Chapter 10: sequences & series

↳ 10.1: sequences

Def:- A sequence is a list of ordered numbers:

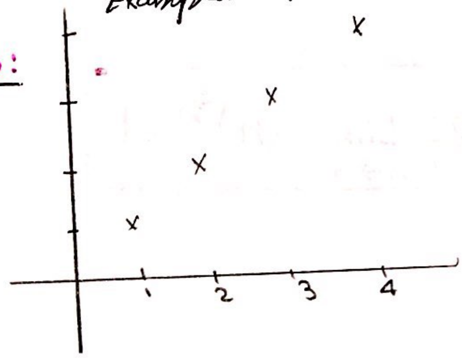
$$a_1, a_2, a_3 \text{ ------ } a_n \text{ ....}$$

first term (a_1) — nth term (a_n)

Domain: the set of positive integers

Graph: sequence can be represented as points

Example: $a_n = n$



Convergent and divergant sequences:

- a_n is called convergent sequence

If $\lim\limits_{n\to\infty} a_n = L$, $-\infty < L < \infty$

If not a_n is divergant

Theories of limits

Th₁ If $\{a_n\} = A$ and $\{b_n\} = B$ converge . Then

$$\lim_{n\to\infty} (\{a_n\} \pm \{b_n\}) = A \pm B$$

$$\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$$

$$\lim_{n\to\infty} (k)(a_n) = Ak$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$$

Th₂: Sandwich Theorem:

If $a_n \leq b_n \leq c_n$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$

Then $\lim\limits_{n\to\infty} b_n = L$

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Th_3: L'hopital rule:-

If $\lim\limits_{n\to\infty} a_n = \frac{\infty}{\infty}$ Then we can use l'hopital Rule:

Limits (Try To Know Them by heart)

1- $\boxed{\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0}$ why? using l'hopital rule

$$\lim_{n\to\infty} \frac{\ln n}{n} = \frac{\infty}{\infty} \quad \text{(differentiate)}$$

$$\to \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \frac{1}{\infty} = 0$$

2- $\boxed{\lim\limits_{n\to\infty} (n)^{\frac{1}{n}} = 1}$ why? using $e^{\ln}$ (7.5 Calculus 141)

$$\lim_{n\to\infty} e^{\ln(n)^{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{\ln n}{n}}$$

$$= e^{\lim\limits_{n\to\infty} \frac{\ln n}{n}} \to \text{l'hopital rule}$$

$$= e^{\lim\limits_{n\to\infty} \frac{1}{n}}$$

$$= e^0 = 1$$

3- $\boxed{\lim\limits_{n\to\infty} (x)^{\frac{1}{n}} = 1}$ $x>0$ why? using $e^{\ln}$

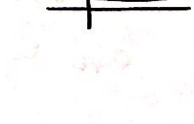
$$\lim_{n\to\infty} e^{\frac{\ln x}{n}} = e^{\lim\limits_{n\to\infty} \frac{\boxed{\ln x}}{n}} \to \text{Constant}$$

$$= e^{\ln x \lim\limits_{n\to\infty} \frac{1}{n}}$$

$$= e^0 = 1$$

4- $\boxed{\lim\limits_{n\to\infty} x^n = 0}$ $|x|<1$ why? (using 7.8 Calculus)

when $-1<x<1$ Then $(x)^n$ graph would be like:-



and when $n\to\infty$ $(x)^n$ goes to zero

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$$\boxed{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x}$$

why? if you use $e^{\ln}$ Then use l'hopital rule you get e^x

$$\lim_{n\to\infty} e^{n\ln(1+\frac{x}{n})} = \lim_{n\to\infty} e^{\frac{\ln(1+\frac{x}{n})}{\frac{1}{n}}}$$

$$= e^{\lim_{n\to\infty}\frac{\ln(1+\frac{x}{n})}{\frac{1}{n}}} \nearrow \frac{0}{0}$$

$$= e^{\lim_{n\to\infty}\frac{(\frac{-x}{n^2})x}{(\frac{-1}{n^2})(1+\frac{x}{n})}}$$

$$= e^{\lim_{n\to\infty}\frac{x}{1+\frac{x}{n}}}$$

$$= e^x$$

$$\boxed{\lim_{n\to\infty}\frac{x^n}{n!} = 0}$$

why? (7.8 calculus H1)

$n!$ is faster Than x^n so the limit $=0$

Recursive Formula:

Sequences are often defined recursively by giving

1- The value of the initial term(s)

2- a rule [recursion formula] so you can calculate later terms

sequences can be: →

→ bounded from above: $a_n \leq M$
upper bound is the answer of $\lim_{n\to\infty}$

M is an upper bound
If no number is less then M then M is a least upper bound.

→ bounded from below $m \leq a_n$
lower bound is answer of $\lim_{n\to\infty}$

m is a lower bound
If no number greater than m is a lower bound then m is a greatest lower bound

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- a bounded sequence is bounded from above and below

Sequences can be :- → non decreasing if $a_{n+1} \geq a_n$
→ non increasing if $a_{n+1} \leq a_n$

- Note:
If $\{a_n\}$ converges then $\{a_n\}$ is bounded

Monotonic sequence: either non decreasing or non increasing

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