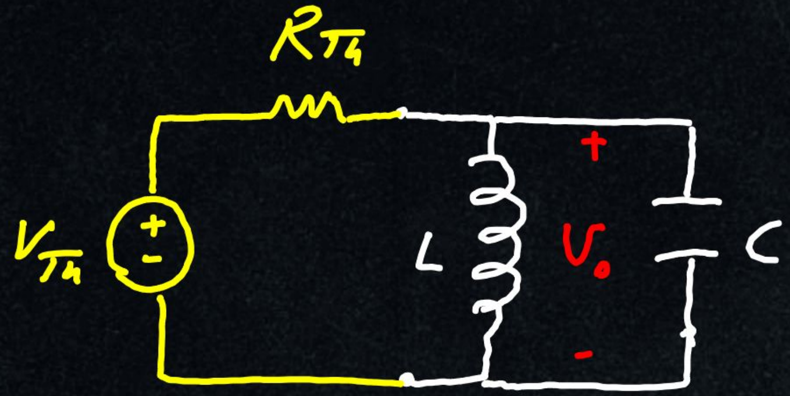
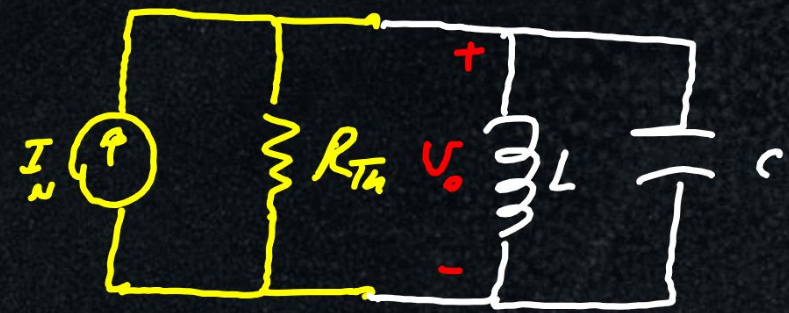
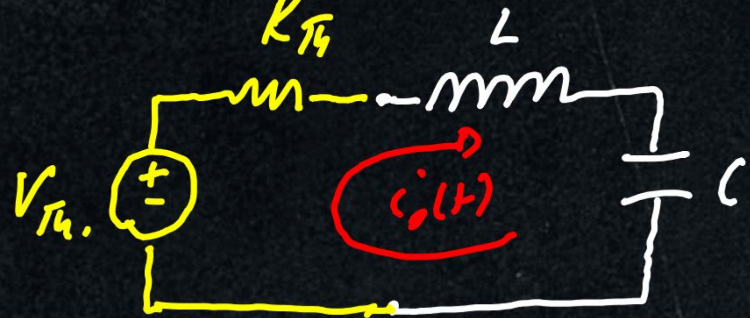
CH 8 :- Second Order RLC Circuits

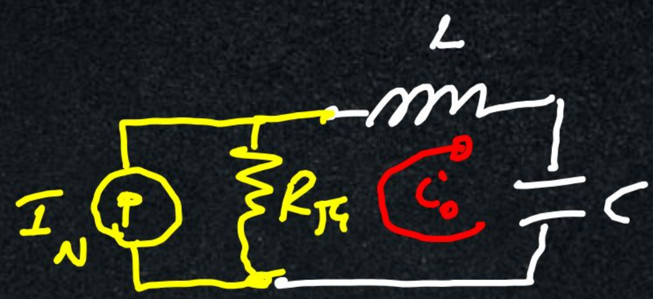
parallel RLC circuit



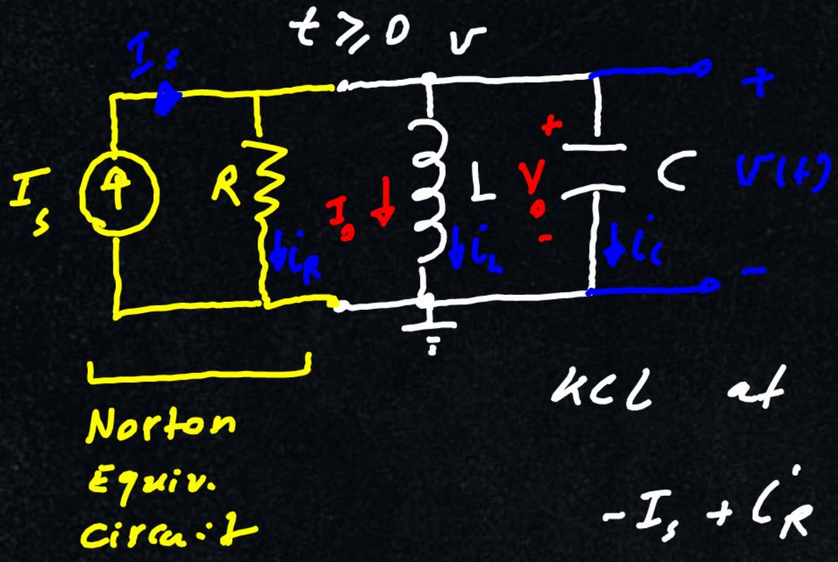


series RLC circuit





① Parallel RLC circuit



I_0 initial inductive current $\left(\frac{1}{2} L I_0^2\right)$

V_0 initial capacitive voltage $\left(\frac{1}{2} C V_0^2\right)$

KCL at v

$$-I_s + i_R + i_L + i_C = 0$$

$$i_R = \frac{v}{R}$$

$$i_C = C\frac{dv}{dt}$$

$$i_L = \frac{1}{L}\int v\,dt + I_0 \quad ; \quad v = L\frac{di_L}{dt}$$

$$I_s = \frac{v}{R} + C\frac{dv}{dt} + \frac{1}{L}\int v\,dt + I_0$$

$$0 = \frac{1}{R}\frac{dv}{dt} + C\frac{d^2v}{dt^2} + \frac{1}{L}v$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \Rightarrow 2^{nd} \text{ order differential equation}$$

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \Rightarrow \text{chara. equation } (*)$$

Roots of equation (*) :-

$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2} \quad ; \quad \alpha = \frac{1}{2RC} \;, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

α ↓ Neper Frequency

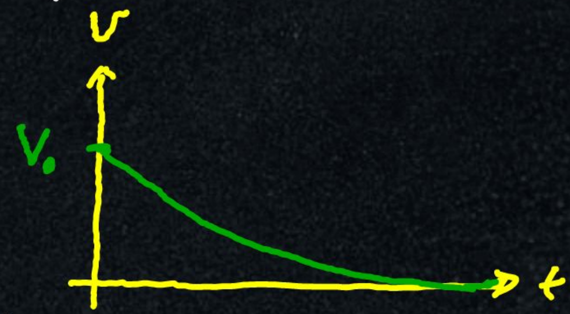
ω_0 ↓ resonant frequency

$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2}$$

Three possible outcomes

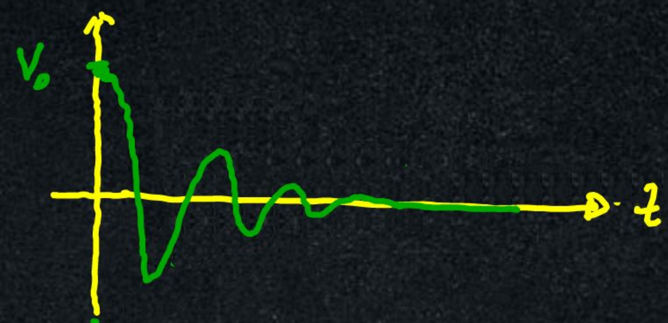
a) $\alpha^2 > \omega_0^2 \Rightarrow s_1 + s_2$ are real + unequal roots

"Over-damped response"



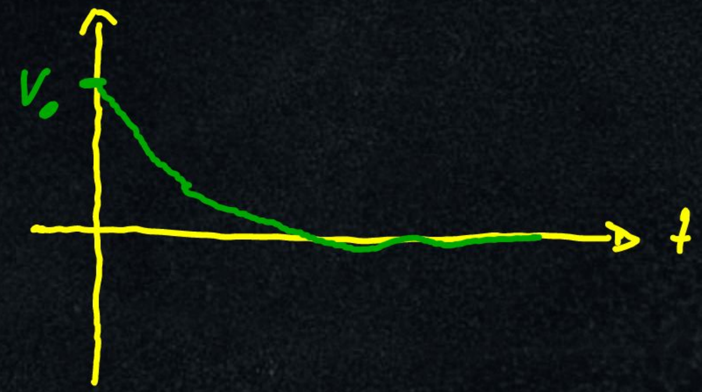
b) $\alpha^2 < \omega_0^2 \Rightarrow s_1 + s_2$ are complex roots

"Under-damped response"



c) $\alpha^2 = \omega_0^2 \Rightarrow$ real + equal roots

critically damped response



(a) Over-damped response ($\alpha^2 > \omega_0^2$)

The solution is given by

$$v(t) = \underline{A_1} e^{s_1 t} + \underline{A_2} e^{s_2 t}, \quad t \geq 0$$

$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2}$$

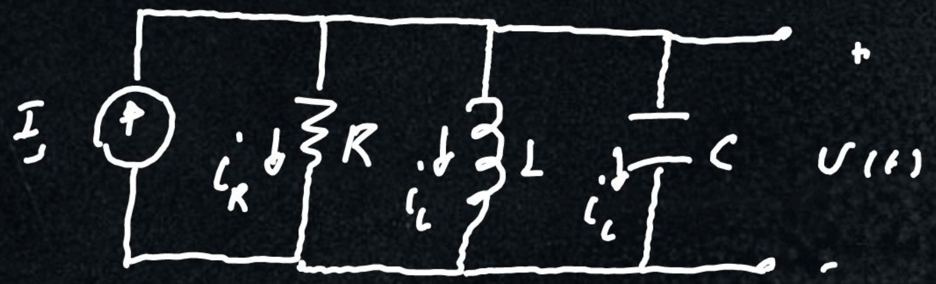
$$\alpha = \frac{1}{2RC}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$v(0^+)$, $\dfrac{dv(0^+)}{dt}$

$$\boxed{v(0^+) = v(0^-) = v(0) = V_0}$$

$$\boxed{\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{1}{C}\left[I_s - \frac{V_0}{R} - I_0\right]}$$

$t \geq 0$



$\underline{V_0}$, $\underline{I_0}$

$$I_s = i_R(0^+) + i_L(0^+) + i_c(0^+)$$

$$i_c(0^+) = I_s - i_R(0^+) - i_L(0^+)$$

$$i_c(0^+) = I_s - \frac{V_0}{R} - I_0$$

✓ $v(0^+) = A_1 + A_2$

✓ $\dfrac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2$

$\Rightarrow$ Solve for A_1 & A_2

Ex:-

$V_0 = 12\,V$

$I_0 = 30\,mA$

$C = 0.2\,\mu F, \quad L = 50\,mH, \quad R = 200\,\Omega$

Find $v(t)$

$$\alpha = \frac{1}{2RC} = \frac{1}{2(200)(0.2\times10^{-6})} = 1.25\times10^{4}\ r/s$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{50\times10^{-3}\times0.2\times10^{-6}}} = 1\times10^{4}\ r/sec$$

$\alpha > \omega_0 \Rightarrow$ over-damped $\qquad v(t) = \underline{A_1} e^{s_1 t} + \underline{A_2} e^{s_2 t}$

$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2} = \left[-1.25 \mp \sqrt{(1.25)^2 - (1)^2}\right]\times10^{4} = \begin{cases} -5000\ r/sec \\ -20{,}000\ r/sec \end{cases}$$

$$v(0^+) = V_0 = 12 \ \underline{V}$$

$$\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{1}{C}\left(\cancelto{0}{I_s} - \frac{V_0}{R} - I_0\right)$$

$$\frac{dv(0^+)}{dt} = \frac{1}{0.2\times10^{-6}}\left[\frac{-12}{200} - 30\times10^{-3}\right] = -450 \ kV/sec$$

$$12 = A_1 + A_2 \ \text{----} \ (1)$$

$$-450{,}000 = \underbrace{-5000}_{s_1} A_1 \underbrace{-20{,}000}_{s_2} A_2 \ \text{---} \ (2)$$

$$\Rightarrow A_1 = -14, \quad A_2 = 26$$

$$\boxed{v(t) = -14e^{-5000t} + 26e^{-20{,}000t}, \ V, \ t \geq 0}$$

(b) Under-damped response ($\alpha < \omega_o$)

$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_o^2} = -\alpha \mp \sqrt{-1(\omega_o^2 - \alpha^2)}$$

$$s_{1,2} = -\alpha \mp j\omega_d \quad ; \quad \omega_d = \sqrt{\omega_o^2 - \alpha^2} \quad \& \quad j = \sqrt{-1}$$

$\omega_d \rightarrow$ damped

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$$

$$v(t) = A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t} = e^{-\alpha t}\left[A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right]$$

$$v(t) = e^{-\alpha t}\left[A_1 \cos(\omega_d t) + jA_1 \sin(\omega_d t) + A_2 \cos(\omega_d t) - jA_2 \sin(\omega_d t)\right]$$

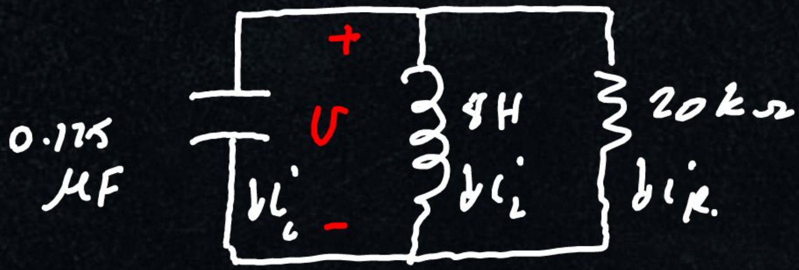
$$v(t) = e^{-\alpha t}\left[\underbrace{(A_1 + A_2)}_{B_1} \cos\omega_d t + \underbrace{j(A_1 - A_2)}_{B_2} \sin\omega_d t\right]$$

$$v(t) = e^{-\alpha t}\left[\beta_1 \cos(\omega_d t) + \beta_2 \sin(\omega_d t)\right]$$

$$v(0^+) = \beta_1 = V_0$$

$$\frac{dv(0^+)}{dt} = \omega_d \beta_2 - \alpha \beta_1 = \frac{1}{C}\left[I_s - \frac{V_0}{R} - I_0\right]$$

EX



$V_0 = 0$

$I_0 = -12.25 \text{ mA}$

Find v, i_R, i_L & i_C

$$\alpha = \frac{1}{2RC} = \frac{1}{2(20\times10^{3})\times(0.175\times10^{-3})} = 200 \text{ r/s}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{8 * 0.175\times10^{-6}}} = 1000 \text{ r/sec}$$

$\alpha < \omega_0 \Rightarrow$ under-damped

$$v(t) = e^{-\alpha t}\left[B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right]$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{(1000)^2-(200)^2} = 980 \text{ r/sec.}$$

$$v(0^+) = V_0 = 0$$

$$\frac{dv(0^+)}{dt} = \frac{1}{C}\left(\cancel{I_s} - \cancel{\frac{V_0}{R}} - \cancel{I_0}\right) = 98{,}000 \text{ V/sec}$$

0.125×10^{-6} 0 0 -12.25×10^{-3}

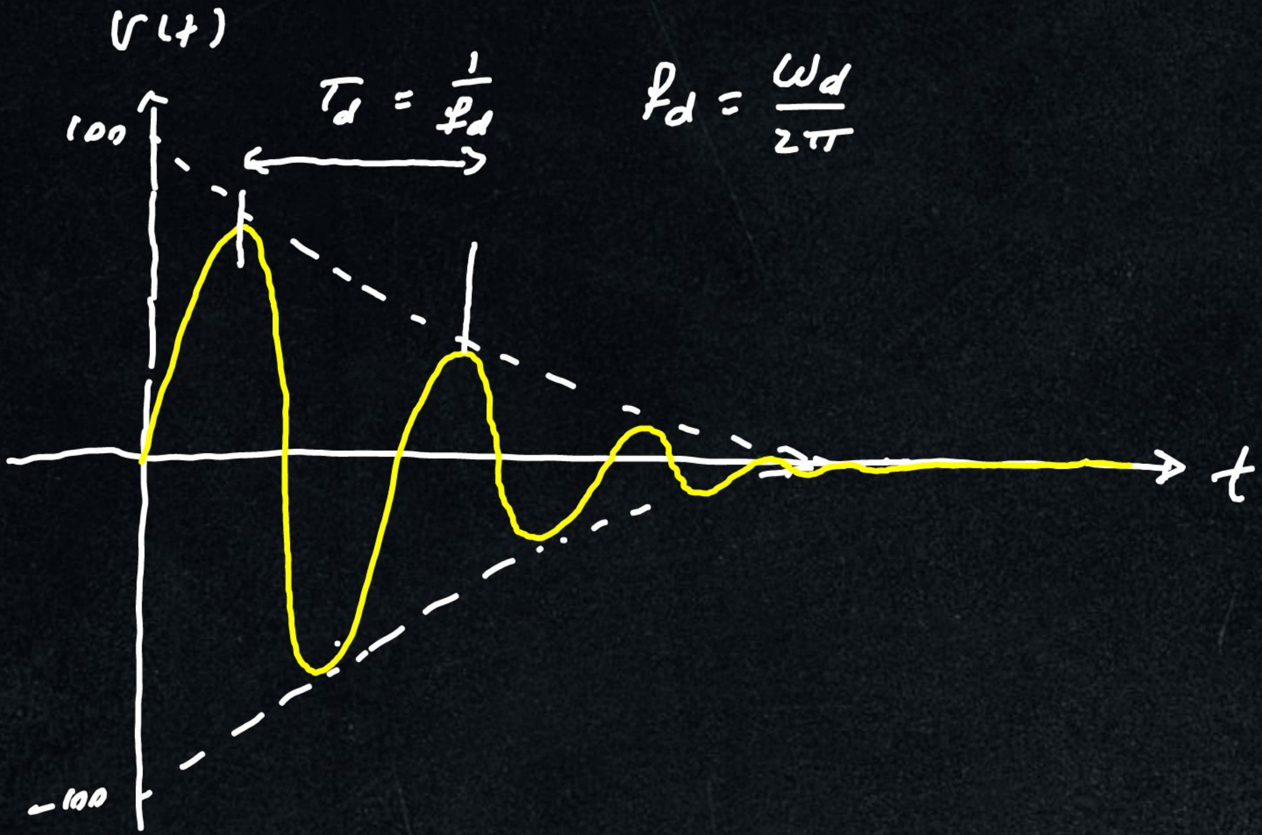
$$v(0^+) = \beta_1 = 0$$

$$\frac{dv(0^+)}{dt} = -\cancel{\alpha\beta_1}^{\,0} + \omega_d\beta_2 = 98{,}000 \Rightarrow \beta_2 = \frac{98000}{980} = 100$$

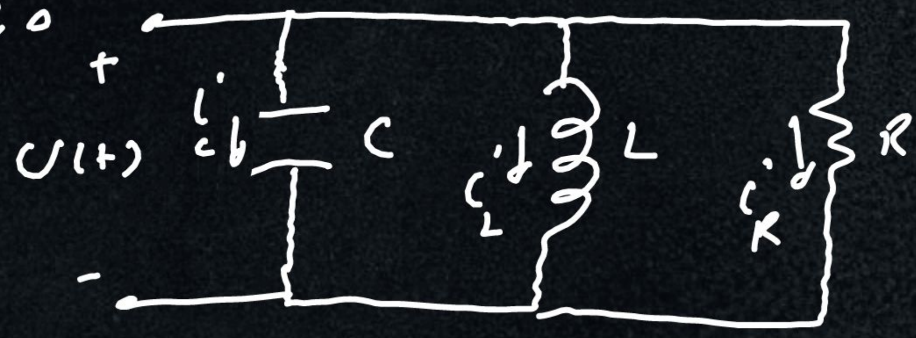
$$v(t) = e^{-200t}\left[100\sin(980t)\right]$$

. .

$$v(t) = 100\, e^{-200t} \sin(980t) \ \underline{V}, \quad t \geq 0$$



$$v(t) = 100 e^{-200t} \sin(980t) \text{ V}, \quad t \ge 0$$



$$i_C = C\frac{dv}{dt}$$

$$i_R = \frac{v}{R}$$

$$i_L = \frac{1}{L}\int v\,dt + I_0$$

$$= -i_R - i_C = -\frac{v}{R} - C\frac{dv}{dt}$$

$$i_R + i_C + i_L = 0$$

$$i_L = -i_R - i_C.$$

$$\int e^{x}\sin x\,dx = \int \frac{e^{x}e^{jx} - e^{x}e^{-jx}}{2j}\,dx$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$= \int \frac{e^{(1+j)x} - e^{(1-j)x}}{2j}\,dx$$

$$= \frac{1}{2j}\int e^{(1+j)x}\,dx - \frac{1}{2j}\int e^{(1-j)x}\,dx$$

parallel Connected RLC

$i_R = \frac{v}{R}$

$i_C = C\frac{dv}{dt}$

$i_L = I_s - i_R - i_C$

$\alpha = \frac{1}{2RC}$

$\omega_0 = \frac{1}{\sqrt{LC}}$

$V(0^+) = V_0$

$\frac{dV(0^+)}{dt} = \frac{1}{C}\left(I_s - \frac{V_0}{R} - I_0\right)$

Overdamped

$\alpha > \omega_0$

$V = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2}$

$V(0^+) = A_1 + A_2$

$\frac{dV(0^+)}{dt} = s_1 A_1 + s_2 A_2$

Underdamped

$\alpha < \omega_0$

$V = e^{-\alpha t}\left[\beta_1 \cos \omega_d t + \beta_2 \sin(\omega_d t)\right]$

$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

$V(0^+) = \beta_1$

$\frac{dV(0^+)}{dt} = \omega_d \beta_2 - \alpha \beta_1$

critically-damp.

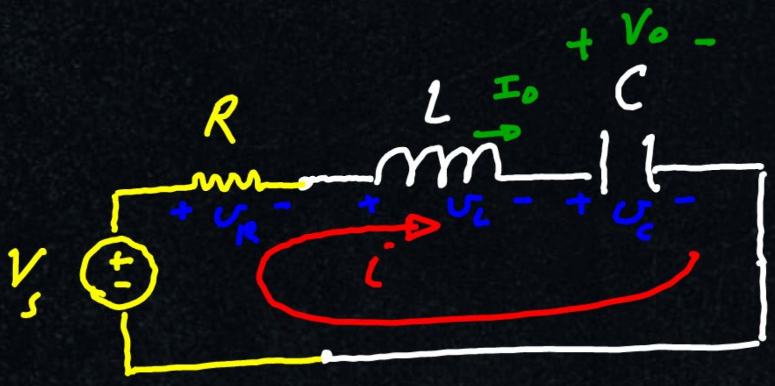
$\alpha = \omega_0$

$V = (D_1 t + D_2)e^{-\alpha t}$

$V(0^+) = D_2$

$\frac{dV(0^+)}{dt} = D_1 - \alpha D_2$

Series connected RLC circuit



V_0: initial capacitive voltage $\left(\frac{1}{2}CV_0^2\right)$

I_0: initial inductive current $\left(\frac{1}{2}LI_0^2\right)$

KVL in the Loop

$$V_s = v_R + v_L + v_C$$

$$v_R = Ri$$

$$v_L = L\frac{di}{dt}$$

$$v_C = \frac{1}{C}\int i\,dt + V_0$$

$$V_s = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt + V_0$$

$$0 = R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{1}{C}i \Rightarrow \boxed{\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0}$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

Charach. equation $\quad s^2 + \frac{R}{L} s + \frac{1}{LC} = 0$

$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{R}{2L} \quad , \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

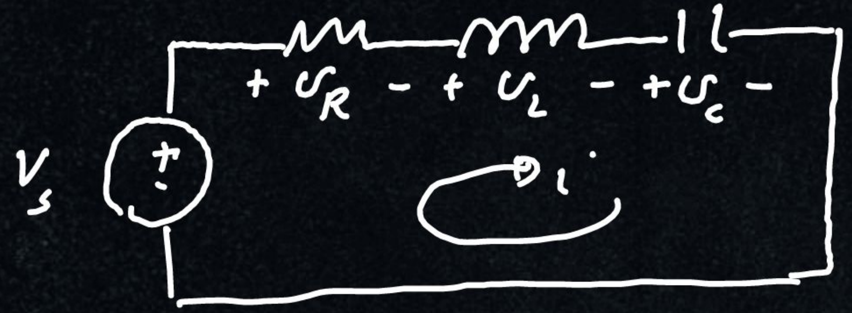
$\alpha > \omega_0 \Rightarrow$ overdamped $\Rightarrow i = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$\alpha < \omega_0 \Rightarrow$ Underdamped $\Rightarrow i = e^{-\alpha t}\left[B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right]$

$\alpha = \omega_0 \Rightarrow$ critically-damped $\Rightarrow i = e^{-\alpha t}\left[D_1 t + D_2\right]$

To find (A_1, A_2), (B_1, B_2), or (D_1, D_2), calculate

$i(0^+)$ + $\frac{di(0^+)}{dt}$



$$\boxed{i(0^+) = i(0^-) = i(0) = I_0}$$

$$v_L = L\frac{di}{dt}$$

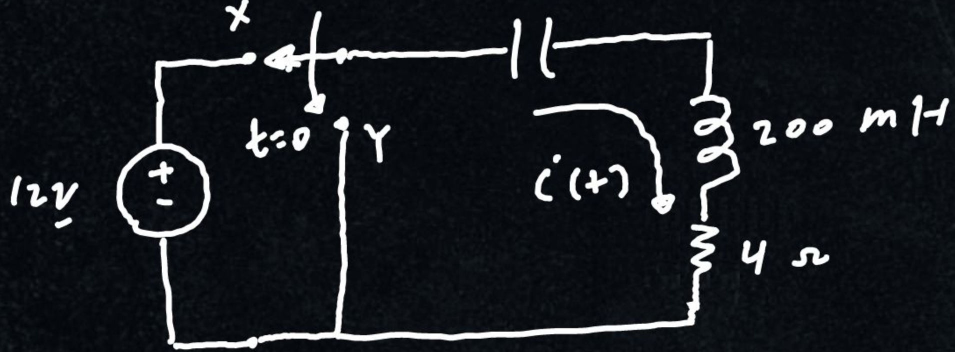
$$V_s = v_R + v_L + v_c$$

$$v_L = V_s - v_R - v_c$$

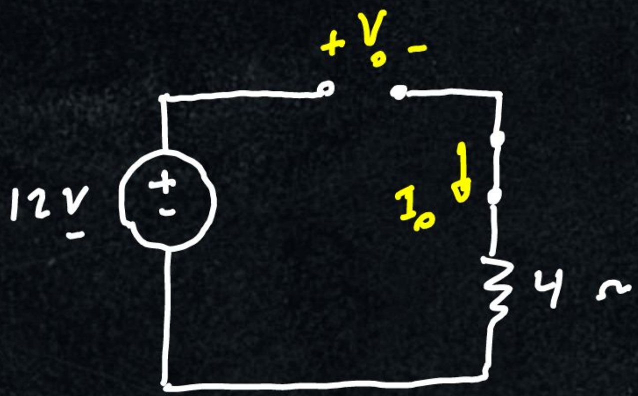
$$v_L(0^+) = L\frac{di(0^+)}{dt} = V_s - v_R(0^+) - v_c(0^+)$$

$$\boxed{\frac{di(0^+)}{dt} = \frac{1}{L}\left[V_s - RI_0 - V_0\right]}$$

Ex :- The switch has been in position x for a long time. At t=0, the switch moves to position Y. Calculate $i(t)$ $t \geq 0$



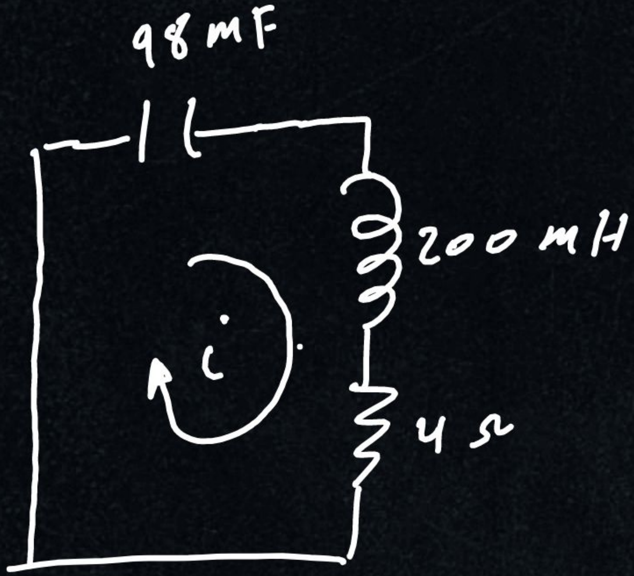
$t = 0^-$ (C is o/c + L is s/c)



$I_0 = 0$

$V_0 = 12V$

$t \geq 0$



$$\alpha = \frac{R}{2L} = \frac{4}{2(0.2)} = 10 \ r/sec$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(200)(98)\times10^{-6}}} = 7.143 \ r/sec$$

$\alpha > \omega_0 \Rightarrow$ overdamped

$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2} = \begin{cases} -3 \ r/s \\ -17 \ r/s \end{cases}$$

$$i = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$i(0^+) = A_1 + A_2 = 0$$

$$A_1 = -A_2 \ \cdots (1)$$

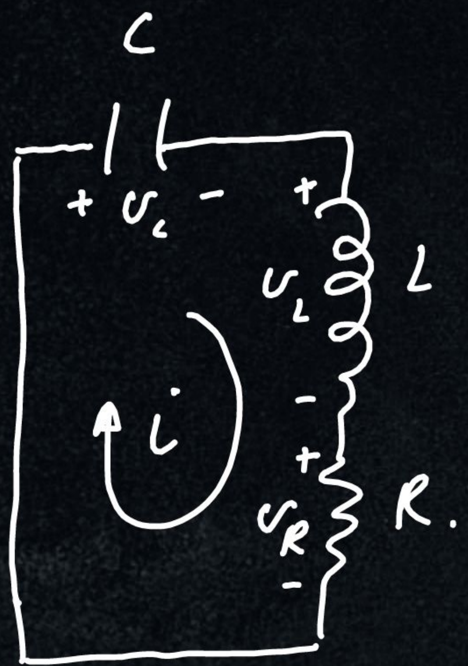
$$i(0^+) = I_0 = 0$$

$$\frac{di(0^+)}{dt} = \frac{1}{L}\left[\cancel{V_s}^{\,0} - R\cancel{I_0}^{\,0} - V_0\right] = \frac{-12}{0.2} = -60 \ A/sec$$

$$\frac{di(0^+)}{dt} = s_1 A_1 + s_2 A_2$$

$$-3A_1 - 17A_2 = -60 \ \cdots (2) \Rightarrow A_1 = -4.29 \ , \ A_2 = 4.29$$

$$i(t) = -4.29\, e^{-3t} + 4.29\, e^{-17t} \ \text{A} \qquad t \geq 0$$

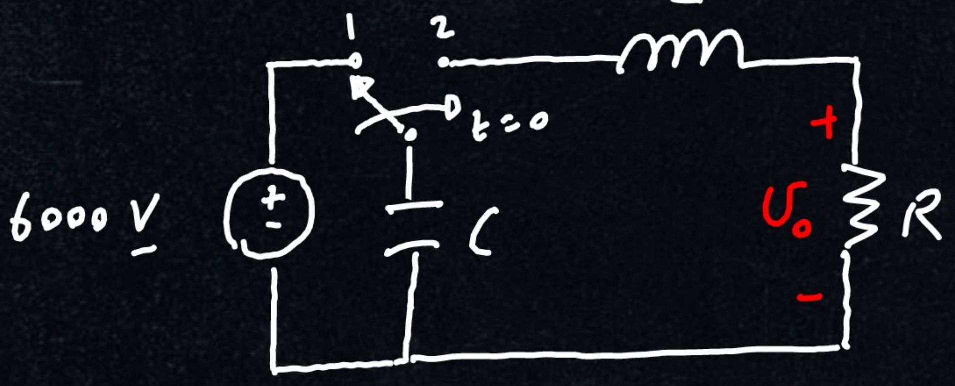


$$v_R = Ri$$

$$v_L = L\frac{di}{dt}$$

$$v_c = -v_R - v_L$$

EX :-



$t \geq 0$

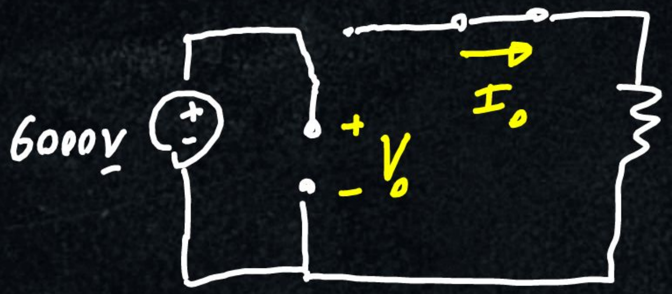
$C = 31\ \mu F$

$R = 50\ \Omega$

$L = 50.3\ mH$

The switch has been in position '1' for a long time. At $t = 0$, the switch moves to position '2'. Find $v_o(t)$ $t \geq 0$.

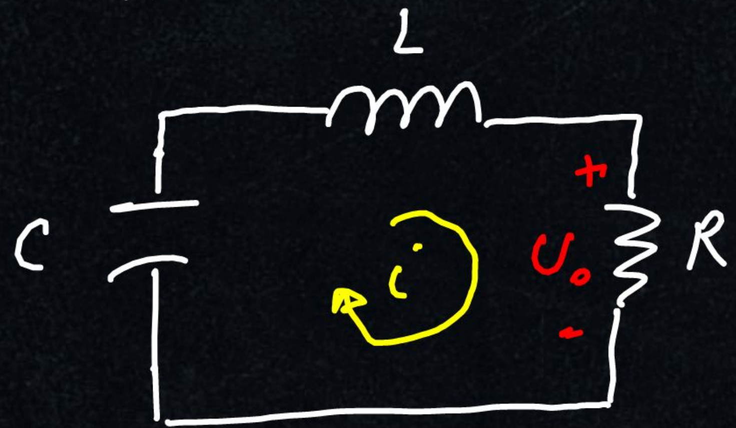
$t = 0^-$ (C is o/c + L is s/c)



$V_o = 6000\ V$

$I_o = 0$

$t \geq 0$



$R = 50\ \Omega$

$L = 50.3\ mH$

$C = 31\ \mu F$

$$\alpha = \frac{R}{2L} = \frac{50}{2(50.3 \times 10^{-3})} = 497\ r/sec$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{50.3 \times 31 \times 10^{-9}}} = 800.82$$

$\alpha < \omega_0 \Rightarrow$ Underdamped

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 627\ r/sec.$$

$$i = e^{-\alpha t}\left[B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)\right]$$

$$i(0^+) = I_0 = 0 = B_1$$

$$\frac{di(0^+)}{dt} = \frac{1}{L}\left[\cancelto{0}{V_s} - R\cancelto{0}{I_0} - V_0\right] = \frac{-6000}{50.3 \times 10^{-3}} = -119.28 \times 10^3\ A/s$$

$$B_2 = \frac{-119.28 \times 10^3}{627} = -190$$

$$= \omega_d B_2 - \alpha B_1 = \omega_d B_2$$