8.1 Integral by parts

$$\int u\,dV = uV - \int V\,du$$

$$u \xrightarrow{\frac{d}{dx}} du \qquad dV \xrightarrow{\int} V$$

$$\int_a^b u\,dv = uV\Big|_a^b - \int_a^b V\,du$$

Exp: ① $\int x \cos x\,dx$

$u = x \qquad dV = \cos x\,dx$

$du = dx \qquad V = \sin x$

S1 $\rightarrow$ $\int u\,dv = uV - \int V\,du$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

② $\int_0^{\pi} x \cos x\,dx$

$$x\sin x\Big|_0^{\pi} + \cos x\Big|_0^{\pi}$$

$$\big((\pi \sin\pi) - (0\sin 0)\big) + (\cos\pi - \cos 0)$$

$$(0 - 0) + (-1 - 1) = -2$$

S2

$$\int x\cos x\,dx = x\sin x + \cos x + C$$

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$\frac{d}{dx}$	$\int$
x $\xrightarrow{(+)}$	$\cos x$
1 $\xrightarrow{(-)}$	$\sin x$
0	$-\cos x$

$$= x\sin x + \cos x + C$$

Exp: $\int \ln x\,dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$$= x\ln x - \int \frac{1}{x}\,x\,dx$$

$$= x\ln x - \int dx = x\ln x - x + C$$

Exp: $\int e^{x}\cos x\,dx$

$\frac{d}{dx}$	$\int$
e^{x} $\xrightarrow{(+)}$	$\cos x$
e^{x} $\xrightarrow{(-)}$	$\sin x$
e^{x} stop $\xrightarrow{\int}$	$-\cos x$

$$\int e^{x}\cos x\,dx = e^{x}\sin x + e^{x}\cos x + \int -e^{x}\cos x\,dx$$

$$2\int e^{x}\cos x\,dx$$

$$= \frac{e^{x}}{2}\left[\sin x + \cos x\right] + C$$

Exp: $\int e^{x}\sin x\,dx$

$\frac{d}{dx}$	$\int$
e^{x} $\xrightarrow{(+)}$	$\sin x$
e^{x} $\xrightarrow{-}$	$-\cos x$
e^{x} $\xrightarrow{\int}$ stope	$-\sin x$

$$= -e^{x}\cos x + e^{x}\sin x + \int e^{x}(-1)\sin x\,dx$$

$$2\int e^{x}\sin x\,dx = e^{x}\left[\sin x - \cos x\right]$$

$$\int e^{x}\sin x\,dx = \frac{e^{x}}{2}\left[\sin x - \cos x\right] + C$$

Exp: ① $\int_1^2 x \ln x \, dx$.

x	$\ln x$
1	$x\ln x - x$
0	$\int x\ln x\,dx - \frac{x^2}{2}$

$$\int_1^2 x \ln x \, dx = x(x\ln x - x)\Big|_1^2 - \int_1^2 x \ln x \, dx + \frac{x^2}{2}\Big|_1^2$$

$$= \left(x^2 \ln x - x^2 + \frac{x^2}{2}\right)\Big|_1^2 - \int_1^2 x \ln x \, dx$$

$$= \left(x^2 \ln x - x^2 + \frac{x^2}{2}\right)\Big|_1^2 - \int_1^2 x \ln x \, dx$$

$$= (4\ln 2 - 2) - \left(0 - \frac{1}{2}\right) - \int_1^2 x \ln x \, dx$$

$$2\int_1^2 x \ln x \, dx = 4\ln 2 - \frac{3}{2}$$

$$\Rightarrow \int_1^2 x \ln x \, dx = 2\ln 2 - \frac{3}{4}$$

Exp. $2\int_1^2 \ln x \cosh(\ln x)\, dx$

$$\cancel{2}\int_1^2 \ln x \left(\frac{e^{\ln x} + e^{-\ln x}}{\cancel{2}}\right) dx$$

$$\Rightarrow \int_1^2 \ln x \left(x + \frac{1}{x}\right) dx$$

$$\Rightarrow \int_1^2 x \ln x \, dx + \int_1^2 \frac{\ln x}{x} \, dx$$

$u = \ln x$
$du = \frac{1}{x} dx$
بالتعويض

$$2\ln 2 - \frac{3}{4} + \frac{\ln^2 2}{2}$$

Exp: $\int e^{\sqrt{3x+9}}\,dx$

$u=\sqrt{3x+9}$

$u^2=3x+9$

$2u\,du=3\,dx$

$= \frac{2}{3}\int e^{u}\, u\, du$

$u \quad e^u$

$1 \xrightarrow{(+)} e^u$

$0 \xrightarrow{(-)} e^u$

$= \frac{2}{3}\left[ue^{u}-e^{u}\right]+c$

$= \frac{2}{3}e^{\sqrt{3x+9}}\left[\sqrt{3x+9}-1\right]+c$

Exp. $\int_0^{\frac{1}{\sqrt{2}}} 2x\sin^{-1}x^2\,dx$

$u=\sin^{-1}x^2 \qquad dv=2x\,dx$

$du=\frac{2x}{\sqrt{1-x^4}}\,dx \qquad v=x^2$

$\Rightarrow \int_0^{\frac{1}{\sqrt{2}}} u\,dv = uv\Big|_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}} v\,du$

$= x^2\sin^{-1}x^2\Big|_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}}\frac{2x^3}{\sqrt{1-x^4}}\,dx$

$w=1-x^4$

$dw=-4x^3\,dx$

$=\left(\left(\frac{1}{2}\sin^{-1}\frac{1}{2}\right)-(0)\right)+\frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}}\frac{-4x^3\,dx}{\sqrt{1-x^4}}$

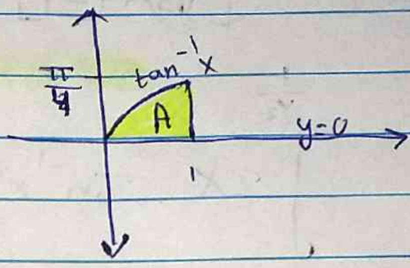
$=\frac{1}{2}\,\frac{\pi}{6}+\frac{1}{2}\int_1^{3/4}\frac{dw}{\sqrt{w}}$

$=\frac{\pi}{12}+\left(\sqrt{w}\Big|_1^{3/4}\right) = \frac{\pi}{12}+\frac{\sqrt{3}}{2}-1$

Exp: Consider the region bounded by $y = \tan^{-1} x$, $x = 1$, $y = 0$

① find the area of this region.

② find the volume of the resulted solid generated by revolving the region about y-axis.



① $A = \int_0^1 \tan^{-1} x \, dx$

$u = \tan^{-1} x \qquad dv = dx$

$du = \frac{1}{1+x^2} \qquad v = x$

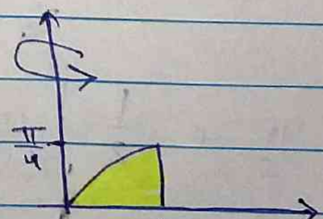
$A = \int_0^1 u \, dv = uv \Big|_0^1 - \int_0^1 v \, du$

$= x \tan^{-1} x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2} dx$

$= \left(\tan^{-1} 1 - 0 \tan^{-1} 0\right) - \frac{1}{2} \ln |1+x^2| \Big|_0^1$

$= \frac{\pi}{4} - \ln \sqrt{2}$

② $V \Rightarrow$ (Shell method)



$V = 2\pi \int_a^b \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{hight}\end{matrix}\right) dx$

$u = \tan^{-1} x \qquad dv = x\,dx$

$du = \frac{1}{1+x^2} dx \qquad v = \frac{x^2}{2}$

$= 2\pi \int_0^1 x \left(\tan^{-1} x\right) dx$

$V = 2\pi \left[\frac{x^2}{2} \tan^{-1} x \Big|_0^1 - 0 - \frac{1}{2} \int \frac{x^2+1-1}{1+x^2} dx \right]$

$$= 2\pi \left[\frac{1}{2}\frac{\pi}{4} - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx \right] = 2\pi \left[\frac{\pi}{8} - \frac{1}{2}\left(1 - \tan^{-1}x \Big|_0^1\right) \right]$$

$$= 2\pi \left[\frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) \right] \#$$

8.2 Trigonometric Integrals:-

$$\int \sin^m x \cos^n x \, dx$$

case(I): If m is odd we use

$\Rightarrow 1 - \sin^2 x = \cos^2 x$

Exp : $I = \int \sin^3 x \cos^2 x \, dx$

$$= \int \sin^2 x \cos^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x)(\cos^2 x) \sin x \, dx$$

$$= \int (\cos^2 x - \cos^4 x) \sin x \, dx$$

let $u = \cos x$

$du = -\sin x \, dx$

$$\therefore = \int (-u^2 + u^4) \, du = \int u^4 - u^2 \, du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

Case (II) : m is even and n is odd.

we use $1-\sin^2 x = \cos^2 x$

or $1-\cos^2 x = \sin^2 x$

Exp: $\int \cos^5 x \, dx$

$= \int \cos^4 x \cos x \, dx = \int (\cos^2 x)^2 \cos x \, dx$

$= \int (1-\sin^2 x)^2 \cos x \, dx$

let $u = \sin x$

$du = \cos x \, dx$

$= \int (1-u^2)^2 \, du = \int (1-2u^2+u^4) \, du$

$= u - \frac{2u^3}{3} + \frac{u^5}{5} + C$

$= \sin x - \frac{2}{3}(\sin x)^3 + \frac{(\sin x)^5}{5} + C$

Exp: $\int \sin^3 x \cos^9 x \, dx = \int \sin^2 x \cos^9 x \sin x \, dx$

$= \int (1-\cos^2 x) \cos^9 x \sin x \, dx$

$= \int (\cos^9 x - \cos^{11} x) \sin x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$\int (-u^9 + u^{11}) \, dx$

Case (III), [m, n] are even :-

w use $\Rightarrow \cos^2 x = \frac{1+\cos 2x}{2}$

$\Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$

Exp: $\int (\sin^4 x) \cos^2 x \, dx$

$= \int \left(\frac{1-\cos 2x}{2}\right)^2 \left[\frac{1+\cos(2x)}{2}\right] dx$

$= \frac{1}{8} \int (1-\cos 2x)(1-\cos 2x)(1+\cos 2x) \, dx$

$= \frac{1}{8} \int (1-\cos 2x)(1-\cos^2 2x) \, dx$

$= \frac{1}{8} \int (1-\cos 2x)(\sin^2 2x)$

$= -\frac{1}{8} \int \sin^2(2x) \cos x \, dx + \frac{1}{8} \int \sin^2(2x) \, dx$

$u = \sin 2x$
$du = 2\cos 2x$

$= -\frac{1}{8} \int u^2 \frac{du}{2} + \frac{1}{8} \int \frac{1-\cos(4x)}{2} dx$

$= -\frac{1}{16} \frac{u^3}{3} + \frac{1}{16} \left(x - \frac{\sin 4x}{4}\right) + C$

$= -\frac{1}{48} (\sin 2x)^3 + \frac{1}{16} x - \frac{\sin 4x}{64} + C$

$$\int \sec^m x \tan^n x \, dx$$

we use $1 + \tan^2 x = \sec^2 x$

or $\tan^2 x = \sec^2 x - 1$

Exp: $\int \tan^3 x \sec^3 x \, dx$

$= \int \tan^2 x \sec^2 x \sec x \tan x \, dx$

$= \int (\sec^2 x - 1) \sec^2 x \tan x \sec x \, dx$

$= \int (u^4 - u^2) \, du$

let $u = \sec x$

$du = \sec x \tan x \, dx$

Exp: $\int \sec^3 x \, dx = \int \sec^2 x \sec x \, dx$

$u = \sec x \qquad dV = \sec^2 x \, dx$

$du = \sec x \tan x \, dx \qquad V = \tan x$

$I = \sec x \tan x - \int \sec x \tan^2 x \, dx$

$I = \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$

$I = \sec x \tan x - \underbrace{\int \sec^3 x \, dx}_{I} + \int \sec x \, dx$

$2I = \sec x \tan x + \ln|\sec x + \tan x|$

$I = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln|\sec x + \tan x| + C$

* Elimating the square roots

$$\int_0^{\pi} \sqrt{1+\sin x} \quad dx$$

$$\int_0^{\pi} \int_0^{\pi} \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$\int_0^{\pi} \sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}$$

$$\int_0^{\pi} \left|\cos\frac{x}{2} + \sin\frac{x}{2}\right|$$

$$\Rightarrow \int_0^{\pi/2} \cos\frac{x}{2} + \sin\frac{x}{2} \; dx$$

$$= \left(2\sin\frac{x}{2} - 2\cos\frac{x}{2}\right)\Big|_0^{\pi}$$

$$= \left(2(1) - 2(0)\right) - \left(2(0) - 2(1)\right)$$

$$= 4$$

8.2 Trigonometric integrals

Question: How to find

$$\int \cos^m x \ \sin^n x \ dx \ ?$$

1 If m odd $\Rightarrow$ $m = 2k+1$

$$\cos^m x = \cos^{2k+1} x = (\cos^2 x)^k * \cos x$$

$u = \sin x$ $\qquad = (1 - \sin^2 x)^k \cos x$

$du = \cos x \ dx$

2 If n odd $\Rightarrow$ $n = 2k+1$

$$\sin^n x = \sin^{2k+1} x = [\sin^2 x]^k \sin x$$

$du = \cos x$

$du = -\sin x \ dx$ $\qquad = [1 - \cos^2 x] \sin x$

3 If n and m are both even then

replace $\Rightarrow$ $\sin^2 x = \frac{1 - \cos 2x}{2}$

$\Rightarrow$ $\cos^2 x = \frac{1 + \cos 2x}{2}$

exp : $\int x \sin^2 x \, dx$

$$= \int x \left(\frac{1-\cos 2x}{2} \right) dx$$

$$= \int \frac{x}{2} dx + \int -\frac{x}{2} \cos 2x \, dx$$

$-\frac{x}{2}$ $\cos 2x$

(x)

$-\frac{1}{2}$ $\frac{1}{2}\sin 2x$

$(-)$

0 $-\frac{1}{4}\cos 2x$

$$\frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C.$$

exp : $\int \sin^3 x \cos^2 x \, dx$

$$= \int \sin^2 x \cos^2 x \sin x \, dx$$

$U = \cos x$

$du = -\sin x \, dx$

$$= \int (1-\cos^2 x)\cos^2 x \sin^2 x \, dx$$

$$= -\int (1-U^2) U^2 \, du$$

$$= \int U^4 - U^2 \, du$$

$$\frac{U^5}{5} - \frac{U^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

exp $\int \sin^2 x \cos^2 x \, dx$

$$\Rightarrow \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$\Rightarrow \frac{1}{4}\int (1-\cos^2 2x)\,dx \Rightarrow \frac{1}{4}\int 1\,dx - \frac{1}{4}\int \cos^2 2x\,dx$$

$$\Rightarrow \frac{1}{4}\int 1\,dx - \frac{1}{4*2}\int (1+\cos 4x)\,dx$$

$$\Rightarrow \frac{1}{4}x - \frac{1}{8}x + \frac{1}{4*8}\sin 4x + C$$

$$\Rightarrow \frac{x}{8} - \frac{1}{4}\frac{1}{8}\sin 4x + C$$

Question : ? How to find ?

$\boxed{1}$ $\int \cos mx \sin nx \, dx$

$\boxed{2}$ $\int \cos mx \cos nx \, dx$

$\boxed{3}$ $\int \sin mx \sin nx \, dx$

* $\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$

* $\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$

* $\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$

Exp: $\int \sin 3x \cos 2x \, dx$

$= \frac{1}{2}\int (\sin(3-2)x + \sin(3+2)x)$

$= \frac{1}{2}\int (\sin x + \sin 5x) \, dx$

$= \frac{1}{2}\left(-\cos x - \frac{1}{5}\cos 5x\right) + c$

Exp: $\displaystyle\int_0^{\pi/6} \sqrt{1+\sin x}\,.dx \;\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}$

$$\int_0^{\pi/6} \frac{\sqrt{(1-\sin^2 x)}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6} \sqrt{\frac{(1-\sin^2 x)}{1-\sin x}}$$

$$= \int_0^{\pi/6} \frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin}}\,dx = \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-\sin x}}\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$\frac{\pi}{6} \to \frac{1}{2}$

$0 \to 0$

$$\int_0^{1/2} \frac{\cos x}{\sqrt{1-u}}\,\frac{du}{\cancel{\cos x}}$$

$$= \int_0^{1/2} \frac{1}{\sqrt{1-u}}\,du \qquad \int_0^{1/2} (1-u)^{-\frac{1}{2}}$$

$$= 2(1-u)^{\frac{1}{2}}\Big|_0^{1/2} = \left(2\left(1-\tfrac{1}{2}\right)^{1/2}\right) - \left(2(1-0)^{1/2}\right)$$

$$= 2\sqrt{\tfrac{1}{2}} - 2$$

$$= 2\left(\sqrt{\tfrac{1}{2}} - 1\right)$$

Exp:- $\int 4\tan^3 x\,dx = 4\int \tan^2 x \tan x\,dx$

$\tan^2 x = \sec^2 x - 1$

$= \int 4\tan x \sec^2 x\,dx - 4\int \tan x\,dx$

$\downarrow$

$u = \tan x$

$du = \sec^2 x\,dx$

$= 4\int u\,du - 4\ln|\sec x| + c$

$= 4\dfrac{u^2}{2} - 4\ln|\sec x| + c$

$= 4\dfrac{\tan^2 x}{2} - 4\ln|\sec x| + c$

Exp: $\int \sec^4 x \, dx = \int \sec^2 x \, \sec^2 x \, dx$

$= \int \sec^2 x \, (1 + \tan^2 x) \, dx$

$u = \tan x$

$du = \sec^2 x$

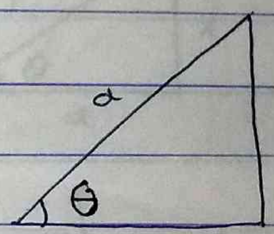
$= \int (1 + u^2) \, du$

$= u + \frac{u^3}{3} + C$

$= \tan x + \frac{1}{3} \tan^3 x + C$

8.3 Trigonometric Substitution :-

1 $x = a\sin\theta$

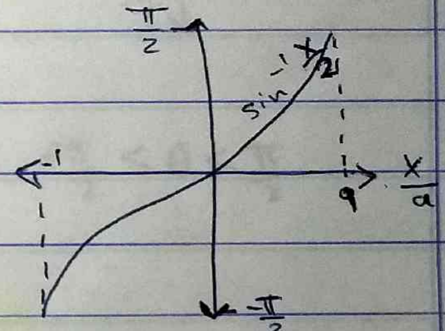


$\sqrt{a^2 - x^2}$

$dx = a\cos\theta \, d\theta$

$\theta = \sin^{-1}\dfrac{x}{a}$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta}$$
$$= \sqrt{a^2(1 - \sin^2\theta)}$$
$$= a\sqrt{\cos^2\theta}$$
$$= a\,|\cos\theta|$$
$$= a\cos\theta$$



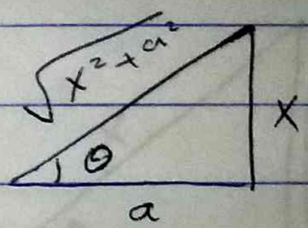
$-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

Ex $\int \sqrt{4 - x^2}\, dx$

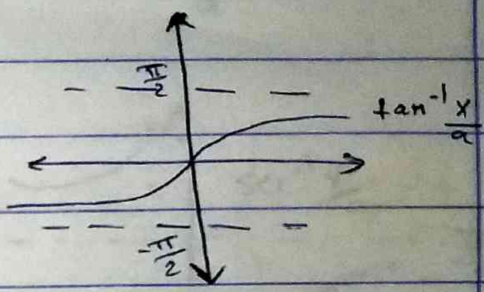
$\boxed{2}$ $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$\theta = \tan^{-1} \frac{x}{a}$



$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta}$$



$$\sqrt{a^2 + x^2} = \sqrt{a^2(1 + \tan^2\theta)}$$

$$= a\sqrt{\sec^2\theta}$$

$$= a\,|\sec\theta|$$

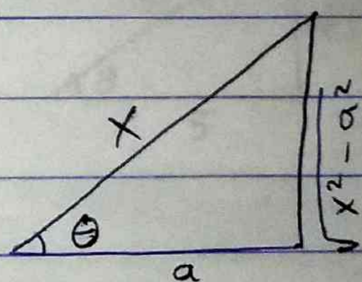
$$= a\sec\theta$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

[3] $x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$\theta = \sec^{-1}\frac{x}{a}$



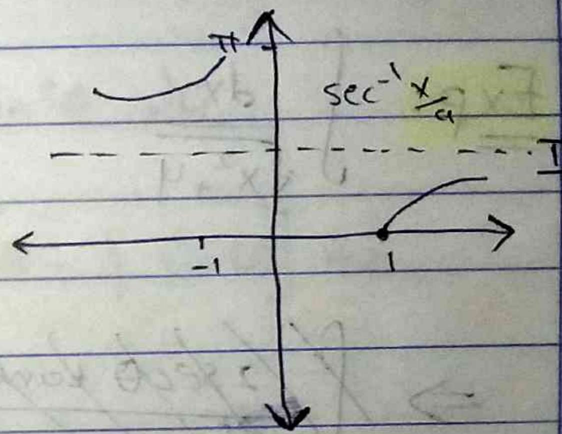
$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2}$

$= \sqrt{a^2(\sec^2\theta - 1)}$

$= a\sqrt{\tan^2\theta}$

$= a\,|\tan\theta|$

$= a\tan\theta$



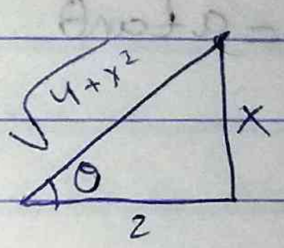
$* \Rightarrow 0 \le \theta < \frac{\pi}{2}$ if $\frac{x}{a} \ge 1$

$\Rightarrow \frac{\pi}{2} < \theta \le \pi$ if $\frac{x}{a} \le -1$

Exp: $\displaystyle\int \frac{dx}{\sqrt{x^2+4}}$

$a = 2$

$x = 2\tan\theta$

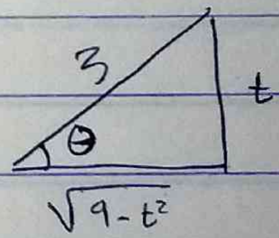


$$\Rightarrow \cancel{\int \frac{2\sec\theta\tan\theta\,d\theta}{\sqrt{\ }}}$$

$$\int \frac{2\sec^2\theta\,d\theta}{2\sec\theta\,d\theta} = \ln\left|\sec\theta + \tan\theta\right| + c$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c$$

Exp: $\displaystyle\int \sqrt{9-t^2}\,dt$



$$= \int 3\cos\theta\, 3\cos\theta\, d\theta$$

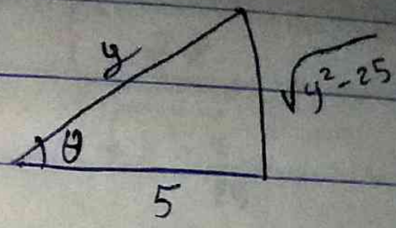
$$= 9\int \cos^2\theta\, d\theta = 9\int \left(\frac{\cos 2\theta + 1}{2}\right) d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]$$

$$= \frac{9}{2}\left[\sin^{-1}\frac{t}{3} + \frac{1}{2}\, 2\sin\theta\cos\theta\right] + c$$

$$= \frac{9}{2}\left[\sin^{-1}\frac{t}{3} + \frac{t}{3}\,\frac{\sqrt{9-t^2}}{3}\right] + c$$

Exp: $\int \frac{\sqrt{y^2-25}}{y^3}\,dy$



$$= \int \frac{\cancel{5}\tan\theta\ \cancel{5}\sec\theta\ \tan\theta\ d\theta}{(5)^{\cancel{3}}\ \sec^{\cancel{3}\,2}\theta}$$

$$\frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\,.d\theta = \frac{1}{5}\int \sin^2\theta\ .d\theta$$

$$= \frac{1}{5}\int\left(\frac{1-\cos2\theta}{2}\right)d\theta = \frac{1}{5}\left[\theta - \frac{1}{2}\sin2\theta\right] + c$$

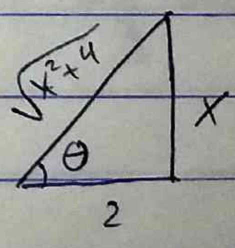
$$= \frac{1}{5}\left(\theta - \sin\theta\cos\theta\right) + c$$

$$= \frac{1}{5}\left[\sec^{-1}y - \frac{\sqrt{y^2-25}}{y}\ \frac{5}{y}\right] + c$$

Exp:

(S1) $\int_{-2}^{2}\frac{dx}{x^2+4} = \frac{1}{2}\ \tan^{-1}\frac{x}{2}\Big|_{-2}^{2} = \frac{\pi}{4}$

(S2) $\int \frac{2\sec^2\theta\ d\theta}{[2\sec\theta]^2}$

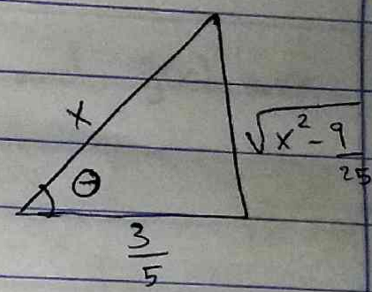


$$\frac{1}{2}\int d\theta$$

$$\frac{1}{2}\theta = \frac{1}{2}\ \tan^{-1}\frac{x}{2}\Big|_{-2}^{2} = \frac{\pi}{4}$$

Exp: $$\int \frac{5\,dx}{\sqrt{25x^2-9}} = \int \frac{5\,dx}{5\sqrt{x^2-\frac{9}{25}}} = \int \frac{dx}{\sqrt{x^2-\frac{9}{25}}}$$

$$\int \frac{\cancel{\frac{3}{5}}\sec\theta\,\cancel{\tan\theta}\;d\theta}{\cancel{\frac{3}{5}}\,\cancel{\tan\theta}}$$



$$\int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left|\frac{5x}{3} + \frac{5\sqrt{x^2-\frac{9}{25}}}{3}\right|$$

Exp: $$\int \sqrt{\frac{x}{1-x^3}}\,dx$$

$u = x^{\frac{3}{2}}$

$du = \frac{3}{2}x^{\frac{1}{2}}\,dx$

$$\frac{2}{3}\int \frac{du}{\sqrt{1-u^2}} = \frac{2}{3}\sin^{-1}\left(\frac{u}{1}\right) + c = \frac{2}{3}\sin^{-1}\left(x^{\frac{3}{2}}\right) + c$$

8.4 Integration of rational functions Using Partial fraction:

- Rational function $\frac{f(x)}{g(x)}$, $f(x)$ and $g(x)$ are polys.

- The idea of using partial fraction is to simplify The rational function into small parts that are easy to integrate.

- If the degree of $f <$ degree of g, and

[1] g can be written as linear, distinct factors then we use Cover Method. to find the Coefficients

[2] g can be written as repeated factors then we don't use Cover Method, but we ~~use~~ simplify the factors to find the coefficients.

* Linear : $x-1$, $2x+3$, x, $1-3x$, ...

- If the degree of $f >$ degree of g then we use Long division $\Rightarrow$ applied [1] or/and [2]

Exp: $\int \frac{dx}{x^2-3x+2} = \int \frac{1}{(x-2)(x-1)} dx$

Cover Method (C.M)

$$\frac{1}{(x-2)(x-1)} = \frac{A}{(x-2)} + \frac{B}{(x-1)}$$

$$A = \frac{1}{2-1} = 1 \quad , \quad B = \frac{1}{1-2} = -1$$

$$\int \frac{dx}{x^2-3x+2} = \int \left(\frac{1}{x-2} + \frac{-1}{x-1} \right) dx$$

$$= \ln|x-2| - \ln|x-1| + c$$

$$= \ln\left|\frac{x-2}{x-1}\right| + c$$

Exp: $\displaystyle\int \frac{dx}{x^3+x^2-2x} = \int \frac{dx}{x(x^2+x-2)} = \int \frac{dx}{x(x+2)(x-1)}$

$$\Rightarrow \frac{1}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$$

$A = \frac{-1}{2}$, $B = \frac{1}{6}$, $C = \frac{1}{3}$

$$\int \frac{dx}{x^3+x^2-2x} = \int \left(\frac{-1/2}{x} + \frac{1/6}{x+2} + \frac{1/3}{x-1} \right) dx$$

$$= -\frac{1}{2} \ln|x| + \frac{1}{6} \ln|x+2| + \frac{1}{3} \ln|x-1| + c$$

Exp: $\displaystyle\int_{-1}^{0} \frac{x^3}{x^2-2x+1}\, dx$

$$x^2-2x+1 \overline{\big)\, x^3 \quad} \quad x+2$$
$$\frac{-x^3 \pm 2x^2 \mp x}{2x^2 - x}$$
$$\frac{-2x^2 \pm 4x + 2}{3x - 2}$$

$$= \int_{-1}^{0} \left(x + 2 + \frac{3x-2}{x^2-2x+1} \right) dx$$

$$= \frac{x^2}{2} + 2x \Big|_{-1}^{0} + \int_{-1}^{0} \frac{3x-2}{x^2-2x+1}\, dx$$

$$\int_{-1}^{0} \frac{(3x-2)}{(x-1)(x-1)}\, dx$$

$$\Rightarrow \frac{3x-2}{(x-1)(x-1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

$$\frac{3x-2}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2}$$

Note: المقام يساوي المقام ∴ البسط يساوي البسط

$A = 3 \qquad B = 1$

Next

$$= \frac{x^2}{2} + 2x \Big|_{-1}^{0} + \int_{-1}^{0} \left(\frac{3}{x-1} + \frac{1}{(x-1)^2} \right) dx$$

$$= (0+0) - \left(\tfrac{1}{2} - 2\right) + \left[3\ln|x-1| - \frac{1}{(x-1)} \right]_{-1}^{0}$$

$$= 2 - 3\ln 2$$

• Note: S_1 $3x - 2 = A(x-1) + B$

$Ax = 3x \Rightarrow A = 3$ معامل X في الجهتين متساوي

$-2 = -A + B$ الثابت في الجهتين متساوي

$B = 1$

S_2 $3x - 2 = A(x-1) + B$

$3x\phi - 2 = A(1-1) + B$ تعويض $X = \phi$

$B = 1$

$A = 3$

S_3 $(3x - 2 = A(x-1) + B)'$ by derivativ

$\boxed{3 = A}$

$B = 1$

• Note: $\Rightarrow$ $\dfrac{1-2x}{(2x-3)^3} = \dfrac{A}{2x-3} + \dfrac{B}{(2x-3)^2} + \dfrac{C}{(2x-3)^3}$

$\Rightarrow$ $\dfrac{1-2x}{(2x^2-3)^2} = \dfrac{Ax+B}{2x^2-3} + \dfrac{Cx+D}{(2x^2-3)^2} + \dfrac{Ex+F}{(2x^2-3)^3}$

$\Rightarrow$ $\dfrac{1-2x}{x(2x^2-3)^3} = \dfrac{G}{x} + \dfrac{Ax+B}{2x^2-3} + \dfrac{Cx+D}{(2x^2-3)^2} + \dfrac{Ex+F}{(2x^2-3)^3}$

8.4 Integration by Partial fraction:

Exp: $\int \frac{x^4}{x^4-1}\,dx = \int \frac{x^4}{(x^2-1)(x^2+1)}\,dx = \int \frac{x^4}{(x-1)(x+1)(x^2+1)}\,dx$

$$\frac{x^4}{(x-1)(x+1)(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{Cx+D}{x^2+1}$$

On cover method $A = \frac{1}{4}$

$B = -\frac{1}{4}$

$$= \frac{\frac{1}{4}}{x-1} - \frac{\frac{1}{4}}{x+1} + \frac{Cx+D}{x^2+1}$$

$$\frac{x^4}{x^4-1} = \frac{1/2}{(x-1)(x+1)} + \frac{Cx+D}{x^2+1}$$

$$\frac{x^4}{\cancel{x^4-1}} = \frac{\frac{1}{2}(x^2+1) + (Cx+D)(x^2-1)}{\cancel{(x^2-1)(x^2+1)}}$$

$$\Rightarrow x^4 = \frac{x^2}{2} + \frac{1}{2} + (C\cancel{x}+D)(x^2-1)$$

$x=0 \Rightarrow 0 = 0 + \frac{1}{2} + D(-1) \Rightarrow D = 1/2$

$\xrightarrow{\frac{d}{dx}}$ $4x^3 = x + 0 + (Cx+D)(2x) + (x^2-1)C$

$x \Rightarrow 0 \Rightarrow 0 = 0 + 0 + 0 - C \Rightarrow C = 0$

$$\int \frac{x^4}{x^4-1}\,dx = \int \frac{1/4}{x-1}\,dx + \int \frac{1/4}{x+1}\,dx + \int \frac{1/2}{x^2+1}\,dx$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}\tan^{-1}x + c$$

Exp: $\int \frac{\sqrt{x+1}}{x}\,dx$

$$y=\sqrt{x+1}$$
$$dy=\frac{1}{2\sqrt{x+1}}\,dx$$
$$2\sqrt{x+1}\,dy=dx$$
$$2y\,dy=dx$$
$$y^2=x+1$$

$$\int \frac{2y^2}{y^2-1}\,dy$$

$$\int \frac{2y^2}{y^2-1}\,dy=\int\left(\frac{1}{y-1}-\frac{1}{y+1}\right)dy=\ln|y-1|-\ln|y+1|+c$$

$$\frac{2y^2}{(y-1)(y+1)}=\frac{A}{y-1}+\frac{B}{y+1}$$
$$A=1\quad,B=-1$$

Exp: $\int \frac{dx}{x(x^2+1)^2}=$

$$\frac{1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Cx+D}{x^2+1}+\frac{Ex+f}{(x^2+1)^2}$$

on cover method $A=1$

$$=\frac{1}{x}+\frac{Cx+D}{x^2+1}+\frac{Ex+f}{(x^2+1)^2}$$

$$\frac{1}{x(x^2+1)^2}=\frac{(x^2+1)^2}{x(x^2+1)^2}+\frac{(Cx+D)\,x(x^2+1)}{x(x^2+1)^2}+\frac{x(Ex+f)}{x(x^2+1)^2}$$

$$1=(x^2+1)^2+x(C\cancel{x}+D)(x^2+1)+x(Ex+f)$$

نشتق $0=2(x^2+1)*2x+2x(Cx^2+Dx)+(x^2+1)(2Cx+D)+2Ex+f$

$x=0\Rightarrow\quad 0=E+f$

نشتق $0=$

8.7 Improper Integrals:

Type II
The interval of integrations contain points of discount for $f(x)$

$$\int_0^2 \frac{dx}{x}$$

Type I
The upper limit or the lower Limit contains ∞ or $-\infty$

$$\int_1^\infty \frac{dx}{x}$$

Q: How to handle Improper Integral of type I?

$f(x)$ is defined on an infinite interval $[a,\infty)$, then

- $$\int_a^\infty f(x).dx = \lim_{b\to\infty}\int_a^b f(x).dx$$
- $$\int_{-\infty}^b f(x).dx = \lim_{a\to-\infty}\int_a^b f(x).dx$$

Def (Improper Integral of type (I) are integrals with infinite limit of integration):

1. f is cont on $[a,\infty) \Rightarrow \int_a^{\infty} f(x)\,dx =$

$$\lim_{b\to\infty} \int_a^b f(x)\,dx$$

2. f is cont on $(-\infty, a] \Rightarrow \int_{-\infty}^{a} f(x)\,dx =$

$$\lim_{b\to-\infty} \int_b^a f(x)\,dx.$$

3. f is cont on $(-\infty,\infty) = \int_{-\infty}^{\infty} f(x)\,dx$

$$= \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

$$= \lim_{b\to-\infty} \int_b^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx$$

- If the limit of the improper integral exists (finite number) then we say that this improper integral **converges**.

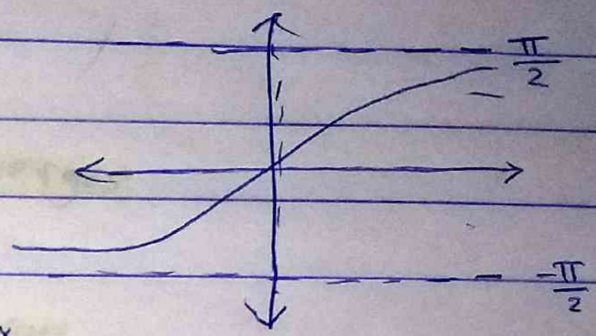
- Otherwise we say that the improper integral **diverges**.

- If $f(x) \geq 0$ $\forall x$ and the limit exists then the improper integral converges to positive number $\Rightarrow$ **Area.**

Exp. ① $\int_0^{\infty} \frac{dx}{x^2+1}$ I

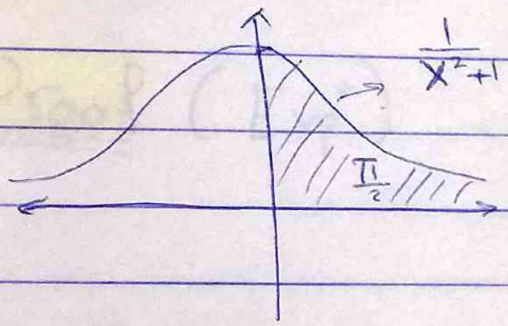
$$= \lim_{b \to \infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b \to \infty} \tan^{-1} x \Big|_0^b = \lim_{b \to \infty} \left[\tan^{-1} b - \tan^{-1} 0 \right]$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$



$\Rightarrow$ converges to $\frac{\pi}{2}$

area since $f(x) \geq 0 \ \forall x$



② $\int_{-\infty}^{0} \frac{dx}{x^2+1} = \frac{\pi}{2}$

③ $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi$

Exp*

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p\le 1 \end{cases}$$

Exp: ① $\displaystyle\int_1^\infty \frac{dx}{x^2} = \frac{1}{2-1} = 1 \Rightarrow$ **Converges**

② $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}} = \infty \Rightarrow$ **diverges**

③ $\displaystyle\int_1^\infty \frac{dx}{x} = \infty \Rightarrow$ **diverges**

Proof (Exp*)

$$\int_1^\infty \frac{dx}{x^p} = \lim_{c\to\infty}\int_1^c \frac{dx}{x^p}$$

$$= \lim_{c\to\infty}\int_1^c x^{-p}\,dx$$

$$= \lim_{c\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^c$$

$$= \lim_{c\to\infty}\left[\frac{c^{1-p}}{1-p} - \frac{1}{1-p}\right]$$

$$= \frac{1}{p-1}\left[1 - \lim_{c\to\infty} c^{1-p}\right]$$

$= \cancel{\infty} \Leftarrow \underline{\underline{p}}$ comp

$$\Rightarrow \int_1^\infty \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p\ge 1 \end{cases}$$

If $p=1 \Rightarrow \displaystyle\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty}\int_1^b \frac{dx}{x}$

$= \displaystyle\lim_{b\to\infty} \ln b - \ln 1 = \infty - 0 = \infty$

Exp: $\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2}\,dx$

$$= \int_{-\infty}^{0} \frac{2x\,dx}{(x^2+1)^2} + \int_{0}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$$

$u = x^2+1$

$du = 2x\,dx$

as $x = 0 \Rightarrow u = 1$

$x = -\infty \Rightarrow u = \infty$

$$= -\int_{1}^{\infty} \frac{du}{u^2} + \int^{\infty} \frac{du}{u^2}$$

$$= \left(\frac{1}{2-1}\right) + \left(\frac{1}{2-1}\right) = -1 + 1 = 0$$

Exp: $\int_{0}^{\infty} \frac{16\tan^{-1}x}{1+x^2}\,dx$

$u = \tan^{-1}x$

$du = \frac{1}{1+x^2}\,dx$

when $x = 0 \rightarrow u = 0$

$x = \infty \rightarrow u = \frac{\pi}{2}$

$$= \int_{0}^{\frac{\pi}{16}} 16\,u\,du$$

$$= 8u^2 \Big|_{0}^{\frac{\pi}{2}}$$

$$8\left[\left(\frac{\pi}{2}\right)^2 - 0\right] = 2\pi^2$$ converges

8.7 Improper Integral of type (II):

Def: Improper Integrals of type II are integrals that become infinite at some point within the limit of Integration (V.Asy)

- If f is discont. at a then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^a f(x)\,dx$$

- If f is discont at b then

$$\int_a^b f(x)\,dx = \lim_{c \to a^-} \int_a^c f(x)\,dx$$

- If f is discont at $c \in [a,b]$ then

$$\int_a^b f(x)\,dx = \lim_{d \to c^-} \int_a^d f(x)\,dx + \lim_{e \to c^+} \int_e^b f(x)\,dx$$

Exp. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

$$= \lim_{c\to 0^+} 2\int_c^1 \frac{dx}{2\sqrt{x}} = \lim_{c\to 0^+} 2\sqrt{x}\Big|_c^1$$

$$2\lim_{c\to 0^+}\left(\sqrt{1}-\sqrt{c}\right)$$

$$2\left[1-\sqrt{0}\right] = 2$$

Exp*: $\displaystyle\int_1^\infty \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p\le 1 \end{cases}$ $\Rightarrow$ Type I

Exp:** $\displaystyle\int_0^1 \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p} & \text{if } p<1 \\ \infty & \text{if } p\ge 1 \end{cases}$ $\Rightarrow$ Type II

$\Rightarrow$ **Exp**$'$: $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 \frac{dx}{x^{1/2}} = \frac{1}{1-\frac{1}{2}} = 2$

Exp: $\displaystyle\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{c\to 0^+}\int_c^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

$$= -2\lim_{c\to 0^+}\int_c^1 \frac{e^{-\sqrt{x}}}{-2\sqrt{x}}\,dx$$

$u = -\sqrt{x}$

$du = \frac{-dx}{2\sqrt{x}}$

$$= -2\lim_{c\to 0^+}\int_{-\sqrt{c}}^{-1} e^{u}\,du$$

$$= -2\lim_{c\to 0^+} e^{u}\Big|_{-\sqrt{c}}^{-1} = -2\left(e^{-1} - e^{-\sqrt{c}}\right)$$

$\lim_{c\to 0^+}$

$$= -2\left[\frac{1}{e} - e^{-\sqrt{c}}\right]$$

$$= -2\left[\frac{1}{e} - 1\right] = 2 - \frac{2}{e}$$

Exp: $\displaystyle\int_0^1 x\ln x\,dx$ Type II

x | $\ln x$

1 $\xrightarrow{+}$ $x\ln x - x$

0 $\xrightarrow{-}$ $\int x\ln x - \frac{x^2}{2}$

$$\int x\ln x\,dx = x^2\ln x - x^2 - \int x\ln x\,dx + \frac{x^2}{2}$$

$$2\int x\ln x\,dx = x^2\ln x - \frac{x^2}{2}$$

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}$$

$$\lim_{c\to 0^+}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]\Big|_c^1 = \lim_{c\to 0^+}\left[\left(0-\frac{1}{4}\right) - \left(\frac{c^2}{2}\ln c - \frac{c^2}{4}\right)\right]$$

$$= -\frac{1}{4} + \lim_{c\to 0^+}\left(\frac{c^2}{4} - \frac{1}{2}c^2\ln c\right)$$

$$-\frac{1}{4} - \frac{1}{2}\lim_{c\to 0^+}\frac{\ln c}{\frac{1}{c^2}} = -\frac{1}{4} - \frac{1}{2}\lim_{c\to 0^+}\frac{\frac{1}{c}}{-2\frac{1}{c^3}} = -\frac{1}{4}$$

Exp : $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$ II

$|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$

$= \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}}$

$= -2 \lim_{c \to 1^-} \int_0^c \frac{dx}{-2\sqrt{1-x}} + 2 \lim_{c \to 1^+} \int_c^2 \frac{dx}{2\sqrt{x-1}}$

$= -2 \lim_{c \to 1^-} \sqrt{1-x} \Big|_0^c + 2 \lim_{c \to 1^+} \sqrt{x-1} \Big|_c^2$

$= -2 \lim_{c \to 1^-} \left[\sqrt{1-c} - 1\right] + 2 \lim_{c \to 1^+} \left[1 - \sqrt{c-1}\right]$

$= 2 + 2 = 4$

Exp $\int_0^\infty \frac{d\theta}{1+e^\theta} = \lim_{b \to \infty} \int_0^b \frac{d\theta}{1+e^\theta} * \left(\frac{e^{-\theta}}{e^{-\theta}}\right)$

$\lim_{b \to \infty} \int_0^b \frac{d\theta \, e^{-\theta}}{1+e^{-\theta}} = \lim_{b \to \infty} -\ln|1+e^{-\theta}| \Big|_0^b$

$= \lim_{b \to \infty} \left(-\ln(1+e^{-b}) + \ln 2\right)$

$= \lim_{b \to \infty} (-0 + \ln 2) = \underline{\ln 2}$

Converge