Jacobian Linearizations, equilibrium points

In modeling systems, we see that nearly all systems are nonlinear, in that the differential equations governing the evolution of the system's variables are nonlinear. However, most of the theory we have developed has centered on linear systems. So, a question arises: "In what limited sense can a nonlinear system be viewed as a linear system?" In this section we develop what is called a "Jacobian linearization of a nonlinear system," about a specific operating point, called an equilibrium point.

Equilibrium Points

Consider a nonlinear differential equation

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

where f is a function mapping $\mathbf{R}^n \times \mathbf{R}^r \rightarrow \mathbf{R}^n$. A point $\bar{x} \in \mathbf{R}^n$ is called an **equilibrium point** if there is a specific $\bar{u} \in \mathbf{R}^r$ (called the **equilibrium input**) such that

$$f(\bar{x}, \bar{u}) = 0_n$$

Suppose $\bar{x}$ is an equilibrium point (with equilibrium input $\bar{u}$). Consider starting the system (1) from initial condition $x(t_0) = \bar{x}$, and applying the input $u(t) \equiv \bar{u}$ for all $t \geq t_0$. The resulting solution $x(t)$ satisfies

$$x(t) = \bar{x}$$

for all $t \geq t_0$. That is why it is called an **equilibrium point**.

Example 1:

Find the Equilibrium points for the following system?

$$\ddot{v} = 2v + v^2\dot{v} + \dot{v}u - 8$$

Let the states as:

$$z_1 = v, \qquad z_2 = \dot{v}$$

Based on that:

$$\dot{z} = f(z,u) \quad (1)$$

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= 2z_1 + z_1^2 z_2 + z_2 u - 8 \end{aligned} \quad (2)$$

To find the Equilibrium point:

$$0 = f(\bar{z},\bar{u}) \quad (3)$$

$$\begin{aligned} 0 &= \bar{z}_2 \\ 0 &= 2\bar{z}_1 + \bar{z}_1^2 \bar{z}_2 + \bar{z}_2 \bar{u} - 8 \end{aligned} \quad (4)$$

Let $\bar{u} = 0$

Therefore: $2 * \bar{z}_1 = 8$ so $\bar{z}_1 = 4$

So the Equilibrium point is $\bar{z}(4,0)$ and $\bar{u}=0$

Deviation Variables

Suppose $(\bar{x}, \bar{u})$ is an equilibrium point and input. We know that if we start the system at $x(t_0) = \bar{x}$, and apply the constant input $u(t) \equiv \bar{u}$, then the state of the system will remain fixed at $x(t) = \bar{x}$ for all t. What happens if we start a little bit away from $\bar{x}$, and we apply a slightly different input from $\bar{u}$? Define deviation variables to measure the difference.

$$\begin{aligned} \delta_x(t) &:= x(t) - \bar{x} \\ \delta_u(t) &:= u(t) - \bar{u} \end{aligned}$$

In this way, we are simply relabling where we call 0. Now, the variables $x(t)$ and $u(t)$ are related by the differential equation

$$\dot{x}(t) = f(x(t), u(t))$$

Substituting in, using the constant and deviation variables, we get

$$\dot{\delta}_x(t) = f\left(\bar{x} + \delta_x(t), \bar{u} + \delta_u(t)\right)$$

This is exact. Now however, let's do a Taylor expansion of the right hand side, and neglect all higher (higher than 1st) order terms

$$\dot{\delta}_x(t) \approx f\left(\bar{x}, \bar{u}\right) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta_x(t) \quad + \quad \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta_u(t)$$

But $f(\bar{x}, \bar{u}) = 0$, leaving

$$\dot{\delta}_x(t) \approx \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta_x(t) \quad + \quad \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta_u(t)$$

This differential equation approximately governs (we are neglecting 2nd order and higher terms) the deviation variables $\delta_x(t)$ and $\delta_u(t)$, **as long as they remain small**. It is a linear, time-invariant, differential equation, since the derivatives of δ_x are linear combinations of the δ_x variables and the deviation inputs, δ_u. The matrices

$$A := \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} \in \mathbf{R}^{n \times n} \quad , \quad B := \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x}\\u=\bar{u}}} \in \mathbf{R}^{n \times r} \qquad (2)$$

$$A(t) \doteq \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{\substack{x=\bar{x}(t)\\u=\bar{u}(t)}} \qquad B(t) \doteq \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_r} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_r} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_r} \end{bmatrix}\right|_{\substack{x=\bar{x}(t)\\u=\bar{u}(t)}}$$

are constant matrices. With the matrices A and B as defined in (2), the linear system

$$\dot{\delta}_x(t) = A\delta_x(t) \quad + \quad B\delta_u(t)$$

is called the **Jacobian Linearization** of the original nonlinear system (1), about the equilibrium point $(\bar{x}, \bar{u})$. For "small" values of δ_x and δ_u, the linear equation **approximately** governs the exact relationship between the deviation variables δ_u and δ_x.

For "small" δ_u (ie., while $u(t)$ remains close to $\bar{u}$), and while δ_x remains "small" (ie., while $x(t)$ remains close to $\bar{x}$), the variables δ_x and δ_u are related by the differential equation

$$\dot{\delta}_x(t) = A\delta_x(t) \quad + \quad B\delta_u(t)$$

If we design a controller that effectively controls the deviations δ_x, then we have designed a controller that works well when the system is operating **near** the equilibrium point $(\bar{x}, \bar{u})$. We will cover this idea in greater detail later. This is a common, and somewhat effective way to deal with nonlinear systems in a linear manner.

This equation represents a linear system, and is the *linearized system* around $\bar{x}(t), \bar{u}(t)$. Similarly h in the output equation (3) can be expanded in Taylor series around $\bar{x}(t), \bar{u}(t)$, obtaining

$$\begin{aligned} y(t) &= h(x(t), u(t)) \\ &= h(\bar{x}(t), \bar{u}(t)) + C(t)(x(t) - \bar{x}(t)) + D(t)(u(t) - \bar{u}(t)) \end{aligned} \tag{3}$$

where $C(t), D(t)$ are the Jacobians of h:

$$C(t) \doteq \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \frac{\partial h_p}{\partial x_2} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix}_{\substack{x = \bar{x}(t) \\ u = \bar{u}(t)}} \quad ; \quad D(t) \doteq \begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \frac{\partial h_1}{\partial u_2} & \cdots & \frac{\partial h_1}{\partial u_r} \\ \frac{\partial h_2}{\partial u_1} & \frac{\partial h_2}{\partial u_2} & \cdots & \frac{\partial h_2}{\partial u_r} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial u_1} & \frac{\partial h_p}{\partial u_2} & \cdots & \frac{\partial h_p}{\partial u_r} \end{bmatrix}_{\substack{x = \bar{x}(t) \\ u = \bar{u}(t)}} . \tag{4}$$

Since $h(\bar{x}(t), \bar{u}(t)) = \bar{y}(t)$, we have

$$y_{\delta}(t) = C(t)x_{\delta}(t) + D(t)u_{\delta}(t) \tag{5}$$

19.3 Tank Example

Consider a mixing tank, with constant supply temperatures T_C and T_H. Let the inputs be the two flow rates $q_C(t)$ and $q_H(t)$. The equations for the tank are

$$\begin{aligned}\dot{h}(t) &= \tfrac{1}{A_T}\left(q_C(t) + q_H(t) - c_D A_o \sqrt{2gh(t)}\right)\\ \dot{T}_T(t) &= \tfrac{1}{h(t)A_T}\left(q_C(t)\left[T_C - T_T(t)\right] + q_H(t)\left[T_H - T_T(t)\right]\right)\end{aligned}$$

Let the state vector x and input vector u be defined as

$$x(t) := \begin{bmatrix} h(t) \\ T_T(t) \end{bmatrix} \quad , \quad u(t) := \begin{bmatrix} q_C(t) \\ q_H(t) \end{bmatrix}$$

$$\begin{aligned}f_1(x,u) &= \tfrac{1}{A_T}\left(u_1 + u_2 - c_D A_o \sqrt{2gx_1}\right)\\ f_2(x,u) &= \tfrac{1}{x_1 A_T}\left(u_1\left[T_C - x_2\right] + u_2\left[T_H - x_2\right]\right)\end{aligned}$$

Intuitively, any height $\bar{h} > 0$ and any tank temperature $\bar{T}_T$ satisfying

$$T_C \le \bar{T}_T \le T_H$$

should be a possible equilibrium point (after specifying the correct values of the equilibrium inputs). In fact, with $\bar{h}$ and $\bar{T}_T$ chosen, the equation $f(\bar{x},\bar{u}) = 0$ can be written as

$$\begin{bmatrix} 1 & 1 \\ T_C - \bar{x}_2 & T_H - \bar{x}_2 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} c_D A_o \sqrt{2g\bar{x}_1} \\ 0 \end{bmatrix}$$

The 2×2 matrix is invertible if and only if $T_C \neq T_H$. Hence, as long as $T_C \neq T_H$, there is a unique equilibrium input for any choice of $\bar{x}$. It is given by

$$\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \frac{1}{T_H - T_C} \begin{bmatrix} T_H - \bar{x}_2 & -1 \\ \bar{x}_2 - T_C & 1 \end{bmatrix} \begin{bmatrix} c_D A_o \sqrt{2g\bar{x}_1} \\ 0 \end{bmatrix}$$

This is simply

$$\bar{u}_1 = \frac{c_D A_o \sqrt{2g\bar{x}_1}\,(T_H - \bar{x}_2)}{T_H - T_C} \quad , \quad \bar{u}_2 = \frac{c_D A_o \sqrt{2g\bar{x}_1}\,(\bar{x}_2 - T_C)}{T_H - T_C}$$

Since the u_i represent flow rates **into** the tank, physical considerations restrict them to be nonegative real numbers. This implies that $\bar{x}_1 \ge 0$ and $T_C \le \bar{T}_T \le T_H$. Looking at the differential equation for T_T, we see that its rate of change is inversely related to h. Hence, the differential equation model is valid while $h(t) > 0$, so we further restrict $\bar{x}_1 > 0$. Under those restrictions, the state $\bar{x}$ is indeed an equilibrium point, and there is a unique equilibrium input given by the equations above.

$$\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{A_T} & \frac{1}{A_T} \\ \frac{T_C - x_2}{x_1 A_T} & \frac{T_H - x_2}{x_1 A_T} \end{bmatrix}$$

The linearization requires that the matrices of partial derivatives be evaluated **at** the equilibrium points. Let's pick some realistic numbers, and see how things vary with different equilibrium points. Suppose that $T_C = 10°, T_H = 90°, A_T = 3\mathtt{m}^2, A_o = 0.05\mathtt{m}, c_D = 0.7$. Try $\bar{h} = 1\mathtt{m}$ and $\bar{h} = 3\mathtt{m}$, and for $\bar{T}_T$, try $\bar{T}_T = 25°$ and $\bar{T}_T = 75°$. That gives 4 combinations. Plugging into the formulae give the 4 cases

1. $\left(\bar{h}, \bar{T}_T\right) = (1\mathtt{m}, 25°)$. The equilibrium inputs are

$$\bar{u}_1 = \bar{q}_C = 0.126 \quad , \quad \bar{u}_2 = \bar{q}_H = 0.029$$

The linearized matrices are

$$A = \begin{bmatrix} -0.0258 & 0 \\ 0 & -0.517 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0.333 & 0.333 \\ -5.00 & 21.67 \end{bmatrix}$$

2. $\left(\bar{h}, \bar{T}_T\right) = (1\mathtt{m}, 75°)$. The equilibrium inputs are

$$\bar{u}_1 = \bar{q}_C = 0.029 \quad , \quad \bar{u}_2 = \bar{q}_H = 0.126$$

The linearized matrices are

$$A = \begin{bmatrix} -0.0258 & 0 \\ 0 & -0.0517 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0.333 & 0.333 \\ -21.67 & 5.00 \end{bmatrix}$$

3. $\left(\bar{h}, \bar{T}_T\right) = (3\mathtt{m}, 25°)$. The equilibrium inputs are

$$\bar{u}_1 = \bar{q}_C = 0.218 \quad , \quad \bar{u}_2 = \bar{q}_H = 0.0503$$

The linearized matrices are

$$A = \begin{bmatrix} -0.0149 & 0 \\ 0 & -0.0298 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0.333 & 0.333 \\ -1.667 & 7.22 \end{bmatrix}$$

4. $\left(\bar{h}, \bar{T}_T\right) = (3\mathtt{m}, 75°)$. The equilibrium inputs are

$$\bar{u}_1 = \bar{q}_C = 0.0503 \quad , \quad \bar{u}_2 = \bar{q}_H = 0.2181$$

The linearized matrices are

$$A = \begin{bmatrix} -0.0149 & 0 \\ 0 & -0.0298 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0.333 & 0.333 \\ -7.22 & 1.667 \end{bmatrix}$$