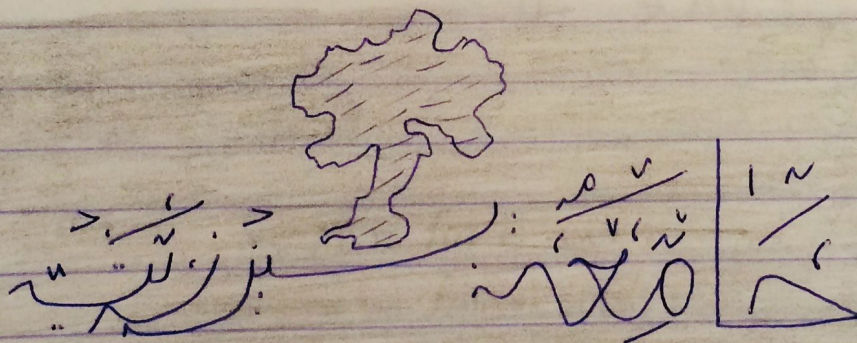
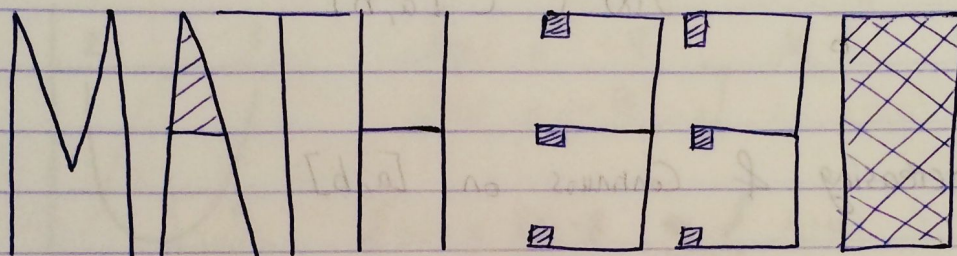


Student's Name : Ayham Hashesh

Instructor : Mahamoud Ghannam

Numerical Methods

طرق التحليل العددي



BITRZET UNIVERSITY

1161301

أيهام حشيش

## \* Chapter 2 \*

How to solve the equation  $f(x) = 0$  numerically? <sup>قِيَمَة تَقْرِيْبِيَّة</sup>  
zeros, roots, solution of  $f(x)$

\* There are five numerical <sup>تكرار</sup> iterative method for solving  $f(x) = 0$

- ① Bisection Method
- ② False - Position Method (Regula - Falsi Method)
- ③ Fixed Point iteration
- ④ Newton's Method (Newton - Raphson method)
- ⑤ The Secant Method

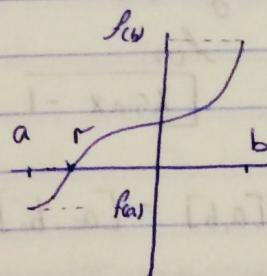
approximation  
Sequence of iteration

$P_0, P_1, P_2, P_3, \dots$  converges  $r$

Ex 1.5, 1.67, 1.77, 1.78, 1.87, 1.95, 1.97, 1.99, 1.999, 1.9999, ...  $\rightarrow 2$   
نقطة

\* Bisection Method نقطة

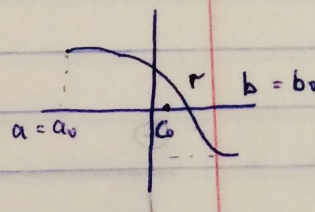
Recall if  $f \in C[a, b]$  &  $f(a)f(b) < 0$   
لأن اقيم موجب واقيم سالب  
الجزء بينهم  $\leftarrow$



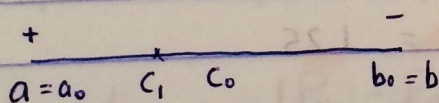
$\Rightarrow \exists$  at least  $r \in (a, b)$  s.t.  
 $f(r) = 0$  ( $r$  root of  $f(x)$ )

$r$ : exact root  $f(r) = 0$

Bisection  $f \in C[a, b]$  &  $[a, b] : [a_0, b_0]$   
 $f(a_0)f(b_0) < 0$



First iteration :  $c_0 = \frac{a_0 + b_0}{2} \Rightarrow f(c_0)$



if  $f(c_1) < 0 \Rightarrow [a_1, b_1] = [a_0, c_1]$

$$c_1 = \frac{a_1 + b_1}{2}$$

نقطة  $c$  على اليسار  $f(c_1) < 0$

if  $f(c_1) > 0 \Rightarrow [a_1, b_1] = [c_1, b_0]$

على اليمين

$$c_1 = \frac{a_1 + b_1}{2}$$

$f(c_1) ?? [a_2, b_2]$

$$c_2 = \frac{a_2 + b_2}{2}$$

Bisection iteration :  $C_n = \frac{a_n + b_n}{2}$  hop

\* Note Rad Mode : Mode Mode 2

Ex Solve the equation  $x \sin x = 1$  in  $[1, 2]$   
using the bisection method. Find three iteration

Solution:  $f(x) = x \sin x - 1 = 0$  OR  $1 - x \sin x = 0$  ✓  
أحد الطرفين يجب أن يكون 1  
لأن هذه الطريقة فقط عندما  $f(x) = 0$

$$[a, b] = [a_0, b_0] = [1, 2] \quad C_0 = \frac{a_0 + b_0}{2} = \frac{1 + 2}{2} = 1.5$$

$$f(1) = -0.15853$$

$$f(2) = 0.81859$$

$$f(1.5) = 0.49624$$

\* Rad Mode \*

نأخذ حد موجب وحد سالب

|   |     |   |
|---|-----|---|
| ⊖ | ⊕   | ⊕ |
| 1 | 1.5 | 2 |

$$[a_1, b_1] = [1, 1.5] \quad C_1 = \frac{1 + 1.5}{2} = 1.25$$

$$f(1.25) = 0.18623$$

$$\Rightarrow [a_2, b_2] = [1, 1.25]$$

$$C_2 = \frac{1 + 1.25}{2} = 1.125$$

\*

## Stopping Methods

① the question determine

② Stop when  $|C_n - C_{n-1}| < \epsilon$  [default]

\*  $\epsilon$ : accuracy (error) &  $|C_n - C_{n-1}| \propto \epsilon$

| $C_k$     | $ C_k - C_{k-1}  \approx \epsilon$ |              |
|-----------|------------------------------------|--------------|
| $C_0$     | /                                  |              |
| $C_1$     | $ C_1 - C_0 $                      |              |
| $C_2$     | $ C_2 - C_1 $                      | $> \epsilon$ |
| $\vdots$  |                                    |              |
| $C_{n-1}$ |                                    |              |
| $C_n$     | $ C_n - C_{n-1} $                  | $< \epsilon$ |

الفروق بين كل متتالية  
تساوي أو أقل من  
stop

$\epsilon \propto C_n$

③ Stop when  $|f(C_n)| < \epsilon$

④ Stop when  $\frac{|P_n - P_{n-1}|}{|P_n|} < \epsilon$

Note  $P_n \equiv C_n$

$\frac{|C_n - C_{n-1}|}{|C_n|}$

Bisection Method Advantage: always Converges

$$\{C_n\}_{n=0}^{\infty} = \{C_0, C_1, C_2, \dots\}$$

$$** \lim_{n \rightarrow \infty} C_n = \mu$$

Disadvantage: too slow

Question

Prove

that  $\lim_{n \rightarrow \infty} C_n = r$  for bisection method

Proof

بالاستقراء ← المطلوب

$$a_0 = a \quad c_1 \quad b = b_0$$

$$a_1 \quad b_1$$

$$b_1 - a_1 = \frac{b-a}{2}$$

$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b-a}{2^2}$$

$$b_3 - a_3 = \frac{b_2 - a_2}{2} = \frac{b-a}{2^3}$$

$$\boxed{b_n - a_n = \frac{b-a}{2^n}} \quad * \textcircled{1}$$

$$a_0 \quad a_n \quad c_n \quad b_n \quad b_0$$

$$0 < |r - c_n| < \frac{b_n - a_n}{2} \quad \textcircled{2}$$

الخطأ التقريبي

Error  $|E|$

upper bound for error for bisection method

by Sandwich Thm  $\lim_{n \rightarrow \infty} |r - c_n| = 0$

$$\lim_{n \rightarrow \infty} r - c_n = 0$$

$$\lim_{n \rightarrow \infty} r = \lim_{n \rightarrow \infty} c_n$$

$$r = \lim_{n \rightarrow \infty} c_n$$

Quiz  $\Rightarrow$  الحيس  $\Rightarrow$  ch1

20-2-2018

Extra 1  $\Rightarrow$  HomeWork (الحيس)

$$\lim_{n \rightarrow \infty} C_n = \infty$$

$$|n - C_n| < \frac{b-a}{2^{n+1}}$$

$|E|$

upper bound for E in bisection

أكبر ايراد ممكن

$$\frac{b-a}{2^{n+1}} < \epsilon$$

كيفية عدد مرات التقريب  
في الباقي سكرته

$$\frac{2^{n+1}}{b-a} > \frac{1}{\epsilon}$$

$$2^{n+1} > \frac{b-a}{\epsilon} \Rightarrow \ln(2^{n+1}) > \ln\left(\frac{b-a}{\epsilon}\right)$$

$$n+1 \ln 2 > \ln\left(\frac{b-a}{\epsilon}\right)$$

نظراً

$$n > \frac{\ln \frac{b-a}{\epsilon}}{\ln 2} - 1$$

used to find # of iteration  
(theoretically) for bisection  
to achieve accuracy  $\epsilon$

Ex Find the # of iteration needed to solve  $x \sin x = 1$  in  $[1, 2]$   
using the bisection method with error less than  $10^{-7}$   
(with accuracy of  $10^{-7}$ )

Sol.

$$n > \frac{\ln \left( \frac{2-1}{10^{-7}} \right)}{\ln 2} - 1 \Rightarrow n > 22.2$$

$$n = 23 \text{ or more}$$

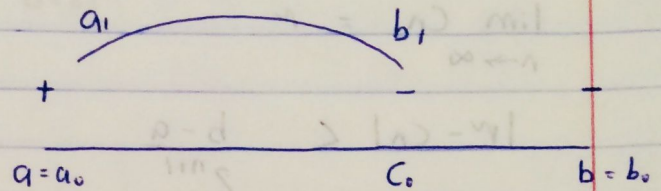
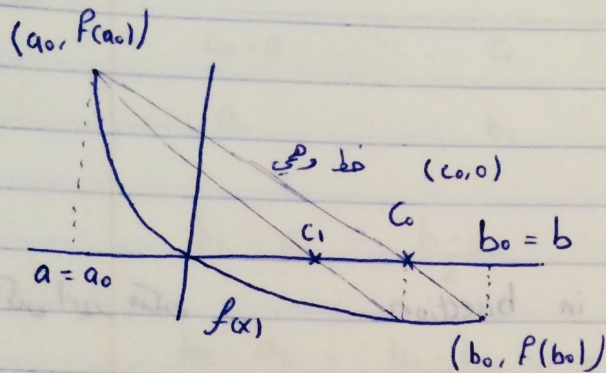
(أي الأقل 23)

اجابة تقريبية

## [2] False Position Method

نفس الطريقة السابقة  
لكن  $C_n$  مختلفة فقط

$\Rightarrow$  need  $f(x) = 0$ ,  $[a, b] = [a_0, b_0]$



الخطوط

now what is  $C_0$  ?

$$\text{Slope of } L = \frac{f(b_0) - f(a_0)}{b_0 - a_0} = \frac{0 - f(b_0)}{C_0 - b_0}$$

$$\frac{C_0 - b_0}{-f(b_0)} = \frac{b_0 - a_0}{f(b_0) - f(a_0)}$$

$$C_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$$

Similarly :  $C_1 = b_1 - \frac{f(b_1)(b_1 - a_1)}{f(b_1) - f(a_1)}$

$$C_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$$

False - Position Formula

Ex  $x \sin x = 1$  in  $[1, 2]$  Solve by using False-Position

Ex Estimate  $\sqrt[4]{12}$  in  $[1, 2]$  using the false-position method  
Find two iterations

Sol نحتاج فترة  $\leftarrow$  الفترة  
نحتاج اقتران  $\leftarrow$  ؟

$$x = \sqrt[4]{12} \Rightarrow x^4 = 12 \Rightarrow \boxed{x^4 - 12 = 0}, [a_0, b_0] = [1, 2]$$

|   |         |   |                            |
|---|---------|---|----------------------------|
| - | x       | + |                            |
| 1 | 1.73333 | 2 | $f(1) = -11$<br>$f(2) = 4$ |

Round-off chopping  $\leftarrow$  (المطلوب)  $\leftarrow$  نختار كله مع بعض

$$C_0 = 2 - \frac{f(2)(2-1)}{f(2)-f(1)} = 2 - \frac{4}{4-(-11)} = 1.73333$$

$f(C_0)$   $\Rightarrow$  نأخذ 1 أو 2 مرة

$$f(C_0) = -2.97331 \Rightarrow [a_1, b_1] = [1.73333, 2]$$

$$C_1 = 2 - \frac{f(2)(2-1.73333)}{f(2)-f(1.73333)} = 1.84703$$

### [3] Fixed-Point iteration (FPI Method)

$$f(x) = 0 \Rightarrow x = p \text{ s.t. } f(p) = 0 \quad ??$$

$$f(x) = 0 \Rightarrow x - \underbrace{g(x)}_{\text{يعطى عدد ثابت}} = 0 \quad \left. \begin{array}{l} f(x) = x - g(x) \\ \Rightarrow x = g(x) \end{array} \right\}$$

$$x = p \Rightarrow f(p) = 0 \Rightarrow p - g(p) = 0 \Rightarrow p = g(p)$$

$x = p$  is a fixed point for  $g(x)$

Def:  $x = p$  is called a Fixed Point for  $g(x)$  if  $g(p) = p$

Fixed p & root  $\Leftrightarrow 0 = p$

Ex Find the fixed pts of  $g(x) = x^2 - 2$

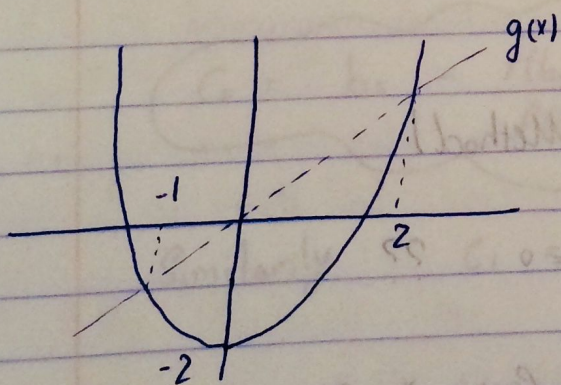
Sol  $x = x^2 - 2 \Rightarrow x^2 - x - 2 = 0$   
 $(x - 2)(x + 1) = 0$   
 $\boxed{x = 2} \quad \boxed{x = -1}$

check:  $g(2) = 2^2 - 2 = 2 \checkmark$

$g(-1) = (-1)^2 - 2 = -1 \checkmark$

Ex  $g(x) = x$  identity function  
 $\Rightarrow x = x \Rightarrow$  F.P.  $\Rightarrow$   $x = x$

\* Geometrically speak, the fixed pts of  $g(x)$  are the intersection point between  $y = g(x)$  and  $y = x$



HW Find fixed pt of  $g(x) = x \ln x - \ln x + 1$

Sol:  $x \ln x - \ln x + 1 = x$

$$\ln x (x-1) + (1-x) = 0$$

$$\Rightarrow (x-1)(\ln x - 1) = 0$$

$$(x=1, \ln x = 1) \Rightarrow (x=e)$$

\* Fixed Point Iteration

كل الشغل على  $g$  من  $f$

$$P_{n+1} = g(P_n), \text{ given } P_0$$

بداية القيمة

Initial (guess) Value

استخدمه لـ  $f(x)$

$$P_1 = g(P_0)$$

$$f(x) =$$

$$P_2 = g(P_1)$$

$$f(1) =$$

$$f(2) =$$

$$\Rightarrow 1.5 = P_0$$

$$P_3 = g(P_2)$$

$$P_n \Big]_{n=0}^{\infty}$$

$$\text{if } \lim_{n \rightarrow \infty} P_n = P \Rightarrow P \text{ fixed pt for } g(x)$$

$$\Rightarrow P \text{ root for } f(x)$$

Ex Consider the following FPI :

$$P_{n+1} = \sqrt{5P_n - 6} \Rightarrow g(x) = \sqrt{5x - 6} \text{ with } P_0 = 2.5$$

Find the first 20 iteration

$$\underline{\text{Sol}} \text{ Calculation : } 2.5 = \sqrt{5 * \text{Ans} - 6}$$

$$P_2 = 2.597604$$

$$P_{40} = 2.9993957$$

$$P_3 = 2.64386$$

$$P_{15} = 2.9449836$$

Ans

$$\lim_{n \rightarrow \infty} P_n = 3$$

check  $g(3) = \sqrt{5(3)-6} = 3$

3 fixed pt of  $g$

$$f(3) = 9 - 5(3) + 6 = 0 \Rightarrow 3 \text{ root of } f(x)$$

if  $P_0 = 6.5 \Rightarrow$  شروع کی 3  
بست میں شرط

if  $P_0 = 2 \Rightarrow$  شروع کی 2

\* FPI:  $g(x)$ ,  $P_0$

22-2-2018

$$P_{n+1} = g(P_n)$$

\*  $g(x)$  is continuous

$$\{P_0, P_1, P_2, \dots\} = P_n \Big|_{n=0}^{\infty}$$

$$\lim_{n \rightarrow \infty} P_n = P$$

$$P = \lim_{n \rightarrow \infty} P_n$$

is  $P \in$

f.p

$\Rightarrow P$  is a Fixed Point of  $g(x)$

Proofs need to show  $g(P) = P$   
 $P_{n+1} = g(P_n)$

$$\lim_{n \rightarrow \infty} P_{n+1} = \lim_{n \rightarrow \infty} g(P_n)$$

$$P = g\left(\lim_{n \rightarrow \infty} P_n\right) \quad \text{cont } g \text{ is c}$$

$$P = g(P)$$

Theorem: given a function  $g(x)$  &  $[a, b]$

if ①  $g(x)$  is continuous in  $[a, b]$

② [2.3] R [2.2, 2.7]

②  $g(x) \in [a, b] \quad \forall x \in [a, b]$  (Range Subset of Domain)

then at least a fixed pt for  $g(x)$  in  $[a, b]$  ممكن أكثر من واحدة

إضافة لذلك

Moreover, if ③  $|g'(x)| \leq K < 1 \quad \forall x \in [a, b]$

then ④  $\exists!$  Fixed Point  $P$  for  $g(x)$  in  $[a, b]$

there exist a unique

⑤ FPI of  $g(x)$  will converge to  $P$  for any  $P_0 \in [a, b]$

at least 1 fixed: شرط

only one fixed: شرط

تأكد

Note  $K$  : upper bound for  $|g'(x)|$  on  $[a, b] = \max_{a \leq x \leq b} |g'(x)|$

Note if  $|g'(x)| > 1 \quad \forall x \in [a, b]$  then FBI  
will not converge to  $P$ , Provided that  $P_0 \neq P$

اثبات Unique Fixed Point : 1 2 3  
اثبات Fixed Point : 1 2

Proofs Suppose [1] & [2]

1 حالة ~ if  $g(a) = a$ , we are done,  $a$  is Fixed Point  
2 حالة ~ if  $g(b) = b$ , = = =

3 حالة ~ if  $g(a) \neq a$  &  $g(b) \neq b$   
$$\begin{array}{ccccc} & a & g(a) & g(b) & b \\ & & \text{اقل} & \text{اكثر} & \end{array}$$

$\Rightarrow g(a) > a$  &  $g(b) < b$

$\Rightarrow g(a) - a > 0$  &  $g(b) - b < 0$

now define  $h(x) = g(x) - x$

Apply Bolzano on  $h(x)$  in  $[a, b]$

$h(x)$  is continuous on  $[a, b]$

$h(a)h(b) < 0$

$\Rightarrow$  at least  $P \in (a, b)$  such that  $h(P) = 0$

$g(P) - P = 0 \Rightarrow g(P) = P$ ,  $P$  Fixed Point for  $g(x)$

Uniqueness  $\Rightarrow$  لا يوجد  $f, p$  لكن  $f$  نفس uniqueness

Now suppose  $3$  is also satisfies

To prove uniqueness  $\Rightarrow$  (assume there are two f.p.  $p$  &  $q$  for  $g(x)$  in  $[a, b]$ )

$$g(p) = p, \quad g(q) = q \quad [\text{Assume we have 2 f.p.}]$$

Apply M.V.T on  $g(x)$  on  $[p, q] \subseteq [a, b]$

$g$  cont on  $[p, q]$

$g'$  exist on  $[p, q]$   $\Rightarrow$  واحد  $f$  من  $p$  الى  $q$

$$\Rightarrow \exists c \in (p, q) \text{ s.t. } g'(c) = \frac{g(q) - g(p)}{q - p}$$

$$g'(c) = 1 \Rightarrow |g'(c)| = 1$$

Contradiction  $\Rightarrow$  implies uniqueness

Now we will show that FPI of  $g(x)$ :  $P_{n+1} = g(P_n)$  will converge to  $p$  for any  $P_0 \in [a, b]$

Apply M.V.T on  $g(x)$  on  $[P_0, p]$

$$\Rightarrow \exists c \in (P_0, p) \text{ s.t.}$$

$$g'(c) = \frac{g(p) - g(P_0)}{p - P_0} \Rightarrow \frac{g'(c)}{1} = \frac{p - P_0}{p - P_0}$$

$$|p - P_1| = |g'(c)| |p - P_0|$$

$$|p - P_1| \leq K |p - P_0|$$

$|P - P_1| < |P - P_0|$  :  $P_1$  is closer to  $P$  from  $P_0$

Similarly, Apply M.V.T on  $g(x)$  on  $[P_1, P]$

$$\Rightarrow |P - P_2| \leq K |P - P_1| \leq K^2 |P - P_0|$$

$\Rightarrow |P - P_2| < |P - P_1|$  FPI will converge to  $P$

$$|P - P_3| \leq K^3 |P - P_0|$$

$$|E| \leftarrow \boxed{|P - P_n| \leq K^n |P - P_0|}$$

upper bound for  $|E|$  for  
FPI of  $g(x)$

\* Theorem for FPI of  $g(x)$  ①  $|P - P_n| \leq K^n |P - P_0|$

$$\boxed{|P - P_n| \leq \frac{K^n |P - P_0|}{1 - K}} \quad \text{upper bound}$$

Used to find number of iteration theoretically for FPI

$$\frac{K^n |P - P_0|}{1 - K} < \epsilon$$

$$K^n < \frac{\epsilon (1 - K)}{|P_1 - P_0|} \Rightarrow n \ln K < \ln \left( \frac{\epsilon (1 - K)}{|P_1 - P_0|} \right)$$

$$\Rightarrow n > \frac{\ln \left( \frac{\epsilon (1 - K)}{|P_1 - P_0|} \right)}{\ln K}$$

Ex

Consider  $g(x) = \sqrt{2+\ln x}$  in  $[1, 2]$

- show that  $g(x)$  has a unique fixed pt in  $[1, 2]$
- show that FPI of  $g(x)$  will converge
- Estimate the fixed pt of  $g(x)$  with error less than  $10^{-2}$ ,  $P_0 = 1.5$
- Find # of iteration needed to estimate f.p with accuracy of  $5 \times 10^{-4}$ ,  $P_0 = 1.5$

Solution (a) (i)  $g$  is clearly continuous

$$(2) g'(x) = \frac{1}{2\sqrt{2+\ln x}} = \frac{1}{2x\sqrt{2+\ln x}}$$

$g$  is increasing in  $[1, 2]$

نسبة ثابت الجذر  
2

على الفترة  $x + \sqrt{2+\ln x}$   
الجذر دائماً +

$$g(1) \leq g(x) \leq g(2)$$

$$\sqrt{2} \leq g(x) \leq \sqrt{2+\ln 2}$$

$$1 < 1.414 \leq g(x) \leq 1.64 < 2 \Rightarrow \exists \text{ Fixed Point Range } S \text{ Domain}$$

$$(3) |g'(x)| = \left| \frac{1}{2x\sqrt{2+\ln x}} \right| = \frac{1}{2x\sqrt{2+\ln x}} \text{ which is decreasing [متناقص]}$$

$$\leq \frac{1}{2(1)\sqrt{2+\ln 1}} = \boxed{0.35355} < 1$$

هنا الوارد فقط لأننا نريد upper

$\Rightarrow (a), (b)$

©  $P_{n+1} = g(P_n) = \sqrt{2 + \ln P_n}$ ,  $P_0 = 1.5$

| $P_k$    | $ P_k - P_{k-1} $ |  |
|----------|-------------------|--|
| 1.5      | —                 |  |
| 1.550956 | 0.05 > 0.01       |  |
| 1.56169  | 0.011 > 0.01      |  |
| 1.56389  | 0.002 < 0.01      |  |

كل مرة ناقص  
داي قليل

④  $n > \frac{\ln \left( \frac{\epsilon(1-K)}{|P_1 - P_0|} \right)}{\ln K} \Rightarrow n > \frac{\ln \left( \frac{5 \times 10^{-4} (1 - 0.35355)}{|1.550956 - 1.5|} \right)}{\ln(0.35355)}$

بالا غير  $P_0$  من قبلها من

$n > 4.867$

$\boxed{n=5}$

$\hat{\pi} = 3.14 \dots$

\* Theorem if  $P$  is a fixed pt for  $g(x)$   $\Rightarrow g(P) = P$   
 then [1]  $|g'(P)| < 1$  then  $P$  is called attractive  
 [2]  $|g'(P)| > 1$  = = = repulsive  
 another f.p. of  $\alpha$  is f.p.  $\Rightarrow$

27-2-2018

if  $|g'(P)| = 1 \Rightarrow$  depend on  $P$

Ex Find the fixed pts of  $g(x) = \frac{x^2}{2} + \frac{1}{x} - \frac{1}{2}$

Then classify them to attractive or repulsive

Solution  $x = g(x) \Rightarrow x = \frac{x^2}{2} + \frac{1}{x} - \frac{1}{2}$

$$2x^2 = x^3 + 2 - x$$

$$x^3 - 2x^2 - x + 2 = 0$$

$$x^2(x-2) - (x-2) = 0$$

$$(x-2)(x^2-1) = 0$$

$$x = 2, x = 1, x = -1$$

$$g'(x) = x - \frac{1}{x^2} \Rightarrow |g'(2)| = |2 - \frac{1}{4}| = \frac{7}{4} > 1$$

$\Rightarrow P = 2$  is repulsive

$|g'(1)| = 0 < 1 \Rightarrow P = 1$  is attractive

$|g'(-1)| = -1 - 1 = -2 \Rightarrow |-2| = 2 > 1 \Rightarrow P = -1$  is repulsive

repulsive may go to  $\overline{P=1}$  because it's a attractive

Ex  $P_0 = -1.5$   $P_1 = 0.5$   
 $FPI \rightarrow \infty$   $FP2 \rightarrow 1$

FPI

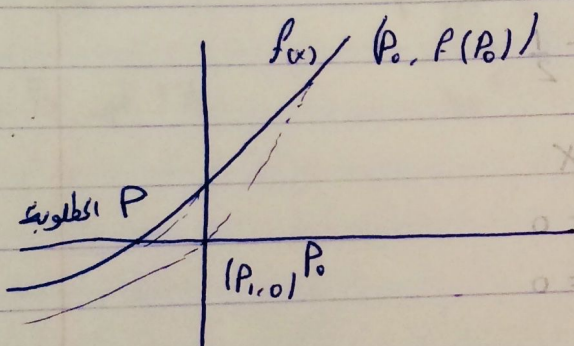
$$P_{n+1} = g(P_n)$$

$$= \frac{P_n^2}{2} + \frac{1}{P_n} - \frac{1}{2}$$

$$\text{equal } -0.5 \quad -0.5 = \frac{\text{Ans}^2}{2} + \frac{1}{\text{Ans}} - \frac{1}{2}$$

#### □ Newton's Method

$$\underline{f(x)} = 0, \quad \underline{P_0} \quad \underline{\text{نقطة}}$$



$$\text{تقريب} \quad P_1 = P_0 - \frac{f(P_0)}{f'(P_0)}$$

$$P_2 = P_1 - \frac{f(P_1)}{f'(P_1)}$$

$$g'(x) = x - \frac{f(x)}{f'(x)}$$

$$P_{n+1} = g(P_n) = P_n - \frac{f(P_n)}{f'(P_n)}$$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

Note Newton's method can be considered an example of FPI with  $g(x) = x - \frac{f(x)}{f'(x)}$