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MATH1321

Calculus 2

Chapter 8.3

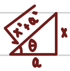
Ahmad Ouri

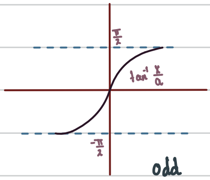
8.3 Trigonometric Substitutions

$\int \frac{dx}{\sqrt{x^2-9}}$, $\int \frac{dx}{\sqrt{9-x^2}}$, $\int \frac{dx}{\sqrt{x^2+9}}$ (How to integrate these integrals ?)

$x = a\tan\theta \longrightarrow \tan\theta = \frac{x}{a}$

$dx = a\sec^2\theta\, d\theta \qquad \theta = \tan^{-1}\frac{x}{a}$





odd

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

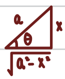
$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2}$

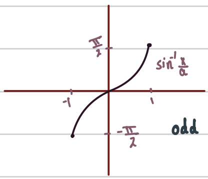
$= \sqrt{a^2(\tan^2\theta+1)} = |a|\,|\sec\theta|$

$= a\sec\theta.$

$x = a\sin\theta \longrightarrow \sin\theta = \frac{x}{a}$

$dx = a\cos\theta\, d\theta \qquad \theta = \sin^{-1}\frac{x}{a}$





odd

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

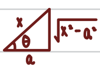
$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta}$

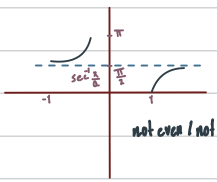
$= \sqrt{a^2(1-\sin^2\theta)} = |a|\,|\cos\theta|$

$= a\cos\theta.$

$x = a\sec\theta \longrightarrow \sec\theta = \frac{x}{a}$

$dx = a\sec\theta\tan\theta\, d\theta \qquad \theta = \sec^{-1}\frac{x}{a}$





not even / not odd

$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2}$

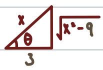
$= \sqrt{a^2(\sec^2\theta-1)} = |a|\,|\tan\theta|$

$= a\tan\theta.$

$0 \leq \theta < \frac{\pi}{2}$ if $\frac{x}{a} \geq 1$

$\frac{\pi}{2} < \theta \leq \pi$ if $\frac{x}{a} \leq -1$

Ex. $\int \frac{dx}{\sqrt{x^2-9}}$ $\qquad x = 3\sec\theta$

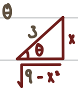


$= \int \frac{\cancel{3}\sec\theta \cancel{\tan\theta}}{\cancel{3\tan\theta}}\, d\theta$

$= \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$

$= \ln\left| \frac{x}{3} + \frac{\sqrt{x^2-9}}{3} \right| + C$ #

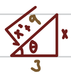
Ex. $\int \frac{dx}{\sqrt{9-x^2}}\, dx$ $\qquad x = 3\sin\theta$



$= \int \frac{3\cos\theta}{3\cos\theta}\, d\theta = \int d\theta$

$= \theta + C = \sin^{-1}\frac{x}{3} + C$ #

Ex. $\int \frac{dx}{\sqrt{x^2+9}}$ $\qquad x = 3\tan\theta$



$= \int \frac{\cancel{3}\sec^{\cancel{2}}\theta}{\cancel{3\sec\theta}}\, d\theta$ $\qquad dx = 3\sec^2\theta\, d\theta$

$= \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$

$= \ln\left| \frac{\sqrt{x^2+9}}{3} + \frac{x}{3} \right| + C$ #

Ex. $\int \sqrt{16-t^2}\, dt$ $\qquad t = 4\sin\theta$

$= \int 4\cos\theta \cdot 4\cos\theta \, d\theta$ $\qquad dt = 4\cos\theta\, d\theta$

$= 16\int \cos^2\theta \, d\theta = 16\int \frac{1+\cos 2\theta}{2}\, d\theta$

$= 8\left(\theta + \frac{\sin 2\theta}{2}\right) + C = 8\left(\theta + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right) + C$

$= 8\theta + 8\sin\theta\cos\theta + C = 8\sin^{-1}\left(\frac{t}{4}\right) + \frac{t\sqrt{16-t^2}}{2} + C$ #

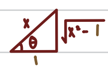
Ex. $\int_{-2}^{2} \frac{dx}{4+x^2}$ $\qquad x = 2\tan\theta$

$= \int \frac{2\sec^2\theta}{4\sec^2\theta}\, d\theta$ $\qquad dx = 2\sec^2\theta\, d\theta$

$= \int \frac{1}{2}\, d\theta = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}\frac{x}{2} \Big|_{-2}^{2}$

$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi}{4}.$

Ex. $\int \frac{x^2}{(x^2-1)^{5/2}}\, dx$ where $x > 1$ $\qquad x = \sec\theta$



$= \int \frac{\sec^2\theta \cdot \sec\theta\tan\theta}{\tan^5\theta}\, d\theta$

$(x^2-1)^{\frac{1}{2}} = \sqrt{x^2-1} = \tan\theta$

$= \int \frac{\sec^3\theta}{\tan^4\theta}\, d\theta = \int \frac{1}{\cancel{\cos^3}\theta} \cdot \frac{\cos^{\cancel{4}\,1}\theta}{\sin^4\theta}\, d\theta$

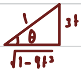
$= \int \frac{\cos\theta}{\sin^4\theta}\, d\theta$ $\qquad u = \sin\theta$

$= \int \frac{1}{u^4}\, d\theta$ $\qquad du = \cos\theta\, d\theta$

$= \frac{u^{-3}}{-3} + C = -\frac{1}{3u^3} + C = -\frac{1}{3\sin^3\theta} + C = \frac{-1}{3\left(\frac{\sqrt{x^2-1}}{x}\right)^3} + C$ #

8.3 Discussion: (8, 10, 12, 14, 18, 24, 26, 29, 33, 38, 45, 46)

Q.8 $\int \sqrt{1-9t^2}\, dt$



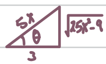
$3t = 1 \sin\theta \qquad \sin\theta = 3t$

$= \int \cos\theta \cdot \cos\theta \frac{d\theta}{3} \qquad 3dt = \cos\theta\, d\theta \qquad \theta = \sin^{-1} 3t$

$= \frac{1}{3}\int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{6}\int 1+\cos 2\theta\, d\theta \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$= \frac{1}{6}\left(\theta + \frac{\sin 2\theta}{2}\right) + C = \frac{\sin^{-1} 3t}{6} + \frac{3t\sqrt{1-9t^2}}{6} + C$ #

Q.10 $\int \frac{5dx}{\sqrt{25x^2-9}}, \; x > \frac{3}{5}$.

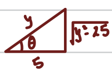


$= \int \frac{\cancel{3}\sec\theta \cancel{\tan\theta}}{\cancel{3}\cancel{\tan\theta}}\, d\theta \qquad 5x = 3\sec\theta \qquad \sec\theta = \frac{5x}{3}$

$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C \qquad 5dx = 3\sec\theta\tan\theta\, d\theta \qquad \theta = \sec^{-1}\frac{5x}{3}$

$= \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C$ # $\qquad 0 \leq \theta < \frac{\pi}{2}$

Q.12 $\int \frac{\sqrt{y^2-25}}{y^3}\, dy, \; y > 5$.



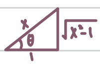
$= \int \frac{5\tan\theta\, 5\cancel{\sec\theta}\tan\theta}{125\sec^{\cancel{3}\,2}\theta}\, d\theta \qquad y = 5\sec\theta \qquad \sec\theta = \frac{y}{5}$

$= \frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\, d\theta = \frac{1}{5}\int \cancel{\cos^2\theta} \cdot \frac{\sin^2\theta}{\cancel{\cos^2\theta}}\, d\theta \qquad dy = 5\sec\theta\tan\theta\, d\theta \qquad \theta = \sec^{-1}\frac{y}{5}$

$= \frac{1}{5}\int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{1}{10}\left(\theta - \frac{\sin 2\theta}{2}\right) + C \qquad 0 \leq \theta < \frac{\pi}{2}$

$= \frac{\sec^{-1} y/5}{10} - \frac{\sin\theta\cos\theta}{10} + C = \frac{\sec^{-1} y/5}{10} - \frac{\sqrt{y^2-25}}{2y^2} + C$ #

Q.14 $\int \frac{2dx}{x^3\sqrt{x^2-1}}, \; x > 1$.



$= \int \frac{2\cancel{\sec\theta}\cancel{\tan\theta}}{\sec^{\cancel{3}}\theta \cancel{\tan\theta}}\, d\theta \qquad x = \sec\theta \qquad \sec\theta = \frac{x}{1}$

$= 2\int \cos^2\theta\, d\theta = \int 1 + \cos 2\theta\, d\theta \qquad dx = \sec\theta\tan\theta\, d\theta \qquad \theta = \sec^{-1} x$

$= \theta + \frac{\sin 2\theta}{2} + C = \sec^{-1} x + \frac{\sqrt{x^2-1}}{x^2} + C$ # $\qquad 0 \leq \theta < \frac{\pi}{2}$

Q.18 $\int \frac{dx}{x^2\sqrt{x^2+1}}$

$= \int \frac{\sec^{\cancel{2}}\theta}{\tan^2\theta \cancel{\sec\theta}}\, d\theta \qquad x = \tan\theta \qquad \tan\theta = \frac{x}{1}$

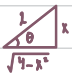
$= \int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\, d\theta \qquad dx = \sec^2\theta\, d\theta \qquad \theta = \tan^{-1} x$

$= \int \frac{\cos\theta}{\sin^2\theta}\, d\theta \qquad u = \sin\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$= \int \frac{1}{u^2}\, du = -\frac{1}{u} + C \qquad du = \cos\theta\, d\theta$

$= -\frac{1}{\sin\theta} + C = -\frac{\sqrt{x^2+1}}{x} + C$ #

Q.24 $\int_0^1 \frac{dx}{(4-x^2)^{3/2}}$



$x = 2\sin\theta$ $\quad$ $\sin\theta = \frac{x}{2}$

$dx = 2\cos\theta\, d\theta$ $\quad$ $\theta = \sin^{-1}\frac{x}{2}$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$= \int \frac{2\cos\theta}{(2\cos\theta)^3}\, d\theta$

$= \frac{1}{4}\int \frac{1}{\cos^2\theta}\, d\theta = \frac{1}{4}\int \sec^2\theta\, d\theta$

$= \frac{1}{4}\tan\theta + C = \frac{1}{4}\cdot\frac{x}{\sqrt{4-x^2}}\Big|_0^1$

$= \frac{1}{4}\cdot\frac{1}{\sqrt{3}} = \frac{1}{4\sqrt{3}}$.

Q.26 Solved in the examples.

Q.27 $\int \frac{8dx}{(4x^2+1)^2}$

$2x = \tan\theta$ $\quad$ $\tan\theta = \frac{2x}{1}$

$2dx = \sec^2\theta\, d\theta$ $\quad$ $\theta = \tan^{-1}2x$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

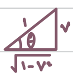
$= 4\int \frac{\sec^2\theta}{\sec^4\theta}\, d\theta$

$= 4\int \cos^2\theta\, d\theta = 2\left(\theta + \frac{\sin 2\theta}{2}\right) + C$

$= 2\theta + 2\sin\theta\cos\theta + C$

$= 2\tan^{-1}2x + 2\cdot\frac{2x}{\sqrt{4x^2+1}}\cdot\frac{1}{\sqrt{4x^2+1}} + C = 2\tan^{-1}2x + \frac{4x}{4x^2+1} + C$ #

Q.33 $\int \frac{v^2\, dv}{(1-v^2)^{5/2}}$



$v = \sin\theta$ $\quad$ $\sin\theta = \frac{v}{1}$

$dv = \cos\theta\, d\theta$ $\quad$ $\theta = \sin^{-1} v$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$= \int \frac{\sin^2\theta\cos\theta}{\cos^5\theta}\, d\theta$

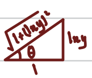
$= \int \frac{\sin^2\theta}{\cos^4\theta}\, d\theta = \int \tan^2\theta\sec^2\theta\, d\theta$

$= \int u^2\, du = \frac{u^3}{3} + C$ $\quad$ $u = \tan\theta$

$= \frac{\tan^3\theta}{3} + C$ $\quad$ $du = \sec^2\theta\, d\theta$

$= \frac{1}{3}\cdot\frac{v^3}{(1-v^2)^{3/2}} + C = \frac{v^3}{3(1-v^2)^{3/2}} + C$ #

Q.38 $\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$



$\ln y = \tan\theta$ $\quad$ $\tan\theta = \frac{\ln y}{1}$

$\frac{dy}{y} = \sec^2\theta\, d\theta$ $\quad$ $\theta = \tan^{-1}(\ln y)$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$= \int \frac{\cancel{y}\sec^2\theta}{\cancel{y}\sec\theta}\, d\theta$

$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\sqrt{1+(\ln y)^2} + \ln y\right|\Big|_1^e = \ln|\sqrt{2}+1| - 0$

$= \ln(\sqrt{2}+1)$.

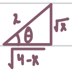
Q.45 $\int \sqrt{\frac{4-x}{x}}\, dx$

$= \int \frac{\sqrt{4-x}}{\sqrt{x}}\, dx$

$= \int \frac{2\cos\theta \cdot 2\cos\theta \cdot 2\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}\, d\theta$

$= 8\int \cos^2\theta\, d\theta = 4(\theta + \sin\theta\cos\theta) + C$

$= 4\sin^{-1}\frac{\sqrt{x}}{2} + \sqrt{4x-x^2} + C$ #



$\sqrt{x} = 2\sin\theta$

$\frac{dx}{2\sqrt{x}} = 2\cos\theta\, d\theta$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$\sin\theta = \frac{\sqrt{x}}{2}$

$\theta = \sin^{-1}\frac{\sqrt{x}}{2}$

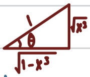
Q.46 $\int \sqrt{\frac{x}{1-x^3}}\, dx$

$= \int \frac{\sqrt{x}}{\sqrt{1-x^3}}\, dx$

$= \int \frac{\cancel{\sqrt{x}}}{\cancel{\cos\theta}} \cdot \frac{2\cancel{\sqrt{x^3}}\,\cancel{\cos\theta}}{3\cancel{x^2}}\, d\theta$

$= \frac{2}{3}\int d\theta = \frac{2}{3}\theta + C$

$= \frac{2}{3}\sin^{-1}\sqrt{x^3} + C$ #



$\sqrt{x^3} = \sin\theta$

$\frac{3x^2}{2\sqrt{x^3}}\, dx = \cos\theta\, d\theta$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$\sin\theta = \frac{\sqrt{x^3}}{1}$

$\theta = \sin^{-1}\sqrt{x^3}$