

Chapter 9

Numerical solution of 1st order ODE's

1st order ODE

$$- y'(t) = f(t, y(t))$$

$$y(t_0) = y_0$$

$$- y' = \frac{t-y}{2}$$

$$y(0) = 1$$

$$- t^2 y' + \sin t y^2 = \cos t$$

$$y(t_1) = y(t_0 + h)$$

$$= y(t_0) + h y'(t_0) + \frac{h^2}{2!} y''(c)$$

$$\rightarrow y(t_1) \approx y(t_0) + h y'(t_0) \text{ with error} = \frac{h^2}{2!} y''(c)$$

$$= y_0 + h f(t_0, y_0) \quad (\text{section 9.2})$$

$$y(t_1) \approx y(t_0) + h y'(t_0) + \frac{h^2}{2} y''(t_0) + \frac{h^3}{3!} y'''(c) \quad (\text{Section 9.4})$$

9.2 Euler method

$$\text{consider } y' = f(t, y)$$

$$y(t_0) = y_0$$

we will approximate the solution using set of points (t_k, y_k)

where

$$\underbrace{y_k}_{\text{Estimate}} = \underbrace{y(t_k)}_{\text{estimation at } t_k}$$

• we will use n subintervals of $[a, b]$

$$h = \frac{b-a}{n}, \quad t_k = a + h k$$

$k=1, \dots, n$

- Using Taylor expansion of $y(t_1)$ at t_0 ,

$$y(t_1) = y(t_0) + h y'(t_0) + \frac{h^2}{2} y''(c_1)$$

$$\rightarrow y_1 = y_0 + h f(t_0, y_0) \text{ with step error} = \frac{h^2}{2} y''(c_1)$$

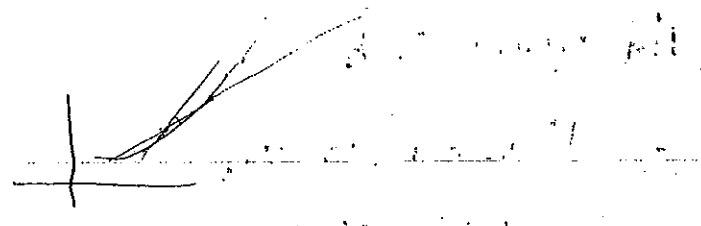
notice that $y_1 \approx y(t_1)$

$$y_2 = y_1 + h f(t_1, y_1)$$

$$y_3 = y_2 + h f(t_2, y_2)$$

⋮

$$y_{n+1} = y_n + h f(t_n, y_n) \text{ Euler method, step error} = \frac{h^2}{2} y''(c_n)$$

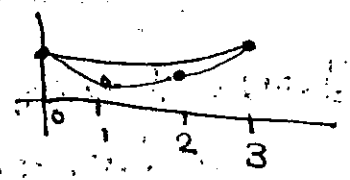


$t_0 = t_0$
 $t_1 = t_0 + h$
 $t_2 = t_0 + 2h$

EXAMPLE

Estimate the solution of $y' = \frac{t-y}{2}$, $y(0) = 1$, on $[0, 3]$

$$y_1 = y_0 + h f(t_0, y_0) = 1 + 1 f(0, 1) = 1 + (-0.5) = 0.5$$



$$y_2 = y_1 + h f(t_1, y_1) = 0.5 + 1 f(1, 0.5) = 0.5 + \frac{1-0.5}{2} = 0.75$$

$$y_3 = y_2 + h f(t_2, y_2) = 0.75 + 1 f(2, 0.75) = 0.75 + \frac{2-0.75}{2} = 1.375$$

$$\begin{aligned} \text{Total error} &= E(y(h), h) \\ &= \frac{y''(c_1) h^2}{2} + \frac{y''(c_2) h^2}{2} + \dots + \frac{y''(c_n) h^2}{2} \\ &= \frac{h^2}{2} (y''(c_1) + y''(c_2) + \dots + y''(c_n)) \\ &= \frac{h^2}{2} (n y''(c)) \\ &= \frac{h^2}{2} \left(\frac{b-a}{h} y''(c) \right) = \frac{(b-a) h}{2} y''(c) \approx ch \end{aligned}$$

$$E(y(h), \frac{h}{2}) = c \left(\frac{h}{2} \right) = \frac{1}{2} ch = \frac{1}{2} E(y(h), h)$$

9.4 Taylor method

Derive a formula of total error $O(h^2)$ to solve

$$y' = \frac{t-y}{2} \text{ on } [0, 3]$$

$$y_0 = 1, \quad h = 1$$

$$y_1 = y_0 + hf'(t_0, y_0) + \frac{h^2}{2} f''(t_0, y_0)$$

$$y(t_1) = \underbrace{y_0 + hf'(t_0) + \frac{h^2}{2} y''(t_0)}_{y_1} + \underbrace{\frac{h^3}{3!} y'''(c)}_{\text{Error}}$$

$$y_1 = y_0 + hf'(t_0, y_0) + \frac{h^2}{2} \frac{df'(t_0, y_0)}{dt}$$

$$\text{Step error} = \frac{h^3}{6} y'''(c)$$

$$y_{k+1} = y_k + hf(t_k, y_k) + \frac{h^2}{2} y''(t_k)$$

$$\text{Total error} = E(y(h), h) = \frac{ch^2}{6} = \frac{y'''(c)(b-a)h^2}{6}$$

• Solving the example

$$y_1 = y_0 + hf(t_0, y_0) + \frac{h^2}{2} y''(t_0)$$

$$= 1 + 1f(0, 1) + \frac{1}{2} y''(0)$$

$$= 1 + \frac{0-1}{2} + \frac{1}{2} \left(\frac{1}{2} - \frac{0-1}{4} \right)$$

$$= 1 - 0.5 + 0.25 + 1/8 = 0.875$$

$$y'(t) = \frac{t-y}{2}$$

$$y''(t) = \frac{1}{2} - \frac{y'}{2}$$

$$= \frac{1}{2} - \frac{(t-y)}{4}$$

$$y_2 = y_1 + hf(t_1, y_1) + \frac{h^2}{2} y''(t_1)$$

$$= y_1 + hf(t_1, y_1) + \frac{h^2}{2} \frac{d}{dt} (f(1, 0.875))$$

$$= 0.875 + hf(1, 0.875) + \frac{1}{2} \left(\frac{1}{2} - \frac{1-0.875}{4} \right)$$

$$= 0.875 + 0.0625 + 0.25 - 0.03125$$

$$y'(t) = f(t, y(t))$$

$$y(t_0) = y_0$$

Use Taylor method of order 4 to estimate the solution of $y' = \frac{t-y}{2}$, $y(0) = 1$ on $[0, 3]$, $h = 1$

$$y(t_1) = y(t_0) + h y'(t_0) + \frac{h^2}{2!} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + \frac{h^4}{4!} y^{(4)}(t_0) + \frac{h^5}{5!} y^{(5)}(c)$$

- total error = $E(y(t_1), h) = Ch^4$ step error

$$y_{k+1} = y_k + h y'(t_k) + \frac{h^2}{2!} y''(t_k) + \frac{h^3}{3!} y'''(t_k) + \frac{h^4}{4!} y^{(4)}(t_k)$$

$$y_1 = y_0 + h y'(t_0) + \frac{h^2}{2} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + \frac{h^4}{4!} y^{(4)}(t_0)$$

$$y'(t) = \frac{t-y}{2}, \quad y'(0) = \frac{0-1}{2} = -\frac{1}{2} \quad (y_0 = 1)$$

$$y''(t) = \frac{1}{2}(1-y') = \frac{1}{2} \left(1 - \frac{t-y}{2}\right) = \frac{1}{2} - \frac{t-y}{4}$$

$$y''(0) = \frac{1}{2} - \left(-\frac{1}{4}\right) = 0.75$$

$$y'''(t) = \frac{1}{2}(-y'') = -\frac{1}{2} \left(\frac{1}{2} - \frac{t-y}{4}\right)$$

$$y'''(0) = -\frac{1}{2}(0.75) = -0.375$$

$$y^{(4)}(t) = -\frac{1}{2} y''' = -\frac{1}{2} \left(-\frac{1}{2} \left(\frac{1}{2} - \frac{t-y}{4}\right)\right)$$

$$y^{(4)}(0) = -\frac{1}{2}(-0.375) = 0.1875$$

$$y_1 = 1 + 1(0.5) + \frac{1}{2}(0.75) + \frac{1}{6}(-0.375) + \frac{1}{24}(0.1875)$$

$$= 0.8203125$$

$$y_2 = y_1 + h y'(t_1) + \frac{h^2}{2} y''(t_1) + \frac{h^3}{3!} y'''(t_1) + \frac{h^4}{4!} y^{(4)}(t_1)$$

$$t_1 = 1, \quad y_1 = 0.8203125$$

$$y_2 = 1.1045$$

$$E(y(b), h) = Ch^4$$

$$E(y(b), h/2) = C(h/2)^4 = Ch^4/16$$

$$E(y(b), 10^{-2}h) = C(10^{-2}h)^4 = C(10^{-8})h^4$$

taylor → one evaluation

• Modified Method :- (Huen's Method)
Euler

$$y'(t) = f(t, y(t))$$

$$y(t_0) = y_0$$

1. $\int_{t_0}^{t_1} y'(t) dt = \int_{t_0}^{t_1} f(t, y(t)) dt \rightarrow$ Using trapezoidal

$$y(t_1) - y(t_0) = \frac{h}{2} (f(t_0, y_0) + f(t_1, y(t_1)))$$

$$\text{Error} = \frac{-h^3 y''(c)}{12}$$

$$y(t_1) \approx y_0 + \frac{h}{2} (f(t_0, y_0) + f(t_1, y(t_1)))$$

↓
نظير قيمة معروفة

$$y(t_1) = y_0 + \frac{h}{2} (f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0)))$$

$$y_{k+1} = y_k + \frac{h}{2} (f(t_k, y_k) + f(t_{k+1}, y_k + hf(t_k, y_k)))$$

total error For Huen's Method = Ch^2

• EXAMPLE

Solve Using Huen's Method with $h=1$

$$y' = \frac{t-y}{2} = f(t, y)$$

$f(t, y)$

$$y(0) = 1$$

$$y_1 = y_0 + \frac{h}{2} (f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0)))$$

$$= 1 + \frac{1}{2} (f(0, 1) + f(1, 1 + (1)f(0, 1)))$$

$$= 1 + \frac{1}{2} (-0.5 + f(1, 1 - 0.5)) = 1 + \frac{1}{2} (-0.5 + \frac{1-0.5}{2}) = 0.875$$

$$\begin{aligned}
 y_2 &= y_1 + \frac{h}{2} (f(t_1, y_1) + f(t_2, y_1 + hf(t_1, y_1))) \\
 &= 0.875 + \frac{1}{2} (f(1, 0.875) + f(2, 0.875 + (1) f(1, 0.875))) \\
 &= 0.875 + \frac{1}{2} \left(\frac{1-0.875}{2} + f(2, 0.875(1-0.875/2)) \right) \\
 &= 1.171875
 \end{aligned}$$

$$y_3 = 1.732422$$

if we have a period of $[0, 0.5]$, $h = \frac{1}{4}$

$$y_1 = 1 + (2)(-0.25) = 0.875$$

on $[0, 1]$, $h = \frac{1}{2}$

$$\begin{array}{ccc}
 0 & 0.5 & 1 \\
 t_0 & t_1 & t_2
 \end{array}$$

$$\begin{array}{ccc}
 0 & 0.25 & 0.5 \\
 t_0 & t_1 & t_2
 \end{array}$$

9.5 RK4: Runge-Kutta Method of order 4

$$y'(t) = f(t, y(t))$$

$$y(t_0) = y_0$$

$$\int_{t_0}^{t_1} y'(t) dt = \int_{t_0}^{t_1} f(t, y(t)) dt$$

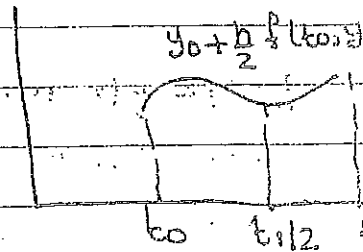
$$y_{n+1} = y_n + \frac{h}{6} (f_0 + 2f_1 + 2f_2 + f_3) \quad \text{where}$$

$$f_0 = f(t_0, y_0)$$

$$f_1 = f(t_0 + h/2, y_0 + h/2 f_0)$$

$$f_2 = f(t_0 + h/2, y_0 + h/2 f_1)$$

$$f_3 = f(t_1, y_0 + h f_2)$$



$$f_1 = \frac{f_1 + f_2}{2}$$

$$f_2(t_{1/2}, y_{1/2}) = y_0 + \frac{h}{2} f_1$$

$$f_1(t_{1/2}, y_{1/2}) = y_0 + \frac{h}{2} f_1$$

EXAMPLE

Solve $y' = \frac{t-y}{2}$, $y(0) = 1$, $[0, 3]$, $h = \frac{1}{4}$

$$y_1 = y_0 + \frac{h}{6} (f_0 + 2f_1 + 2f_2 + f_3)$$

$$f_0 = f(t_0, y_0) = f(0, 1) = -0.5$$

$$f_1 = f(t_0 + h/2, y_0 + h/2 f_0) = f\left(\frac{1}{8}, 1 + \frac{1}{8}(-0.5)\right) = -0.40625$$

$$f_2 = f(t_0 + h/2, y_0 + h/2 f_1) = f\left(\frac{1}{8}, 1 + \frac{1}{8}(-0.40625)\right) = -0.4121094$$

$$f_3 = f(t_1, y_0 + h f_2) = f(1, 1 + 1(-0.4121094)) = -0.3234863$$

$$y_1 = 1 + \frac{1}{4} \left((-0.5) + 2(-0.40625) + 2(-0.4121094) + (-0.3234863) \right)$$

$$= 0.8974915 \quad \text{comparing to the exact } 0.8974917$$

$$y_2 = y_1 + h/6 (f_0 + 2f_1 + 2f_2 + f_3)$$

$$f_0 = f(t_1, y_1) = f(114, 0.8974915) = \dots$$

$$f_1 = f(t_1 + h/2, y_1 + h/2 f_0) = f(117, 0.8974915 + 118 f_0) =$$

$$f_2 = f(t_2, y_1 + h f_1) =$$

$$f_3 = f(t_2 + h, y_1 + 2h f_2) =$$

$$y_2 =$$