})$$

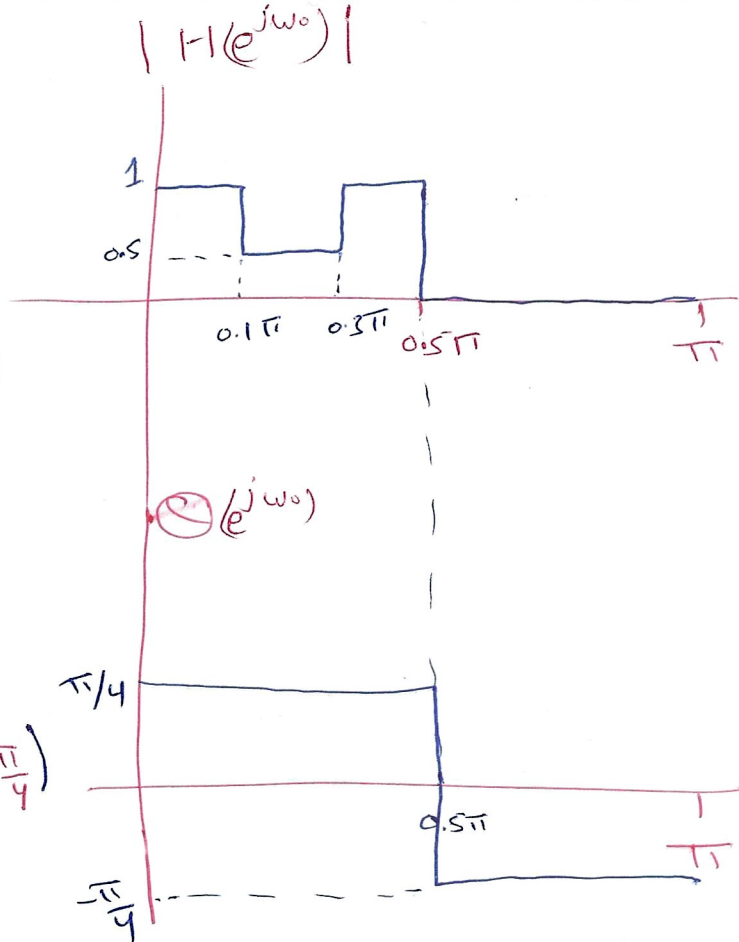
$$x_2[n] = 4 \cos(0.4 \pi n)$$

$$y_2[n] = (1) 4 \cos(0.4 \pi n + \frac{\pi}{4})$$

$$x_3[n] = 3 \cos(0.8 \pi n)$$

$$y_3[n] = (0) (3) \cos(0.8 \pi n - \frac{\pi}{4})$$

$$= 0$$



Examples
consider the following system described by the following Difference equation:

$$y[n] - y[n-1] = x[n]$$

assume the system is Causal and Stable.

From previous example $h[n] = a^n u[n] \rightarrow$ causality

$0 < a < 1 \rightarrow$ Stability

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$\uparrow$ magnitude $\downarrow$ phase

$$= \frac{1}{1 - a e^{-j\omega}}$$

$$|H(e^{j\omega})|^2 = \frac{1}{1 - a e^{j\omega}} \cdot \frac{1}{1 - a e^{-j\omega}}$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$0 < |a| < 1$$

Sum of geometric series

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$= \frac{1}{1 - a [\cos(\omega) - j\sin(\omega)]}$$

$$= \frac{1 \angle 0}{1 - a \cos(\omega) + ja \sin(\omega)}$$

$$= \frac{1 \angle 0}{\sqrt{(1 - a \cos(\omega))^2 + (a \sin(\omega))^2} \angle \tan^{-1}\left(\frac{a \sin(\omega)}{1 - a \cos(\omega)}\right)}$$

$$= \frac{1 \angle 0}{\sqrt{1 - 2a \cos(\omega) + a^2 \cos^2(\omega) + a^2 \sin^2(\omega)} \angle \tan^{-1}\left(\frac{a \sin(\omega)}{1 - a \cos(\omega)}\right)}$$

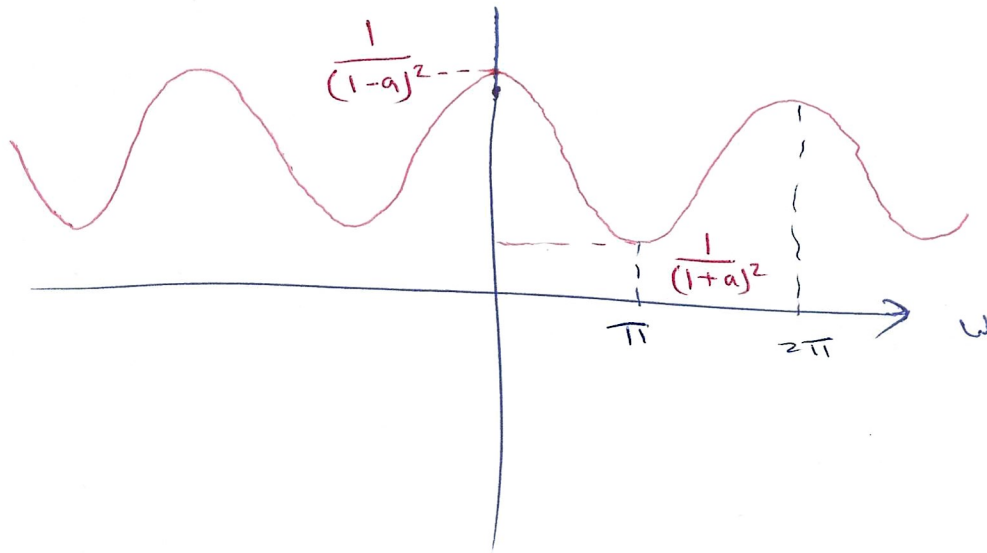
$$|H(e^{j\omega})| = \frac{1}{\sqrt{1 - 2a \cos(\omega) + a^2}}, \quad \angle H(e^{j\omega}) = -\tan^{-1}\left(\frac{a \sin(\omega)}{1 - a \cos(\omega)}\right)$$

$$|H(e^{j\omega})|^2 = \frac{1}{1 - 2a \cos(\omega) + a^2}$$

easier to sketch

Uploaded By: Malak Obaid

$$|H(e^{j\omega})|^2$$



the response is low pass filter

the frequency response is continuous

Suppose $x[n] = 2 \cos(0.2\pi n)$, find $y[n]$

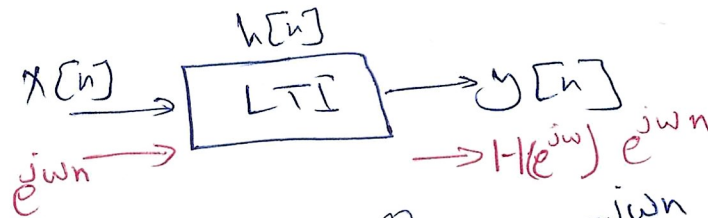
$$y[n] = () 2 \cos(0.2\pi n +)$$

from frequency response

$$\angle H e^{j\omega} = -\tan^{-1} \left(\frac{a \sin \omega}{1 - a \cos \omega} \right)$$

- * Properties of Frequency Response
- continuous
 - periodic

Fourier Transform for DT Systems



$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} \quad (\text{The frequency response of the system})$$

$\equiv$ Fourier transform for DT systems

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

, ω is a continuous variable
 n is a discrete variable

to prove the relation

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n} d\omega$$

$$= \sum_{k=-\infty}^{\infty} h[k] \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-k)} d\omega$$

$$= \sum_{k=-\infty}^{\infty} h[k] \delta[n-k]$$

$$= h[n]$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-k)} d\omega = \frac{\sin \pi(n-k)}{\pi(n-k)}$$

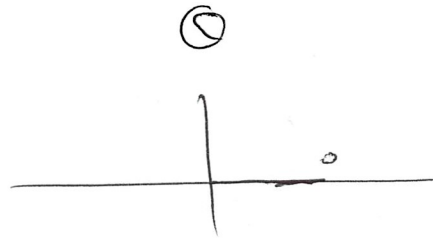
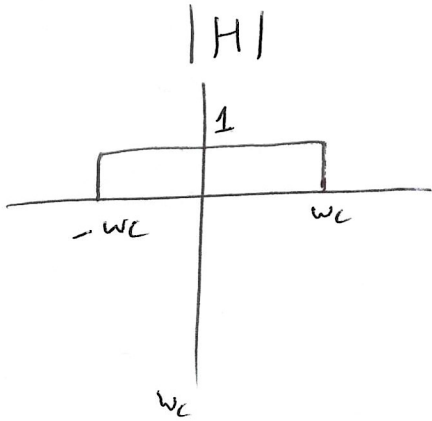
$$= \frac{\sin \pi x}{\pi x} = \text{sinc}(\pi x)$$

$$= \begin{cases} 1, & n=k \\ 0, & \text{o.w} \end{cases}$$

$$= \delta(n-k)$$

Examples Ideal low pass Filter

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c \leq \omega < 2\pi \end{cases}$$



$$h[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 e^{j\omega n} d\omega = \frac{1}{2\pi j n} \left(e^{j\omega_c n} - e^{-j\omega_c n} \right)$$

$$= \frac{\sin \omega_c n}{\pi n}$$

* Theorems of Fourier transform for DT

1- Linearity

$$\mathcal{F}[a x_1[n] + b x_2[n]] = a X_1(e^{j\omega}) + b X_2(e^{j\omega})$$

2- Time shifting

$$\mathcal{F}[x[n - n_d]] = X(e^{j\omega}) e^{-j\omega n_d}$$

3- Frequency shifting

$$\mathcal{F}[x[n] e^{j\omega_0 n}] = X(e^{j(\omega - \omega_0)})$$

4- Time reversal

$$x[n] \xrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$x[-n] \xrightarrow{\mathcal{F}} X(e^{-j\omega})$$

$$X^*(e^{j\omega})$$

$$\left\{ \begin{array}{l} |X(e^{j\omega})| = |X(e^{-j\omega})| \\ \text{even function} \\ \text{but the phase is} \\ \text{odd} \end{array} \right.$$

5- Differential in frequency

$$n X[n] \xleftrightarrow{F} j \frac{dX(e^{j\omega})}{d\omega}$$

6- Parseval's theorem

$$E = \sum_{n=-\infty}^{\infty} |X[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

$|X(e^{j\omega})|^2$ is energy spectral density

7- Convolution theorem

$$\begin{aligned} F(X[n] * h[n]) &= X(e^{j\omega}) H(e^{j\omega}) \\ &= Y(e^{j\omega}) \end{aligned}$$

8- Modulation and windowing theorem

$$X[n] w[n] \xleftrightarrow{F} X(e^{j\omega}) * W(e^{j\omega})$$

Example: Consider the following system

$$y[n] = x[n] \cos(\omega_0 n)$$

Evaluate $Y(e^{j\omega})$

Solution:-

$$Y(e^{j\omega}) = \mathcal{F}[x[n] \cos(\omega_0 n)]$$

$$= \mathcal{F}\left[\frac{1}{2} x[n] e^{j\omega_0 n} + \frac{1}{2} x[n] e^{-j\omega_0 n}\right]$$

Linearity

$$= \mathcal{F}\left[\frac{1}{2} x[n] e^{j\omega_0 n}\right] + \mathcal{F}\left[\frac{1}{2} x[n] e^{-j\omega_0 n}\right]$$

$$= \frac{1}{2} X(e^{j(\omega - \omega_0)}) + \frac{1}{2} X(e^{j(\omega + \omega_0)})$$

frequency shifting

using modulation theorem

$$\begin{aligned} \mathcal{F}\left[\frac{1}{2} x[n] e^{j\omega_0 n}\right] &= \frac{1}{2} \mathcal{F}[x[n] e^{j\omega_0 n}] \\ &= \frac{1}{2} \mathcal{F}[x[n]] * \mathcal{F}(e^{j\omega_0 n}) \end{aligned}$$

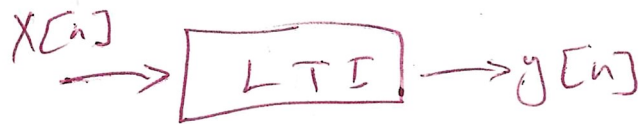
constant

Given that $\mathcal{F}(1) = \delta(\omega)$

$$\therefore \mathcal{F}(e^{j\omega_0 n}) = \delta(\omega - \omega_0)$$

$$\begin{aligned} \therefore \mathcal{F}\left[\frac{1}{2} x[n] e^{j\omega_0 n}\right] &= \frac{1}{2} \mathcal{F}[x[n]] * \delta(\omega - \omega_0) \\ &= \frac{1}{2} X(e^{j(\omega - \omega_0)}) \end{aligned}$$

Example: Consider the following system



If $x[n] = \delta[n - nd]$, find $Y(e^{j\omega})$

$$y[n] = x[n] * h[n]$$

$$= \delta[n - nd] * h[n]$$

$$= h[n - nd]$$

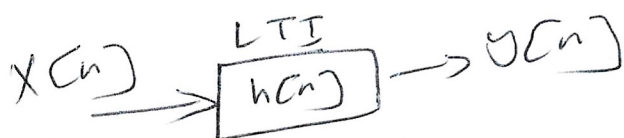
time shifting

$$Y(e^{j\omega}) = F(h[n - nd]) = H(e^{j\omega}) e^{-j\omega nd}$$

Example: Evaluate the frequency response of

the following LTI system with input/output relation given by the following difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2]$$



$$y[n] = x[n] * h[n]$$

$$F(y[n]) = F[x[n] * h[n]]$$

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$F(y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2])$$

$$Y(e^{j\omega}) - \frac{1}{2}Y(e^{j\omega})e^{-j\omega} = X(e^{j\omega}) + 2X(e^{j\omega})e^{-j\omega} + X(e^{j\omega})e^{-j2\omega}$$

$$Y(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega}] = X(e^{j\omega})[1 + 2e^{-j\omega} + e^{-j2\omega}]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

Example 0- consider the following system $y[n] - y[n-1] = x[n]$
evaluate $H(e^{j\omega})$

$$y[n] - y[n-1] = x[n]$$

$$F(y[n] - y[n-1] = x[n])$$

$$Y(e^{j\omega}) - Y(e^{j\omega})e^{-j\omega} = X(e^{j\omega})$$

$$\left\{ \begin{array}{l} Y(e^{j\omega})[1 - e^{-j\omega}] = X(e^{j\omega}) \\ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - e^{-j\omega}} \end{array} \right.$$

Example 8 Determine the Fourier transform of the sequence

$$x[n] = a^n u[n-5]$$

to evaluate $X(e^{j\omega})$ the following steps should be considered

$$x[n] = a^n u[n-5]$$

$$x[n] = a^n u[n-5] \frac{a^{-5}}{a^{-5}}$$

$$= \frac{1}{a^{-5}} e^{n-5} u[n-5]$$

$$= a^5 e^{n-5} u[n-5]$$

↑
scaling

↑
time shifting

$$X(e^{j\omega}) = a^5 \left(\frac{1}{1 - a e^{-j\omega}} \right) e^{-j5\omega}$$

$$x[n] = a^n u[n]$$

$$X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

Example 2- Determine the impulse response of the high pass filter with delay

$$H_{HP}(e^{j\omega}) = \begin{cases} e^{-j\omega nd} & \omega_c < |\omega| < \pi \\ 0 & |\omega| < \omega_c \end{cases}$$

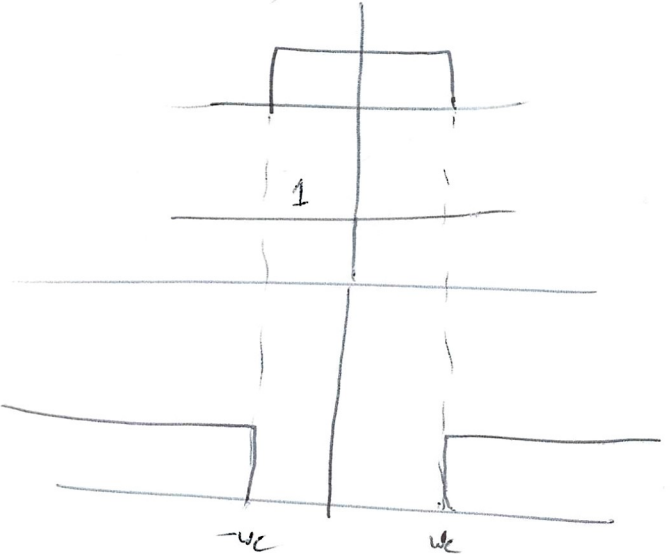
for ideal High pass filter

$$H_{HP}(e^{j\omega}) = 1 - H_{LP}(e^{j\omega})$$

for non ideal

$$H_{HP}(e^{j\omega}) = e^{-j\omega nd} (1 - H_{LP}(e^{j\omega}))$$

$H(e^{j\omega})$ ideal LPF



$$h_{HP}[n] = \delta[n - nd] - h_{LP}(n - nd)$$

$$= \delta(n - nd) - \frac{\sin(\omega_c(n - nd))}{\pi(n - nd)}$$

Inverse Fourier transform

Example 8- Suppose that

$$X(e^{j\omega}) = \frac{1}{(1 - ae^{-j\omega})(1 - be^{-j\omega})}$$

Evaluate $X[n]$

$$\frac{1}{(1 - ae^{-j\omega})(1 - be^{-j\omega})} = \frac{A}{1 - ae^{-j\omega}} + \frac{B}{1 - be^{-j\omega}}$$

$$1 = A(1 - be^{-j\omega}) + B(1 - ae^{-j\omega})$$

$$1 - be^{-j\omega} = 0$$

$$e^{-j\omega} = \frac{1}{b}$$

$$1 = A(0) + B\left(1 - a\left(\frac{1}{b}\right)\right)$$

$$1 = B\left(1 - \frac{a}{b}\right)$$

$$1 = B\left(\frac{b-a}{b}\right) \Rightarrow B = \frac{b}{b-a}$$

$$1 - a e^{j\omega} = 0$$

$$e^{-j\omega} = \frac{1}{a}$$

$$1 = A \left(1 - b \left(\frac{1}{a} \right) \right) + B(0)$$

$$1 = A \left(1 - \frac{b}{a} \right)$$

$$1 = A \left(\frac{a-b}{a} \right) \Rightarrow A = \frac{a}{a-b}$$

$$X[e^{j\omega}] = \frac{a}{a-b} \cdot \frac{1}{1 - a e^{j\omega}} + \frac{b}{b-a} \cdot \frac{1}{1 - b e^{-j\omega}}$$

$$= \frac{a}{a-b} a^n u[n] + \frac{b}{b-a} b^n u[n]$$