Signals and Systems

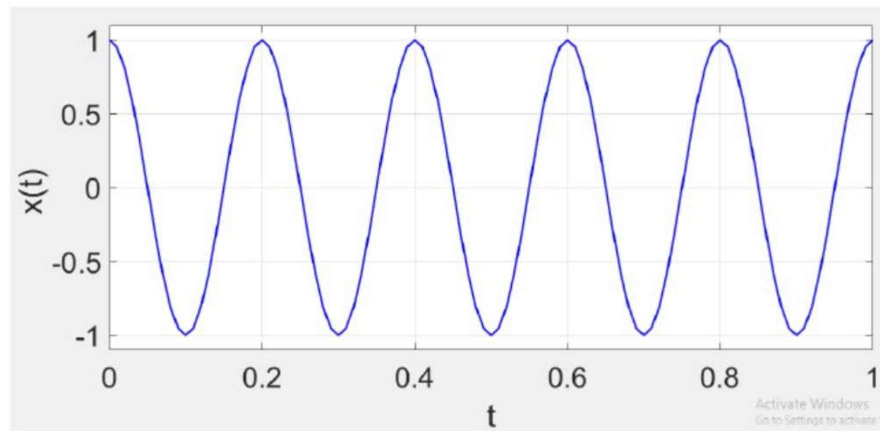
Quick Review

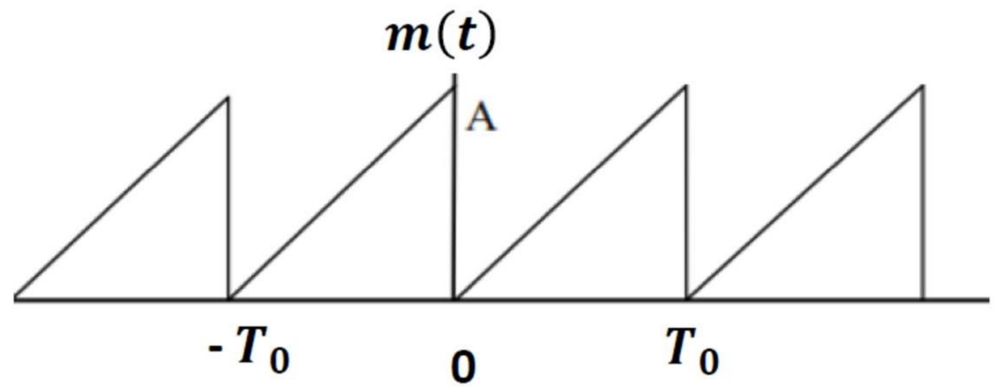
Signal Classifications

- **Definition:** A signal may be defined as a single valued function of time that conveys information.
- Depending on the feature of interest, we may distinguish four different classes of signals:
  - Periodic and Non-periodic Signals
  - Deterministic and Random Signals
  - Analog and Digital Signals
  - Energy and Power Signals

Classification of Signals: Periodic and Non-periodic

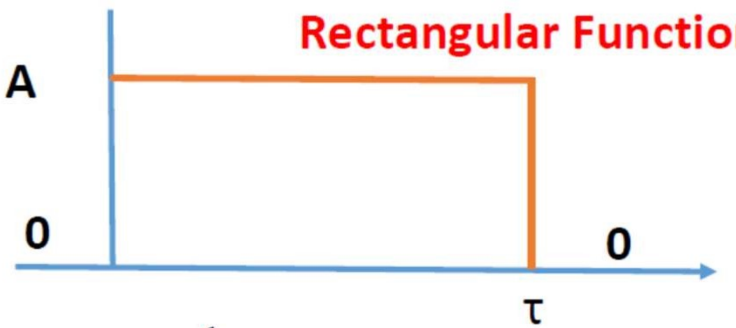
- **A periodic signal** g(t) is a function of time that satisfies the condition
$$g(t) = g(t + T_0), \forall t.$$
- The smallest value of T_0 that satisfies this condition is called the period of $g(t)$.
- **Example**: The sinusoidal signal $x(t) = \cos(2\pi(5)t)$ is periodic with period $\boldsymbol{T_0 = 1/5}$.
- The reciprocal of the period is the **fundamental frequency** $f_0 = \frac{1}{T_0}$. In this example, $\boldsymbol{f_0 = 5}$ **Hz**.
- **Example**: The saw-tooth function shown is another example of a periodic signal.
- $m(t) = \frac{A}{T_0}t, \quad 0 \le t \le T_0$
- If $T_0 = 0.001\ sec$, then the fundamental frequency $\boldsymbol{f_0 = 1000}$ **Hz**



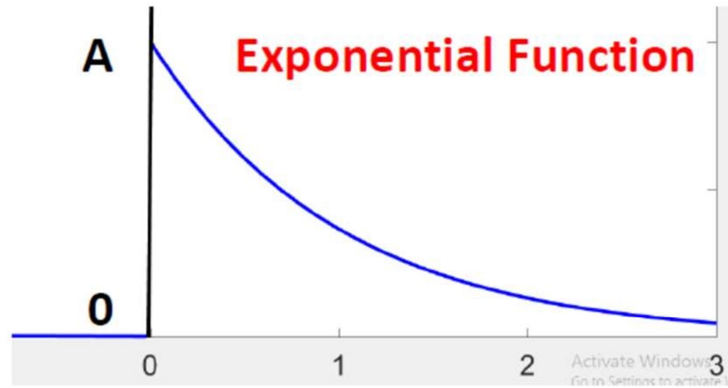


Non-periodic Signals

- A non-periodic signal g(t) is one for which there does not exist a T_0 for which the condition $g(t) = g(t + T_0)$ is satisfied, i.e., the signal does not repeat itself each T_0 .



$$g(t) = \begin{cases} A, & 0 \le t \le \tau \\ 0, & otherwise \end{cases}$$



$$g(t) = \begin{cases} A\exp(-\alpha t), & 0 \le t < \infty \\ 0, & t < 0 \end{cases}$$

$$g(t) = \begin{cases} A, & t > 0 \\ 0 & t < 0 \end{cases}$$

Deterministic and Random Signals

- **A deterministic signal** is one about which there is no uncertainty with respect to its value at any time. It is a completely specified function of time.
- **Deterministic Signal Example:** $x(t) = Ae^{-at}u(t)$; $A = 1$ and α is a constant.
- **A random signal** is one about which there is some degree of uncertainty before it actually occurs. (It is a function of a random variable)
- **Random Signal Example:** $x(t) = Ae^{-at}u(t)$; α is a constant and A is a random variable with the following probability density function (two possible realizations shown below)

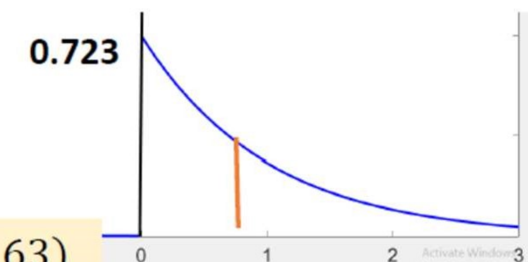
$$f_A(a) = \begin{cases} 1 & 0 \le a \le 1 \\ 0 & otherwise \end{cases}$$

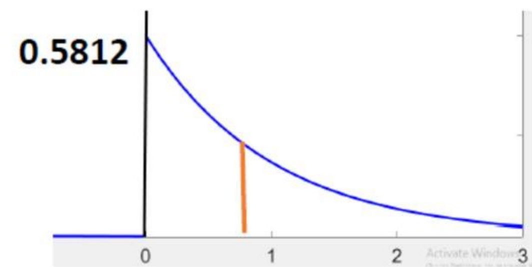
- **Random Signal Example:** $x(t) = \cos(2\pi f_c t + \Theta)$; f_c is a constant and Θ is a random variable uniformly distributed over the interval $(0, 2\pi)$ with the following probability density function (pdf).

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & otherwise \end{cases}$$

$x(t) = \cos(2\pi f_c t + 30)$

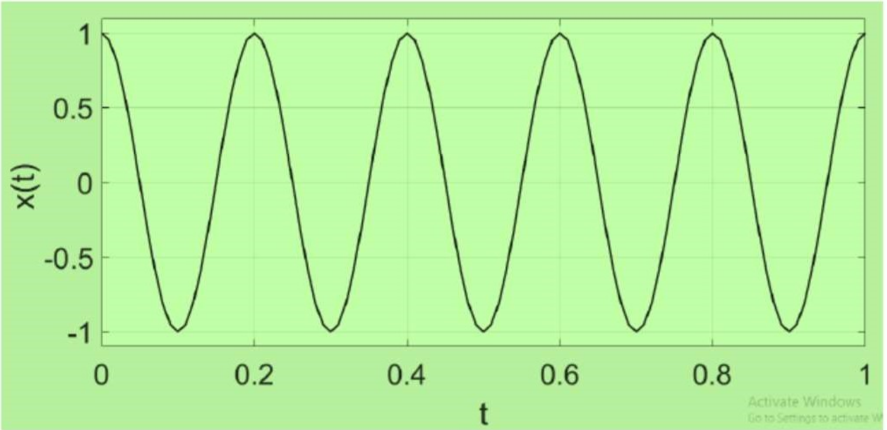
$x(t) = \cos(2\pi f_c t + 63)$

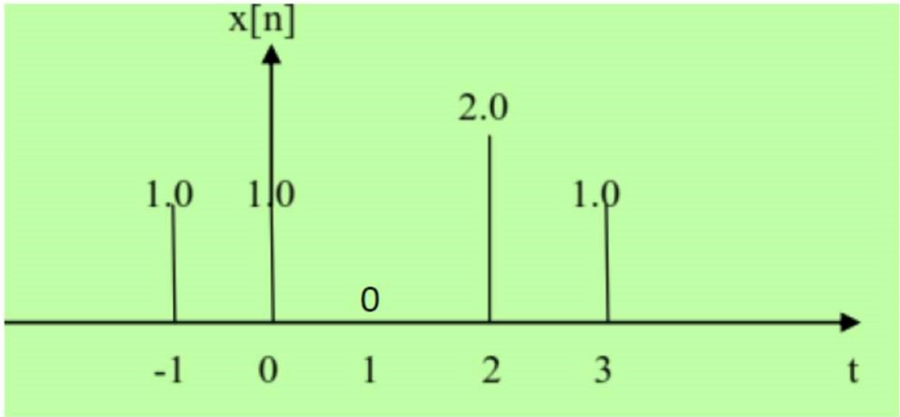




Analog and Digital Signals

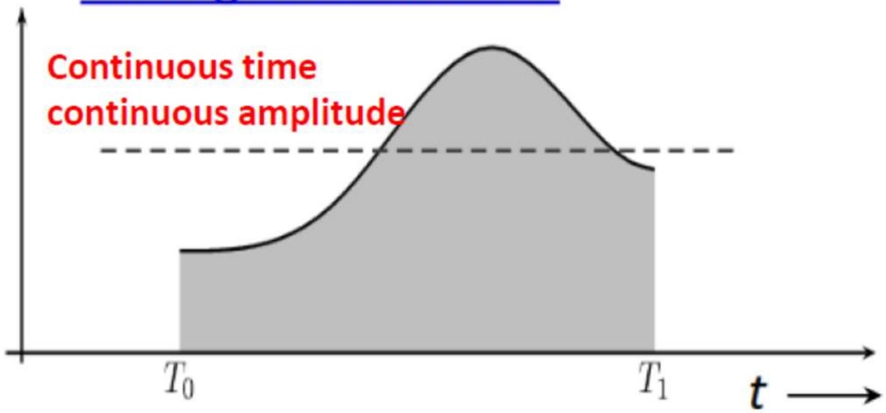
- In an **analog signal** the amplitude takes on any value within a defined range of continuous values.
- **Example**: The sinusoidal signal $x(t) = A\cos 2\pi f_0 t$, $-\infty < t < \infty$, is an example of an analog signal.
- ***A digital signal*** : The values assumed by the signal belong to a finite and countable set.
- **Example**: The sequence x[n] shown below is an examples of a digital signal. The amplitudes are drawn from the finite set {1, 0, 2}.





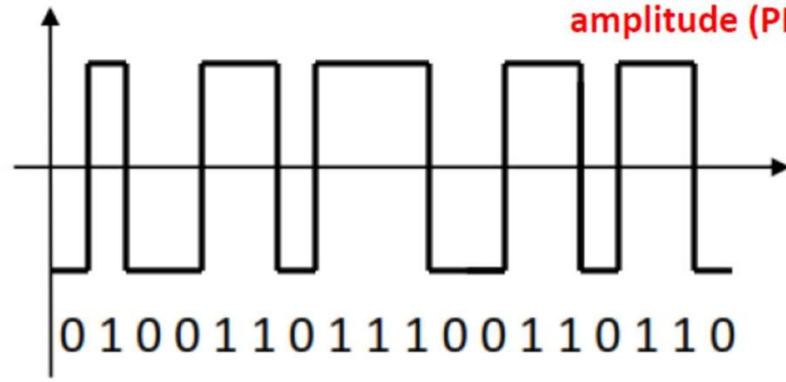
Analog and Digital Signals: Continuous Valued and Discrete Valued

Analog & continuous

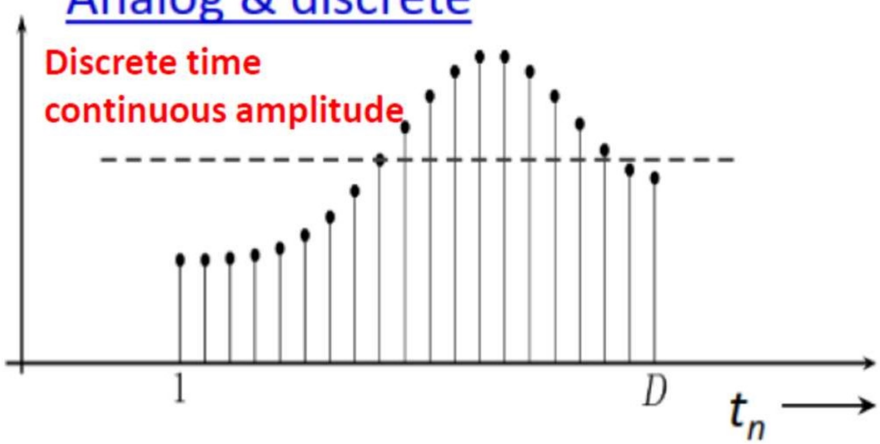


Digital & continuous

Continuous time discrete amplitude (PNRZ)

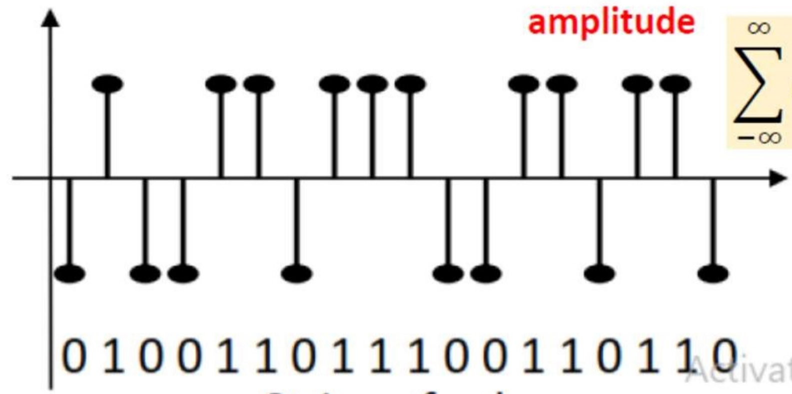


Analog & discrete



Digital & discrete

discrete time discrete amplitude



String of values

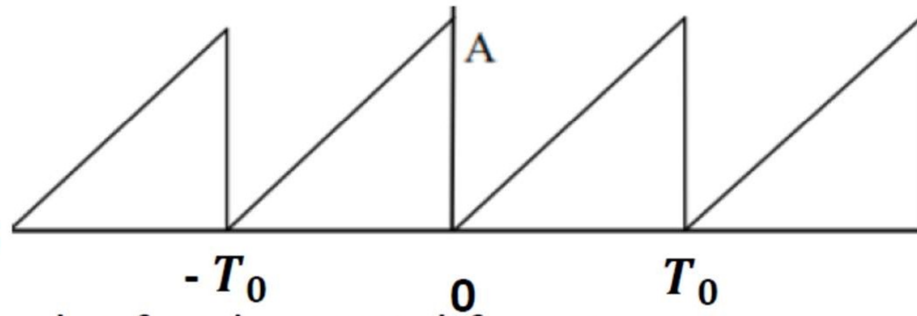
$$\sum_{-\infty}^{\infty} a_n \delta(t - kT_s)$$

Average Value of a Signal

- ***The average value of a signal*** $g(t)$ over an observation interval of 2T centered at the origin is:

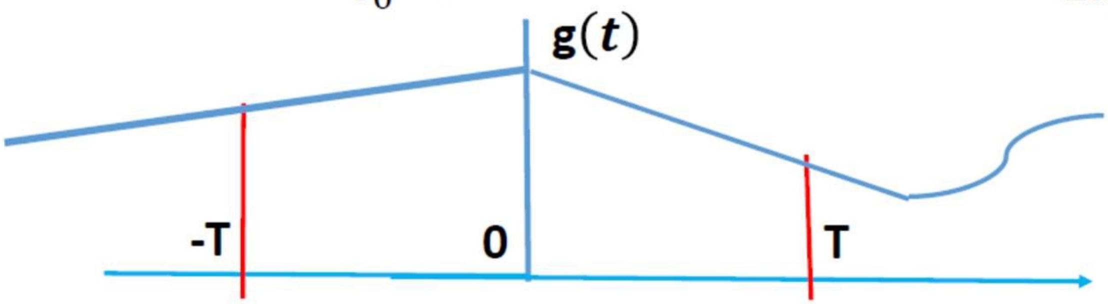
  $$g_{av} = \frac{1}{2T}\int_{-T}^{T} g(t)\, dt$$

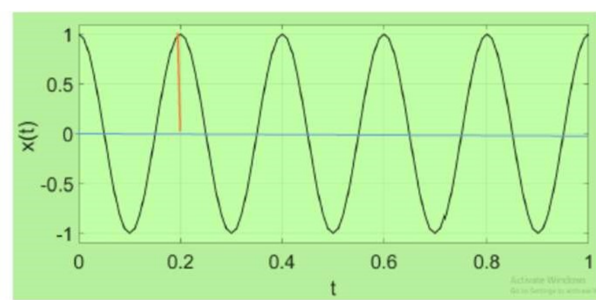
- The average value of a periodic signal $g(t)$ is



  $$\boldsymbol{g_{av} = \frac{1}{T_0}\int_{0}^{T_0} g(t)\, dt}$$; T_0 is the period; f_0 is the fundamental frequency.

- **Example**: Find the average value of the sinusoidal signal
- $x(t) = A\cos 2\pi f_0 t\ , -\infty < t < \infty$
- **Solution**: $x_{av} = \frac{1}{T_0}\int_{0}^{T_0} A\cos 2\pi f_0 t dt = -\frac{A\sin 2\pi f_0 t}{2\pi f_0}\Big|_{0}^{T_0} = 0$





Energy and Power Signals

- ***The instantaneous power*** in a signal g(t) is defined as that power dissipated in a 1-Ω resistor, i.e.,

$$p(t) = |g(t)|^2$$

- ***The average power*** over an observation interval of 2T centered at the origin is:

$$P_{av} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 \, dt$$

- The total energy of a signal $g(t)$ is

$$E = \lim_{T\to\infty} \int_{-T}^{T} |g(t)|^2 \, dt$$

- A signal g(t) is classified as **energy signal** if it has a finite energy, i.e,$0 < E < \infty$.
- A signal g(t) is classified as **power signal** if it has a finite power, i.e, $0 < P_{av} < \infty$.
- The average power in a periodic signal $g(t)$ is

$$\boldsymbol{P_{av} = \frac{1}{T_0} \int_{0}^{T_0} |g(t)|^2 \, dt}$$; T_0 is the period; f_0 is the fundamental frequency.

Energy and Power Signals

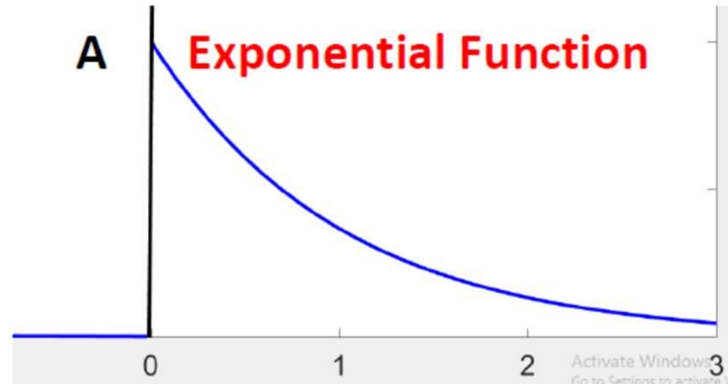
- **Example: Consider the Exponential Pulse**

$g(t) = Ae^{-\alpha t}u(t)$. **Is it an energy or a power signal?**

Solution: Let us first find the energy in the signal

$$E = \int_0^{\infty} A^2\, e^{-2\alpha t} dt = A^2 \frac{-e^{-2\alpha t}}{2\alpha} \Big|_0^{\infty} = \frac{A^2}{2\alpha}.$$

Since E is finite, then $g(t)$ is an energy signal.



- **Example: Consider the Rectangular Pulse**

$$g(t) = \begin{cases} A, & 0 < t < \tau \\ 0, & o.w \end{cases}$$ **Is it an energy or a power signal?**

- **Solution**: Let us first find the energy in the signal

$E = \int_0^{\tau} A^2\, dt = A^2\tau$. This is an energy signal since E is finite.

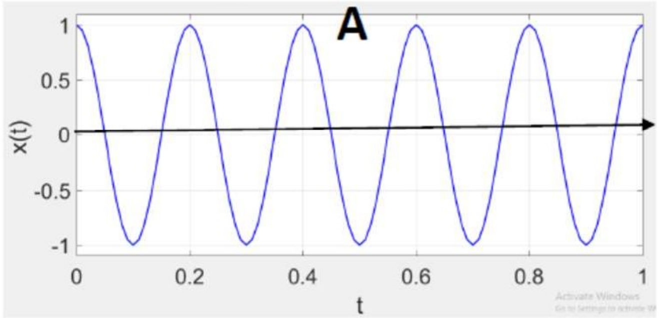
Energy and Power Signals

Example: Consider the Periodic Sinusoidal Signal

- $g(t) = A cos(2\pi f_0 t),\ -\infty < t < \infty$; **Is it an energy or a power signal**
- Since $g(t)$ is periodic, then

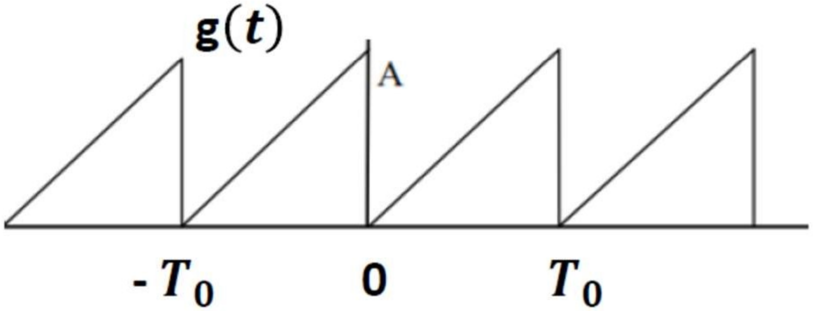
$$P_{av} = \frac{1}{T_0}\int_0^{T_0} A^2 \cos^2 \omega t\ dt = \frac{A^2}{T_0}\int_0^{T_0}(\frac{1+\cos 2\omega t}{2}) = (\frac{A^2}{T_0}) . (\frac{T_0}{2}) \Rightarrow \boldsymbol{P_{av} = \frac{A^2}{2}}.$$

- Here, P_{av} is finite. Therefore, $g(t)$ is a power signal.



Example: Consider the Periodic Saw-tooth Signal

- $g(t) = \frac{A}{T_0}t,\ 0 \le t \le T_0$. **Is it an energy or a power signal?**
- Let us evaluate the average power in $g(t)$
- $P_{av} = \frac{1}{T_0}\int_0^{T_0} \frac{A^2}{{T_0}^2} t^2 dt = \frac{1}{T_0}\frac{A^2}{{T_0}^2}\frac{t^3}{3}\Big|_0^{T_0} = \frac{A^2 {T_0}^3}{3\ {T_0}^3} = \frac{A^2}{3}.$
- Here, P_{av} is finite. Therefore, $g(t)$ is a power signal.



Energy and Power Signals

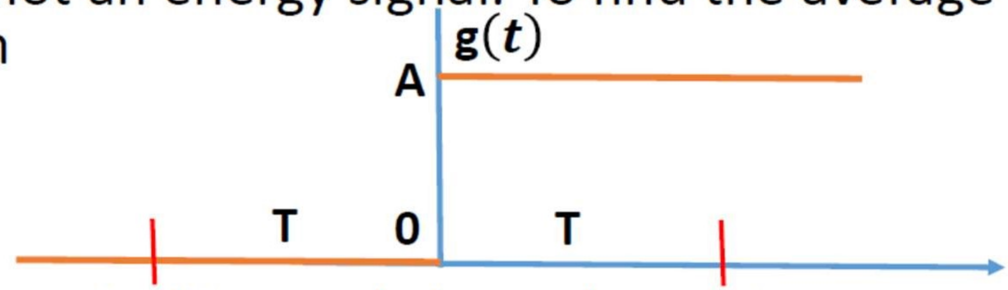
Example: The Unit Step Function $g(t) = Au(t)$

- This is a non-periodic signal. Let us first try to find its energy

$$E = \int_0^{\infty} A^2 \, dt \rightarrow \infty$$

- Sine E is not finite, then $g(t)$ is not an energy signal. To find the average power, we employ the definition

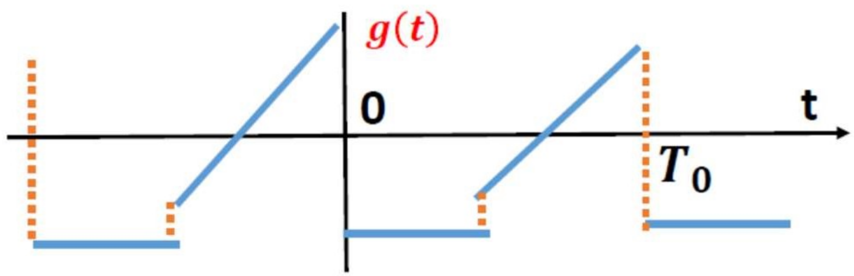
$$P_{av} = \lim_{T\rightarrow\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 \, dt,$$



- where 2T is chosen to be a symmetrical interval about the origin.
- $$P_{av} = \lim_{T\rightarrow\infty} \frac{1}{2T} \int_0^{T} A^2 \, dt = \lim_{T\rightarrow\infty} \frac{A^2 T}{2T} = \frac{A^2}{2}.$$
- So, even-though $g(t)$ is non-periodic, it turns out that it is a power signal.
- **Remark**: This is an example where the general rule (periodic signals are power signals and non-periodic signals are energy signals) fails to hold.

Fourier Series

- Let $g(t)$ be a periodic function of time with period $T_0 = \frac{1}{f_0}$ such that
  - The function g(t) is absolutely integrable over one period, i.e.,, $\int_0^{T_0} |g(t)|\,dt < \infty$
  - Any discontinuities in g(t) are finite (the amount of jump at points of discontinuity is finite).
  - g(t) has only a finite number of discontinuities and only a finite number of maxima and minima in the period
- When these conditions (called the **Dirichlet's conditions**) apply, g(t) may be expanded in a trigonometric Fourier series of the form
- $\boldsymbol{g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}$, where,
  - $a_0 = \frac{1}{T_0}\int_0^{T_0} g(t)\,dt$; (dc or average value)
  - $a_n = \frac{2}{T_0}\int_0^{T_0} g(t)\cos n\omega_0 t\,dt$
  - $b_n = \frac{2}{T_0}\int_0^{T_0} g(t)\sin n\omega_0 t\,dt$



- These conditions are sufficient (but not necessary)
- In this representation, we can associate with g(t) a FS. This does not mean equality.
- At points where g(t) is continuous, the FS converges to the function g(t)
- At a point of discontinuity t_0, the FS converges to $\frac{1}{2}\left(g(t_0 -) + g(t_0 +)\right)$

Coefficients of the Fourier Series

- $g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n\ cos\, n\omega_0 t + b_n\, sin\, n\omega_0 t)$, (1)
- Orthogonality Relations: You can easily verify the following relations:
- $\int_0^{T_0} \cos n\omega_0 t \cos m\omega_0 t = \begin{cases} \frac{T_0}{2} & , n = m \\ 0 & n \neq m \end{cases}$,
- $\int_0^{T_0} \sin n\omega_0 t \sin m\omega_0 t \ = \begin{cases} \frac{T_0}{2} & , n = m \\ 0 & n \neq m \end{cases}$
- $\int_0^{T_0} \sin n\omega_0 t \cos m\omega_0 t = 0$ for all n and m.

- To get a_0, we integrate both sides of (1) with respect to t over one period.
- $\int_0^{T_0} g(t)\, dt \ \sim \ \int_0^{T_0} a_0\, dt + \int_0^{T_0}[\sum_{n=1}^{\infty}(a_n\ cos\, n\omega_0 t + b_n\, sin\, n\omega_0 t)]\, dt$
- **Result:** $a_0 = \frac{1}{T_0}\int_0^{T_0} g(t)\, dt$; (dc or average value)

Coefficients of the Fourier Series

- $\boldsymbol{g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \ cos\, n\omega_0 t + b_n \ sin\, n\omega_0 t)}$, (1)
- To get a_n, we multiply both sides of (1) by $\cos m\omega_0 t$, integrate over one period and use the orthogonality relations.
- $\int_0^{T_0} g(t)\, \boldsymbol{cos\, m\omega_0 t}\, dt \ \sim \ \int_0^{T_0} a_0\, \boldsymbol{cos\, m\omega_0 t}\, dt$

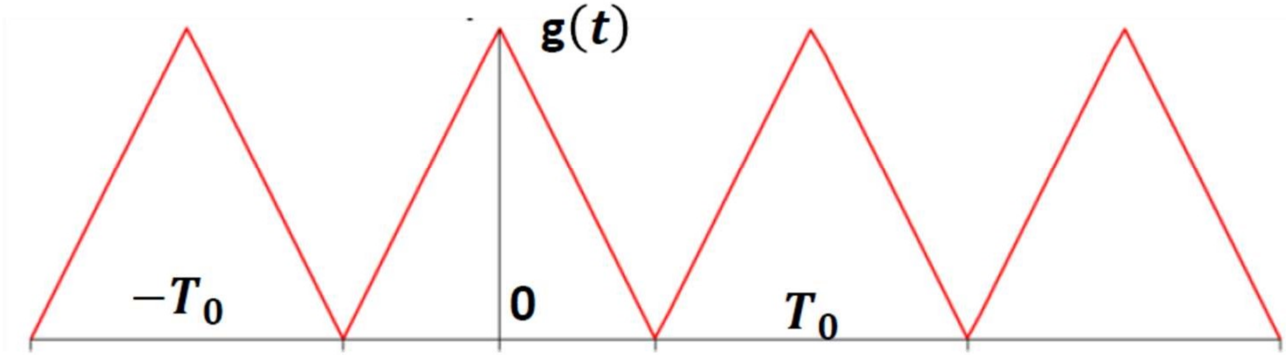
  $+ \int_0^{T_0} [\sum_{n=1}^{\infty}(a_n \ cos\, n\omega_0 t + b_n\, sin\, n\omega_0 t)\, \boldsymbol{cos\, m\omega_0 t}]\, dt$
- **Result**: $a_n = \frac{2}{T_0}\int_0^{T_0} g(t) \cos n\omega_0 t\, dt$
- To get b_n, we multiply both sides of (1) by $\sin m\omega_0 t$, integrate over one period and use the orthogonality relations.
- $\int_0^{T_0} g(t)\, \boldsymbol{sin\, m\omega_0 t}\, dt \ \sim \ \int_0^{T_0} a_0\, \boldsymbol{sin\, m\omega_0 t}\, dt$

  $+ \int_0^{T_0} [\sum_{n=1}^{\infty}(a_n \ cos\, n\omega_0 t + b_n\, sin\, n\omega_0 t)\, \boldsymbol{sin\, m\omega_0 t}]\, dt$
- **Result:** $b_n = \frac{2}{T_0}\int_0^{T_0} g(t) \sin n\omega_0 t\, dt$

Example: Existence of Fourier Series

- The Dirichlet conditions apply to the waveform given below.
- The function g(t) is absolutely integrable, i.e., $\int_0^{T_0} |g(t)|\, dt < \infty$.
- The function g(t) is continuous over the period (no discontinuities)
- Has one maximum and one minimum within one period.
- Therefore, **the FS exists**. Moreover, the **FS converges to g(t) at all points**. That is,

- $\boldsymbol{g(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}$; Note the equality sign

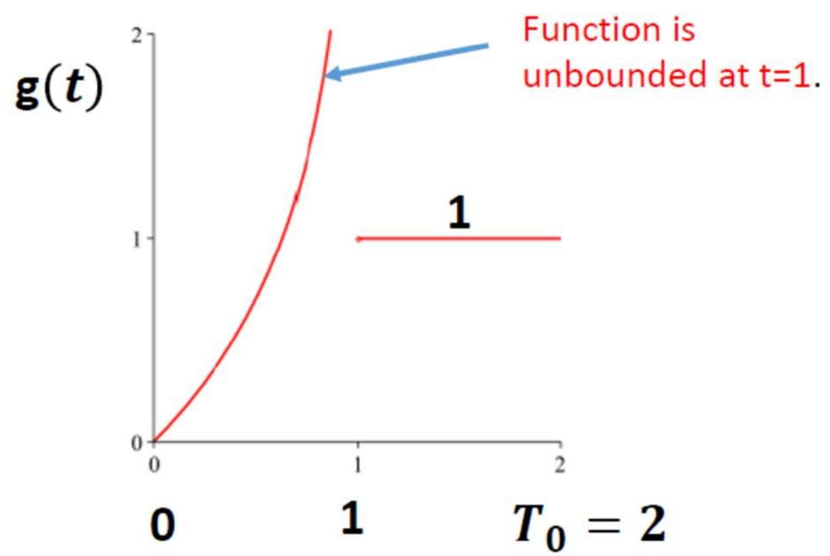


Example: Existence of Fourier Series

- Let g(t), defined over one period, be given by

$$g(t) = \begin{cases} -\ln(1-t), & 0 < t < 1 \\ 1, & 1 < t < 2 \end{cases}$$

- $\lim_{t \to 1}\big(g(t)\big) = -\ln(1-t) \to \infty$
- the function g(t) has a discontinuity. However, this discontinuity is infinite.
- **Therefore, the FS does not exists**



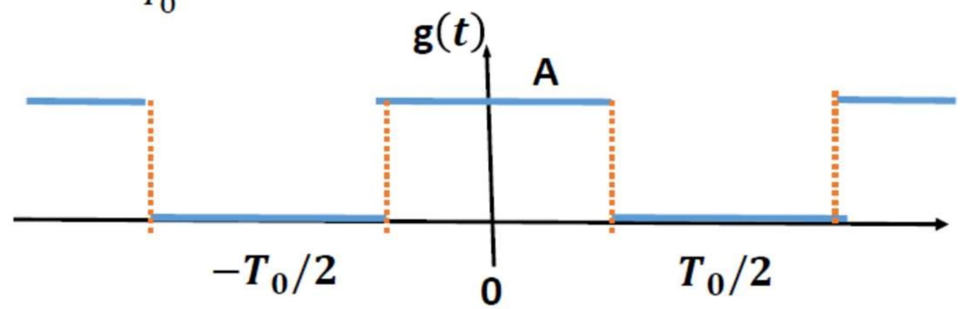
Example: Fourier Series Coefficient Evaluation

- **Example**: Find the trigonometric Fourier series of the periodic rectangular signal defined over one period T_0 as:

$$g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}$$

- **Solution**: The FS is given as $g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$
- $a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, dt = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} A\, dt = A/2$
- $b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\sin(\frac{2\pi n}{T_0}t)\, dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A\sin(\frac{2\pi n}{T_0}t)\, dt = 0$
- $a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos(\frac{2\pi n}{T_0}t)\, dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A\cos(\frac{2\pi n}{T_0}t)\, dt$
- $a_n = \begin{cases} \frac{2A}{n\pi}, & n = 1, 5, 9, \ldots \\ \frac{-2A}{n\pi}, & n = 3, 7, 11, \ldots \\ 0, & n = 2, 4, 6 \ldots \end{cases}$

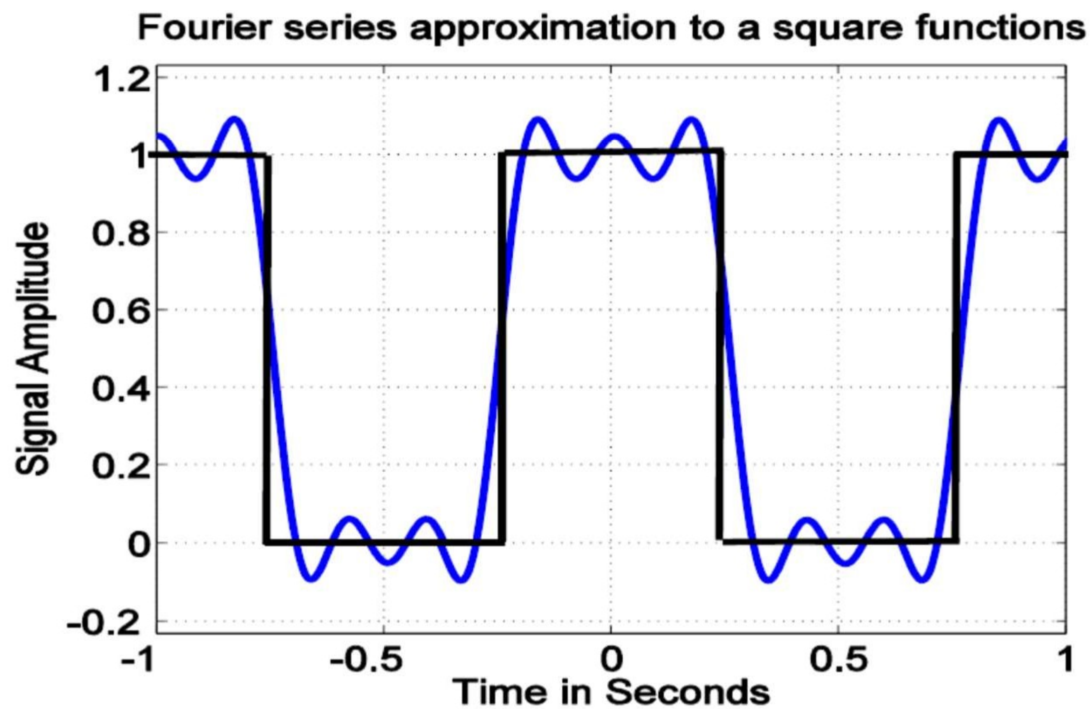
Dirichlet conditions apply. Therefore, a FS exists



Example: Convergence of Fourier Series

- The first four terms in the expansion of $g(t)$ are:
- $\tilde{g}(t) = \frac{A}{2} + \frac{2A}{\pi}\{\cos(2\pi f_0 t) - \frac{1}{3}\cos(2\pi 3 f_0 t) + \frac{1}{5}\cos(2\pi 5 f_0 t)\}$
- The function $\tilde{g}(t)$ along with $g(t)$ are plotted in the figure for $-1 \leq t \leq 1$ assuming $A = 1$ and $f_0 = 1$

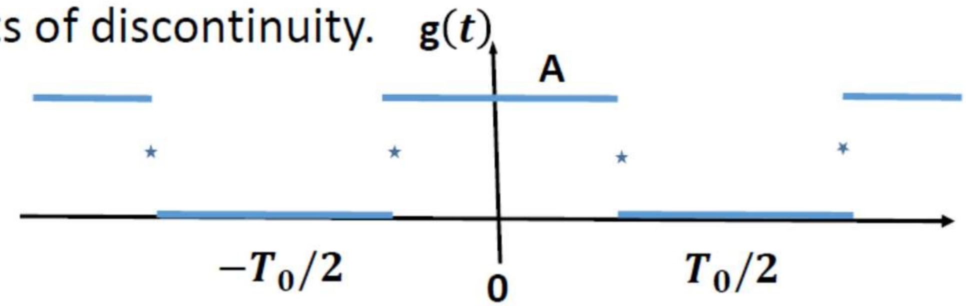
Comments: As more terms are added to $\tilde{g}(t)$, $\tilde{g}(t)$ becomes closer to $g(t)$ and in the limit as $n \to \infty$, $\tilde{g}(t)$ becomes equal to $g(t)$ at all points except at the points of discontinuity.



Convergence of the Fourier Series

The Fourier series of the signal $g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}$ is given by

- $g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$. The FS is shown in the figure below
- $a_0 = A/2$, $b_n = 0$,
- $a_n = \begin{cases} \frac{2A}{n\pi}, & n = 1, 5, 9, \ldots \\ \frac{-2A}{n\pi}, & n = 3, 7, 11, \ldots \\ 0, & n = 2, 4, 6 \ldots \end{cases}$
- The FS converges to g(t) at all points where g(t) is continuous
- Converges to A/2, the average value at points of discontinuity.



Fourier Cosine and Sine Series

- Let $g(t)$ be a periodic function of time with period $T_0 = \frac{1}{f_0}$ such that its FS exists.
- **Fourier Cosine Series**:
- Let g(t) be an even function of t, then
- $b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\sin(\frac{2\pi n}{T_0}t)\,dt = 0$
- $a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos(\frac{2\pi n}{T_0}t)\,dt = \frac{4}{T_0}\int_{0}^{T_0/2} g(t)\cos(\frac{2\pi n}{T_0}t)\,dt$
- The FS becomes a Fourier cosine series $\boldsymbol{g(t) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t}$
- **Fourier Sine Series**:
- Let g(t) be an odd function of t, then
- $a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\,dt = 0,\;\; a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos(\frac{2\pi n}{T_0}t)\,dt = 0$
- $b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\sin(\frac{2\pi n}{T_0}t)\,dt = \frac{4}{T_0}\int_{0}^{T_0/2} g(t)\sin(\frac{2\pi n}{T_0}t)\,dt$
- The FS becomes a Fourier sine series $\boldsymbol{g(t) \sim \sum_{n=1}^{\infty} b_n \sin n\omega_0 t}$

Complex Form of the Fourier Series

The Fourier series can also be expressed in the complex form:

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where, $C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t} dt.$

- Note that C_n is a complex valued quantity, which can be written as
- $C_n = |C_n| e^{j\theta n}$
- The plot of $|C_n|$ versus frequency is called the ***Discrete Amplitude Spectrum***.
- The plot of θ_n versus frequency is called the ***Discrete Phase Spectrum***.
- The term at f_0 is referred to as the fundamental frequency. The term at $2f_0$ is referred to as the second order harmonic, the term at $3f_0$ is referred to as the third order harmonic and so on.

Parseval's Power Theorem

- The average power of a periodic signal g(t) is given by:

$$P_{av} = \frac{1}{T_0}\int_0^{T_0}|g(t)|^2\,dt = \sum_{n=-\infty}^{\infty}|C_n|^2 = |C_0|^2 + 2\sum_{n=1}^{\infty}|C_n|^2$$

- $= |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2$
- **Proof:** $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$, where, $C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t}dt$.
- $|g(t)|^2 = g(t)g^*(t) = \left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right)\left(\sum_{m=-\infty}^{\infty} C_m^* e^{-jm\omega_0 t}\right)$
- $\frac{1}{T_0}\int_0^{T_0}|g(t)|^2 dt = \frac{1}{T_0}\int_0^{T_0}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} C_n\, C_m^*\, e^{j(n-m)\omega_0 t}dt$
- Orthogonality: $\int_0^{T_0} e^{j(n-m)\omega_0 t}dt = \begin{cases} T_0 & , n = m \\ 0 & n \neq m \end{cases}$
- $\frac{1}{T_0}\int_0^{T_0}|g(t)|^2 dt = \sum_{n=-\infty}^{\infty}|C_n|^2 = |C_0|^2 + 2\sum_{n=1}^{\infty}|C_n|^2$

Power Spectral Density

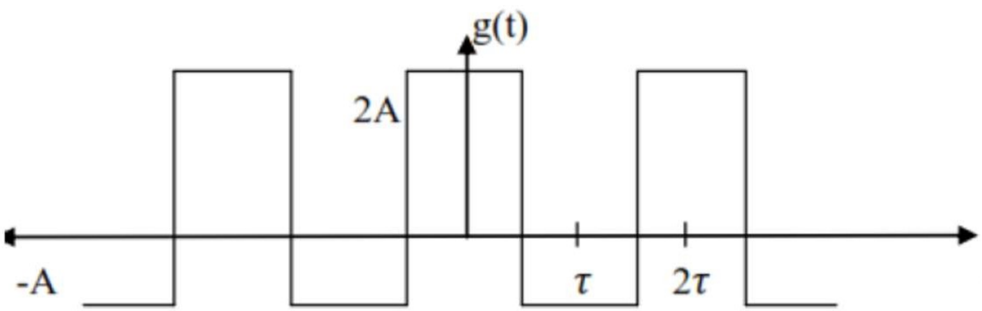
- The plot of $|\mathrm{C_n}|^2$ versus frequency is called the ***power spectral density*** (PSD).
- It displays the power content of each frequency (spectral) component of a signal.
- For a periodic signal, the PSD consists of discrete terms at multiples of the fundamental frequency.
- The next example demonstrate these properties

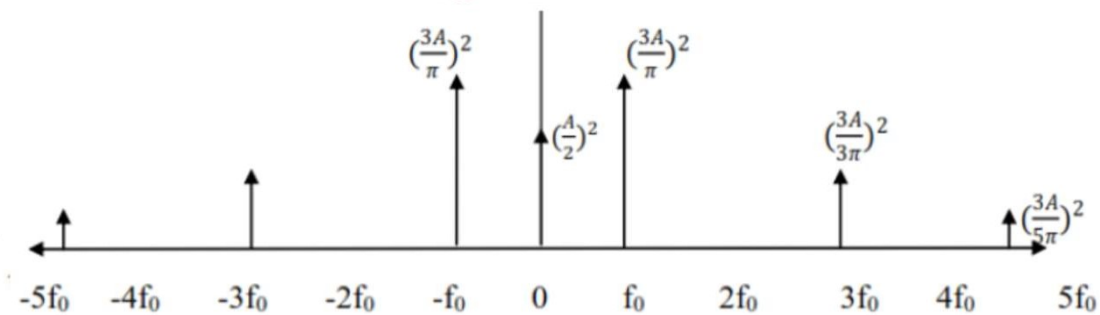
Power Spectral Density

- **Example:** Find the power spectral density of the $g(t)$ shown in the figure.
- Here, we need to find the complex Fourier series expansion, where the period $T_0 = 2\tau$
- $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t};$ $\quad C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t} dt$
- $C_n = \begin{cases} \frac{A}{2}, & n = 0 \\ \frac{3A}{|n|\pi}, & n = \pm1, \pm5, \pm9, \ldots \\ \frac{-3A}{|n|\pi}, & n = \pm3, \pm7, \pm11, \ldots \\ 0, & n = \pm2, \pm4, \ldots \end{cases}$

$$|C_n|^2 = \begin{cases} (\frac{A}{2})^2, & n = 0 \\ (\frac{3A}{n\pi})^2, & n: odd \\ 0, & n: even \end{cases}$$

$$S_g(f) = \sum_{n=-\infty}^{\infty} |C_n|^2\, \delta(f - nf_0)$$





Fourier Transform

- Let $g(t)$ be a function of time t. The Fourier transform maps the function g(t) into another function G(f) defined into the frequency domain. The Fourier transform is defined as:

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft}\, dt$$

- The inverse Fourier transform is defined as

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft}\, df$$

- Conditions for existence (Dirichlet conditions, which are the same as those for the FS)
  - The function g(t) is absolutely integrable, i.e.,, $\int_{0}^{T_0} |g(t)|\, dt < \infty$.
  - Any discontinuities in g(t) are finite
  - g(t) has only a finite number of discontinuities and only a finite number of maxima and minima in any finite interval.
- **Remarks**:
  - These conditions are sufficient but not necessary
  - A **weaker sufficient condition** for existence is $\int_{-\infty}^{\infty} |g(t)|^2\, dt < \infty$ (g(t) is an energy signal). This is the finite-energy condition that is satisfied by all physically realizable waveforms.
  - Generally, physical waveforms encountered in engineering practice are Fourier transformable.
  - The Fourier transform can be derived from the Fourier series by allowing the period T_0 to go to infinity, but this will not be covered in this presentation.

Fourier Transform: Amplitude and Phase Spectrum

Observations: $\boldsymbol{G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\ dt}$

- G(f) is a complex function of frequency f, which can be expressed as:

  $G(f) = |G(f)|j^{\theta(f)}$

- The function $G(f)$ is often referred to as the spectrum of $g(t)$.
  - $|G(f)|$: is the ***continuous amplitude spectrum*** of $g(t)$, (an even function of f).
  - $\theta(f)$: is the ***continuous phase spectrum*** of $g(t)$, (an odd function of f).

- **Notation**:
  - To denote that $G(f)$ is the Fourier transform of $g(t)$, we write $G(f) = \Im(g(t))$
  - To denote that $g(t)$, *is the inverse Fourier transform of* $G(f)$, we write $g(t) = \Im^{-1}(G(f))$
  - Sometimes, the following notation is used for a Fourier transform pair g$(t) \leftrightarrow G(f)$.

Rayleigh Energy Theorem

Rayleigh Energy Theorem: The energy in a signal g(t) is given by:

$$E = \int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

- The proof of this result is the same as that for Parseval's power theorem
- The function $|G(f)|^2$ is called the ***energy spectral density***. It depicts the range of frequencies over which the signal energy extends and the frequency bands which are significant in terms of their energy contents.
- For a non-period signal energy signal, the **energy spectral density is a continuous function of f.**

A General Form of the Rayleigh Energy Theorem

- For two energy functions $g(t)$ and $v(t)$, the following result holds:

$$\int_{-\infty}^{\infty} g(t) v(t)^* \, dt = \int_{-\infty}^{\infty} G(f) V(f)^* \, df$$

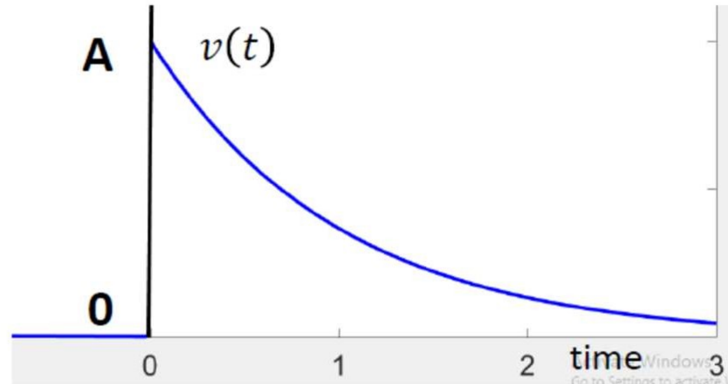
Example: Exponential Pulse

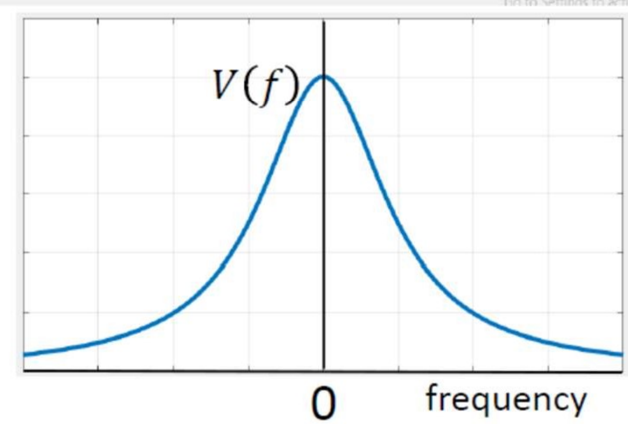
- $v(t) = \begin{cases} A\,e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$
- $\boldsymbol{E = \int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{0}^{\infty} A^2\, e^{-2bt} dt = (A^2 / 2b)}$ **, F.T exists**
- $V(f) = \int_0^{\infty} v(t) e^{-j2\pi f t}\ dt = \int_0^{\infty} A e^{-bt}\, e^{-j2\pi f t} dt$

  $V(f) = A \int_0^{\infty} e^{-(b+j2\pi f)t}\ dt = A \frac{e^{-(b+j2\pi f)t}}{-(b+j2\pi f)} \Big|_0^{\infty} = \frac{A}{b+j2\pi f}.$
- $V(f) = \frac{A}{b+j2\pi f},$
- $|V(f)| = \frac{A}{(b^2+(2\pi f)^2)^{1/2}}$
- The energy spectral density is: $S_v(f) = |V(f)|^2 = \frac{A^2}{b^2+(2\pi f)^2}$

Exercise: For the given v(t), verify Rayleigh Energy Theorem:

$$\boldsymbol{E = \int_{-\infty}^{\infty} |v(t)|^2\, dt = \int_{-\infty}^{\infty} |V(f)|^2 df}$$

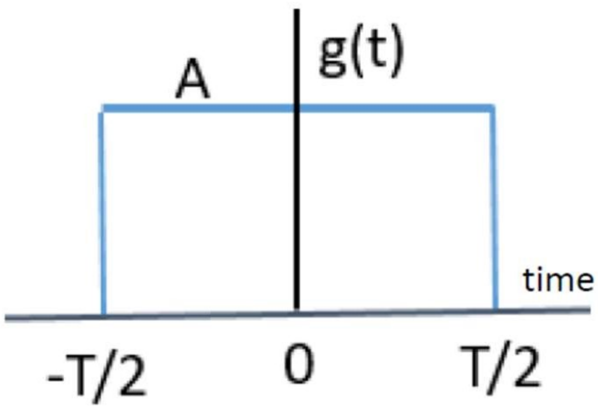


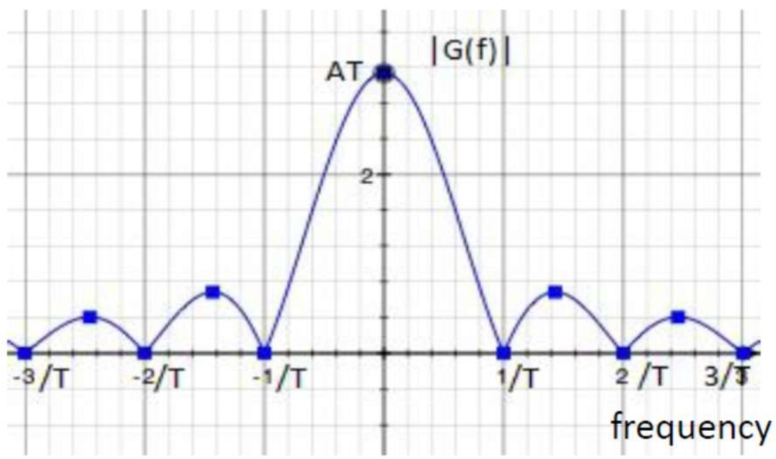


- **Remark**: The signal $v(t)$ is called a ***baseband signal*** since the signal occupies the low frequency part of the spectrum. That is, the energy in the signal is found around the zero frequency. When the signal is multiplied by a high frequency carrier, the spectrum becomes centered around the carrier and the modulated signal is called a ***bandpass signal***.

Example: The Rectangular Pulse $g(t) = Arect(\frac{t}{T})$

- $G(f) = \int_{-T/2}^{T/2} Ae^{-j2\pi ft}dt = \frac{A}{\pi f}\sin\pi fT \qquad , \quad AT\frac{\sin\pi fT}{\pi fT} \triangleq AT\, sincfT$
- $|\boldsymbol{G(f)}| = \boldsymbol{AT}\ |\boldsymbol{sincfT}|$
- The maximum of $|G(f)|$ occurs at $f = 0$ since $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Also, $G(f) = 0$ when $sin(\pi fT) = 0$, which occurs at the points that satisfy $\pi fT = n\pi, \Rightarrow fT = n, or\ f = \frac{n}{T}, n = \pm 1, \pm 2, \pm 3, \ldots$





Properties of the Fourier Transform

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} \, dt$$

- **Linearity (superposition)**

  Let $g_1(t) \leftrightarrow G_1(f)$ and $g_2(t) \leftrightarrow G_2(f)$, then

  $c_1 g_1(t) + c_2 g_2(t) \leftrightarrow c_1 G_1(f) + c_2 G_2(f)$; c_1, c_2 are constants

- **Time Scaling**

$g(t) \leftrightarrow G(f)$	$g(at) \leftrightarrow \dfrac{1}{\lvert a \rvert} G(f/a)$

- **Duality**

$g(t) \leftrightarrow G(f)$	$G(t) \leftrightarrow g(-f)$

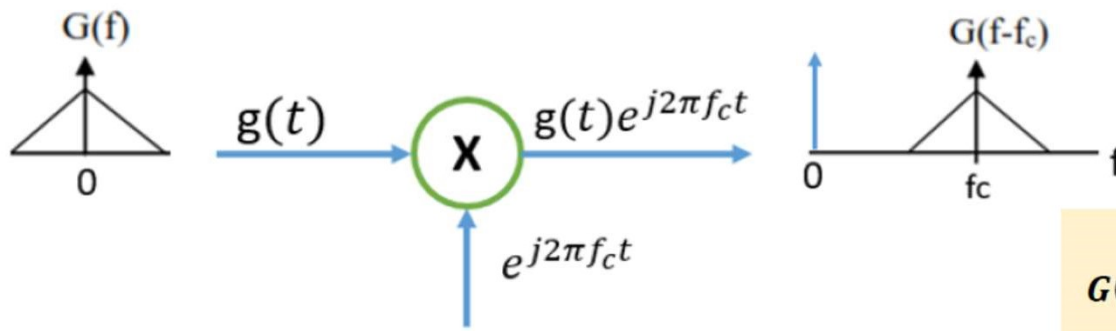
- **Time Shifting**

$g(t) \leftrightarrow G(f)$	$g(t - t_0) \leftrightarrow G(f) e^{-j2\pi f t_0}$
Delay in time domain corresponds to a phase shift in frequency domain	

Properties of the Fourier Transform

- **Frequency Shifting**

$g(t) \leftrightarrow G(f)$	$g(t)e^{j2\pi f_c t} \leftrightarrow G(f - f_c)$; f_c is constant

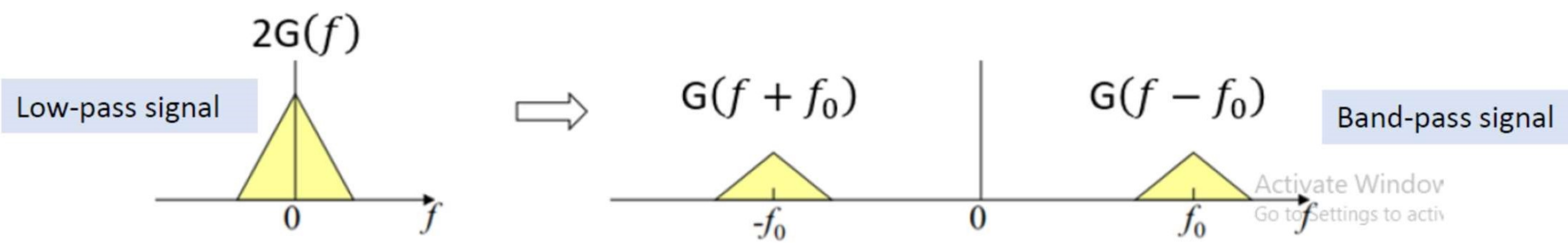


$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}\, dt$$

Frequency Shifting Property of the Fourier Transfer

- **Modulation Property**

$g(t) \leftrightarrow G(f)$	$2g(t)\cos(2\pi f_0 t) \leftrightarrow G(f - f_0) + G(f + f_0)$; f_0 is a constant



Properties of the Fourier Transform

- **Area under $G(f)$**

$g(t) \leftrightarrow G(f)$	$g(t=0)= \int_{-\infty}^{\infty} G(f)df$
The value $g(t=0)$ is equal to the area under its Fourier transform function	

- **Area under $g(t)$**

$g(t) \leftrightarrow G(f)$	$G(0) = \int_{-\infty}^{\infty} g(t)dt$
The area under a function g(t) is equal to the value of its Fourier transform G(f) at f = 0, where $G(0)$ implies the presence of a dc component.	

$$\boldsymbol{g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\, df}$$

$$\boldsymbol{G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}}$$

Properties of the Fourier Transform

- **Differentiation in the Time Domain**

If $g(t)$ and its derivative $g'(t)$ are Fourier transformable, then,

$g'(t) \leftrightarrow (j2\pi f)G(f)$

i.e., differentiation in the time domain $\Longrightarrow$ multiplication by j$2\pi f$ in the frequency domain.

(Differentiation in the time domain enhances high frequency components of a signal)

Also, $\frac{d^n g(t)}{dt^n} \leftrightarrow (j2\pi f)^n G(f)$

$$\boldsymbol{g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\, df}$$

- **Integration in the Time Domain**

$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f} G(f)$; assuming $G(0) = 0$

i.e., integration in the time domain corresponds to division by (j$2\pi f$) in the frequency domain. This amounts to low pass filtering, where high frequency components are attenuated due to filtering.

When $G(0) \neq 0$, the above result becomes:

$$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f} G(f) + \frac{1}{2}\, G(0)\delta(f).$$

Properties of the Fourier Transform

- **Multiplication of two signals in the time domain**

$$g_1(t)\, g_2(t) \leftrightarrow \int_{-\infty}^{\infty} G_1(\lambda)\, G_2(f-\lambda) d\lambda = G_1(f) * G_2(f)$$

Multiplication of two signals in the time domain is transformed into the convolution of their Fourier transforms in the frequency domain.

- **Convolution of two signals in the time domain**

$$g_1(t) * \; g_2(t) \leftrightarrow G_1(f) G_2(f)$$

Convolution of two signals in the time domain is transformed into a multiplication of their Fourier transforms in the frequency domain

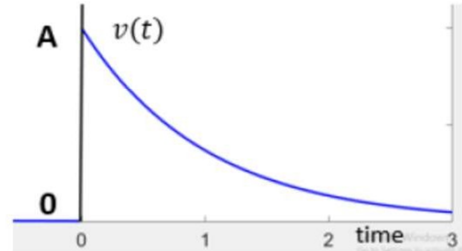
- **Multiplication by t in the time domain corresponds to differentiation in the** frequency domain

$$\Im\{\mathbf{tg}(t)\} = \frac{j}{2\pi}\frac{dG(f)}{df}$$

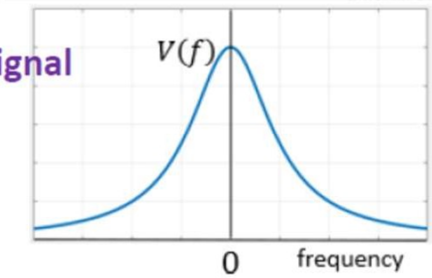
$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}$$

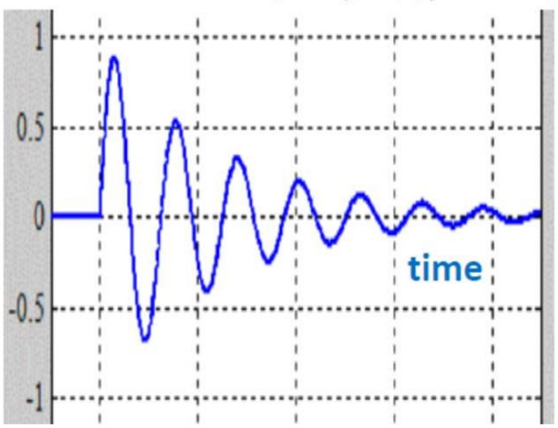
Examples: The RF Negative Exponential Pulse

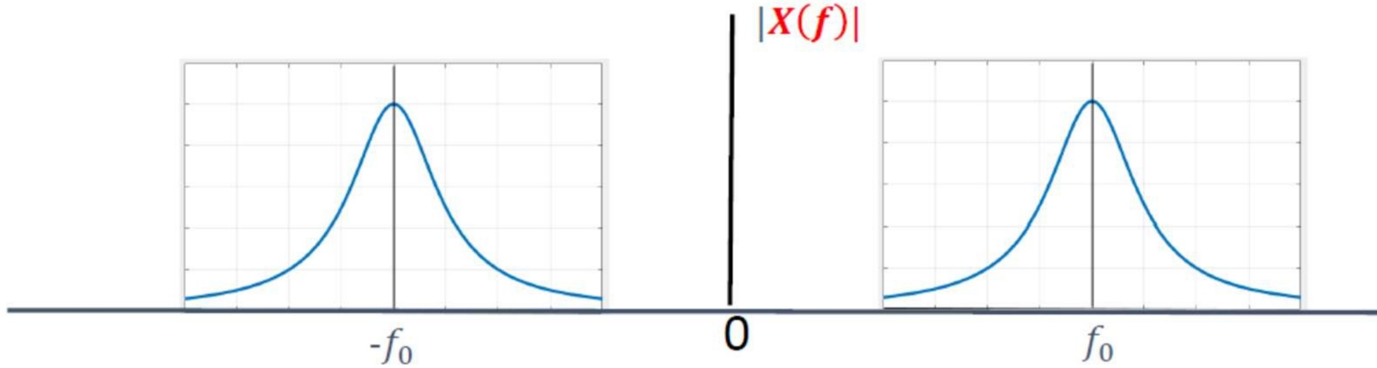
- **Example**: Find the Fourier transform of $x(t) = A\,e^{-bt}cos(2\pi f_0 t),$ t > 0
- **Solution**: Note that x(t) can be expressed as
- $\mathbf{x(t)} = \mathbf{g}(t)cos(2\pi f_0 t),\ g(t) = \begin{cases} A\,e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$
- $G(f) = \left(\frac{A}{b+j2\pi f}\right)$
- Use the modulation property
- $X(f) = \frac{1}{2}\{G(f - f_0) + G(f + f_0)\}$
- $X(f) = \frac{1}{2}\left\{\frac{A}{b+j2\pi(f-f_0)} + \frac{A}{b+j2\pi(f+f_0)}\right\};$ **Band-pass signal**



Baseband signal







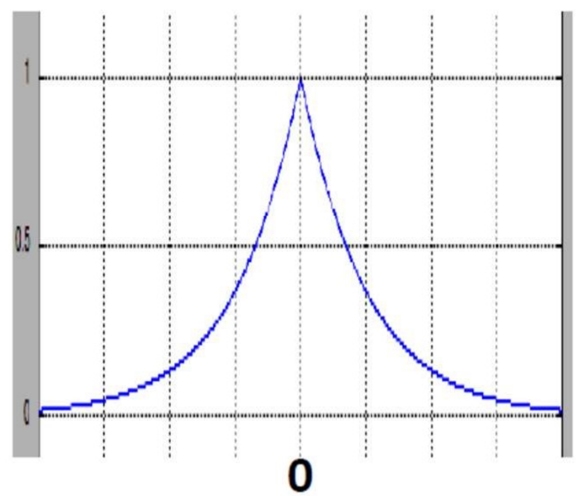
Example: double-sided exponential pulse

- **Example**: Find the Fourier transform of the double-sided exponential pulse

$$\mathbf{g}(t) = Ae^{-b|t|}, -\infty < t < \infty$$

- **Solution**: You can easily find that the energy in g(t) is finite, and hence the F.T. exists.

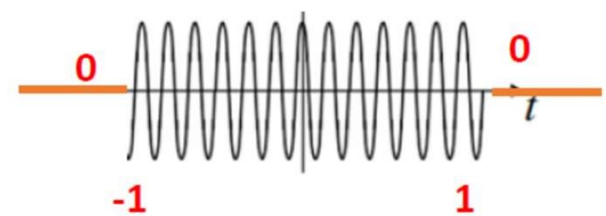
- $\mathrm{G}(f) = \int_{-\infty}^{0} Ae^{bt} e^{-j2\pi f t}\, dt + \int_{0}^{\infty} Ae^{-bt}\, e^{-j2\pi f t} dt$
- $\mathrm{G}(f) = \frac{A}{b-j2\pi f} + \frac{A}{b+j2\pi f} = \frac{2bA}{b^2+(2\pi f)^2}$



Examples: Fourier Transform of an RF Pulse

Find the Fourier transform of the RF pulse $x(t) = \cos(2\pi f_0 t)\,; 1 \le t \le 1$,

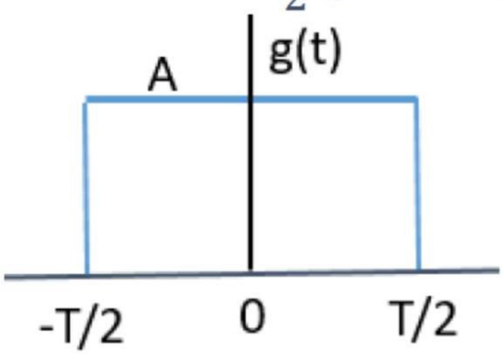
Solution: x(t) can be viewed as a product of the rectangular pulse and the cosine function $x(t) = \mathrm{g(t)}\cos(2\pi f_0 t)$, where

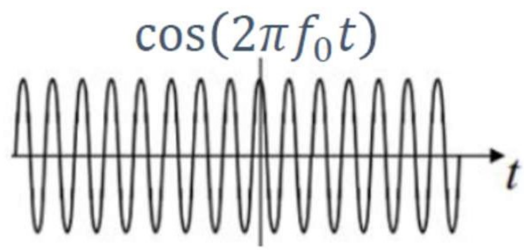


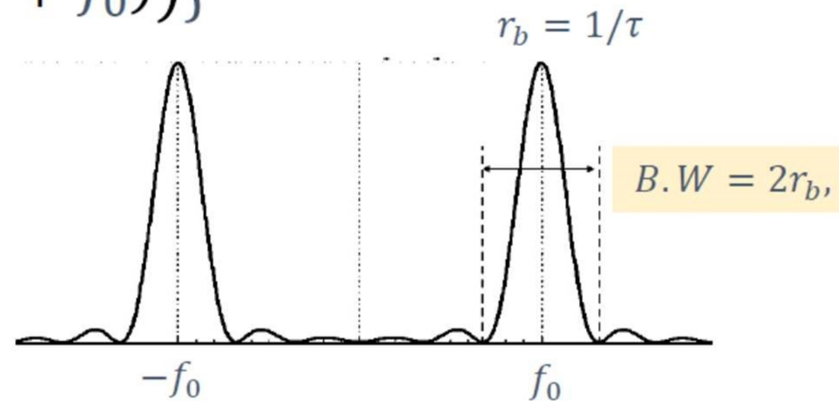
$$g(t) = u(t+1) - u(t+1) = rect(\frac{t}{2})$$

$$\mathrm{G}(f) = AT\ sincfT = 2sinc(2f)$$

- $X(f) = \frac{1}{2}\{X(f - f_0) + X(f + f_0)\}$
- $X(f) = \frac{1}{2}\{2sinc\big(2(f - f_0)\big) + 2sinc\big(2(f + f_0)\big)\}$







Examples: Fourier Transform of the doublet pulse

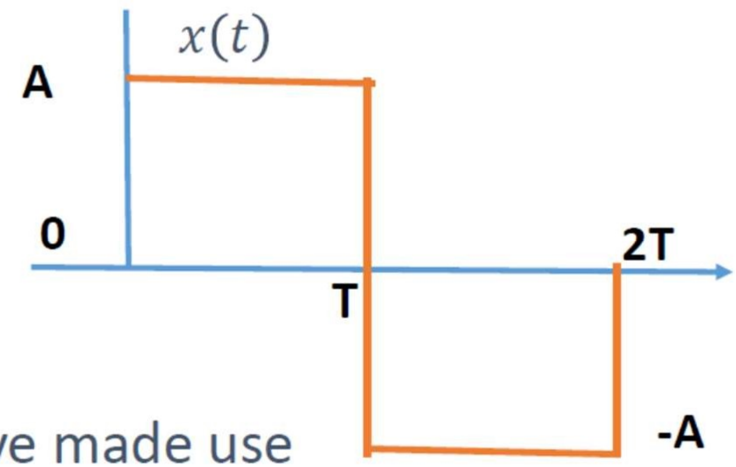
Find the Fourier transform of the pulse $x(t)$ shown in the figure

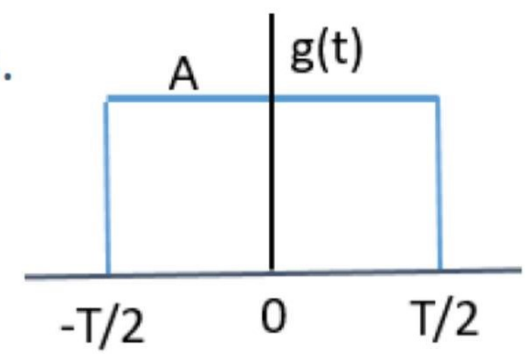
Solution: x(t) can be expressed in terms of the rectangular pulse g(t) as

$$x(t) = g(t - T/2) - g(t - 3T/2)$$

$$X(f) = G(f)e^{-\frac{j2\pi fT}{2}} - G(f)e^{-\frac{j2\pi f3T}{2}}$$

- $X(f) = G(f)e^{-j2\pi fT}(e^{\frac{j2\pi fT}{2}} - e^{-\frac{j2\pi fT}{2}})$
- $X(f) = G(f)e^{-j2\pi fT}(j2)sin(\frac{2\pi fT}{2})$
- Remark: Note that in this example, we have made use of the linearity and time shifting properties.





Examples: Fourier Transform of the triangular pulse

Find the Fourier transform of the pulse $y(t)$ shown in the figure

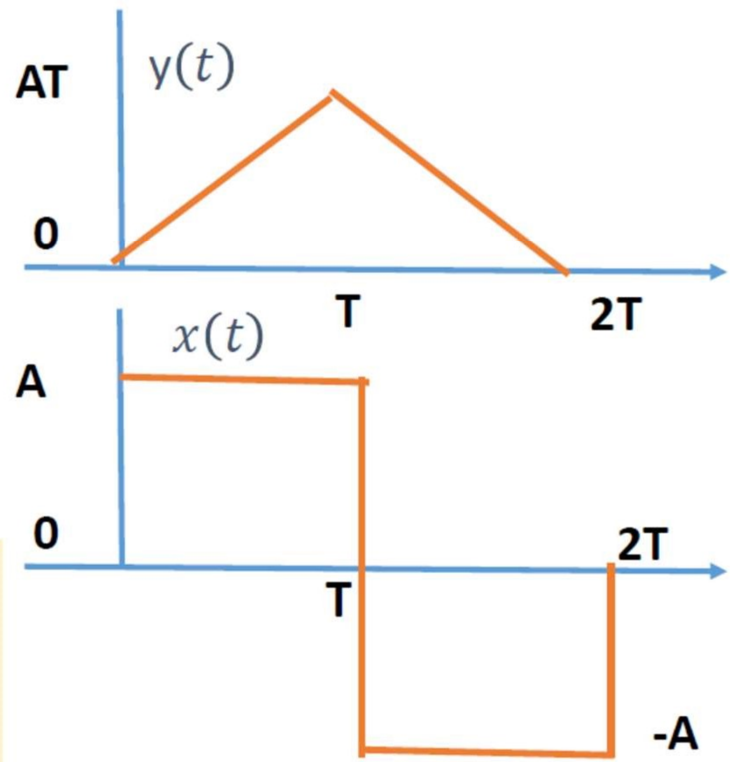
Solution: If we differentiate y(t), we get x(t) of the previous example

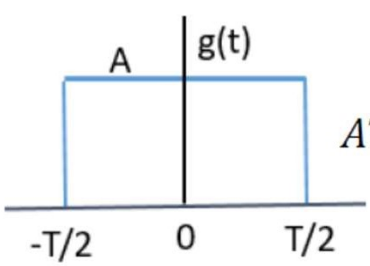
$$\frac{dy(t)}{dt} = x(t).\ \ \text{Taking the F.T of both sides,}$$

$$j2\pi f Y(f) = X(f)$$

$$Y(f) = \frac{X(f)}{j2\pi f} = \frac{G(f)e^{-j2\pi fT}(j2)sin(\frac{2\pi fT}{2})}{j2\pi f}$$

$$G(f) = \frac{TG(f)e^{-j2\pi fT}sin(2\pi fT)}{\pi fT} = AT^2(sincfT)^2e^{-j2\pi fT}$$





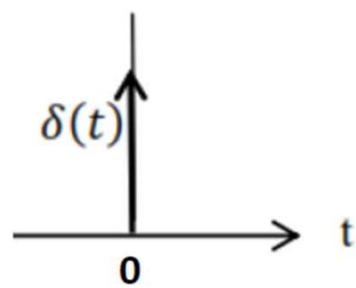
$$AT\frac{\sin \pi fT}{\pi fT} \triangleq AT\ sincfT$$

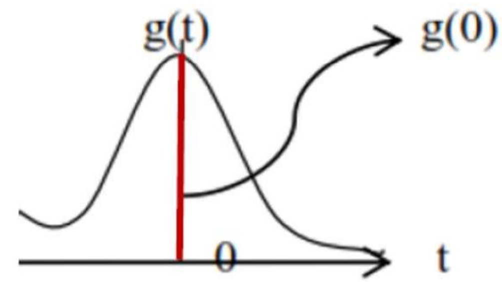
Same result can be obtained by realizing that y(t)=g(t)*g(t) and using the convolution property Y(f)=G(f).G(f) and then using the time shifting property

- The Fourier transform maps the function g(t) into another function G(f) defined into the frequency domain. In this lecture, we define the Fourier transform, present its properties and solve many examples illustrating the use of these properties. We also present Rayleigh energy theorem.

Fourier Transform of Power Signals

- For a **non-periodic (energy) signal** g(t), the Fourier transform exists when
- $E = \int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$ (sufficient condition for existence)
- so that $G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$ **exists**
- **For power signals**, the integral $\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$ **does not exist**.
- However, one can still finds the Fourier transform of power signals by employing the delta function. This function is defined next.
- **Dirac – Delta Function (Impulse Function)**





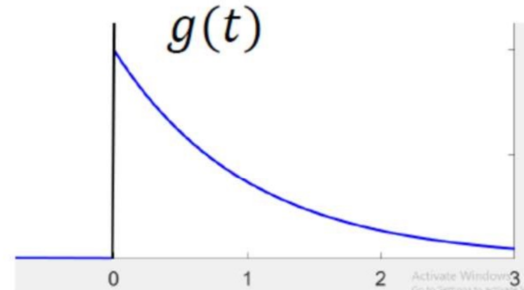
This function is defined as

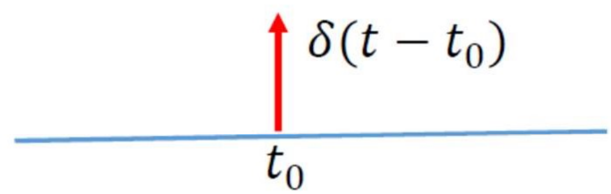
$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

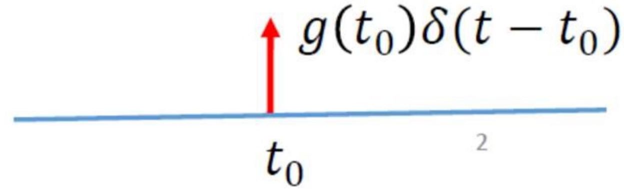
- such that: $\int_{-\infty}^{\infty} \delta(t)dt = 1$ and $\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0)$
- Here, g(t) is a continuous function of time. The second property, known as the sifting property, shows that the delta function samples the function g(t) at the time of its occurrence.

Some Properties of the Delta Function

- $g(t)\delta(t-t_0) = g(t_0)\delta(t-t_0)$; (Multiplication)
- $\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$; (Sifting or sampling property)
- $\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$
- $\delta(t) * g(t) = g(t)$
- $\delta(t) = \frac{du(t)}{dt} \quad \Rightarrow u(t) = \int_{-\infty}^{t} \delta(t)dt$
- $\delta(t) = \delta(-t)$; an even function of its argument.
- Fourier transform: $\Im\{\delta(t)\} = 1$
- $\Im\{\delta(t-t_0)\} = e^{-j2\pi f t_0}$







Applications of the Delta Function

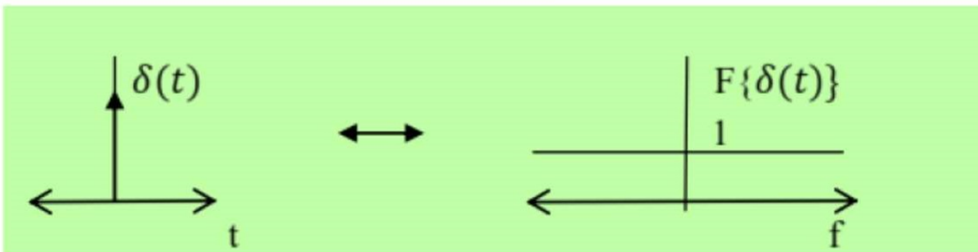
- **Fourier transform of the delta function**
- $\Im\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}dt = 1$. This follows from the sifting property

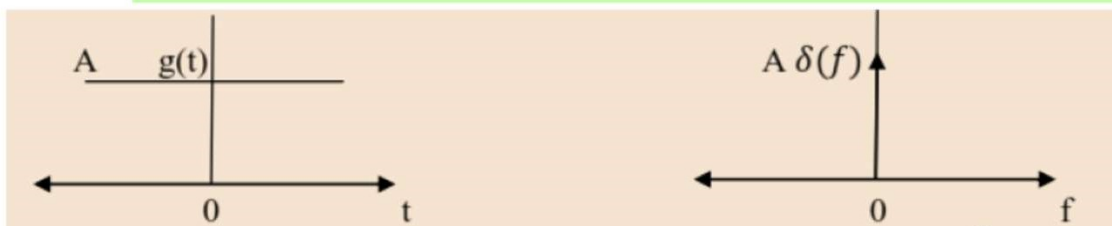
  $\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0) = 1$
- $\Im\{\delta(t-t_0)\} = e^{-j2\pi ft_0}$; (using the time delay property $\Im\{g(t-t_0)\} = G(f)e^{-j2\pi ft_0}$
- **DC or a Constant Signal**
- Since $\Im\{\delta(t)\} = 1$, then by the duality property $\Im\{1\} = \delta(f)$

$g(t) \leftrightarrow G(f)$

$G(t) \leftrightarrow g(-f)$

- Note how the time-bandwidth relationship holds for this pair. A narrow pulse in time extends over a large frequency spectrum).
- Also, the transform of a dc signal is an impulse at f =0.





Applications of the Delta Function

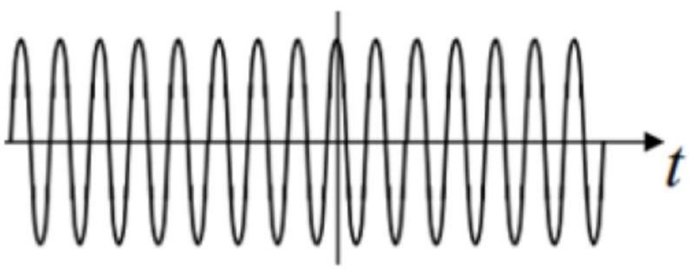
- **Complex Exponential Function**
- $\Im\left\{Ae^{j2\pi f_c t}\right\} = A\delta(f - f_c)$;
- follows from the duality property, since $\Im\{\delta(t - t_0)\} = e^{-j2\pi f t_0}$

$\boldsymbol{g(t) \leftrightarrow G(f)}$

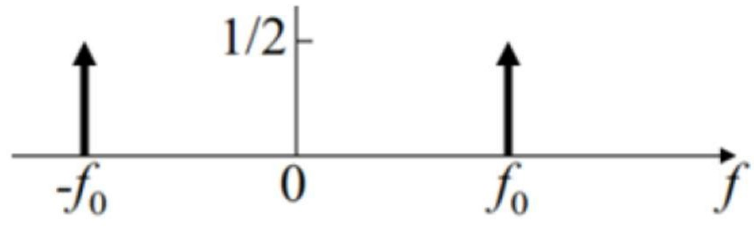
$\boldsymbol{G(t) \leftrightarrow g(-f)}$

- **Sinusoidal Functions**
- $\Im\{\cos 2\pi f_0 t\} = \Im\frac{1}{2}\left\{Ae^{j2\pi f_c t} + Ae^{-j2\pi f_c t}\right\} = \frac{1}{2}\{\delta(f - f_0) + \delta(f + f_0)\}$
- $\Im\{\sin 2\pi f_0 t\} = \Im\frac{1}{j2}\left\{Ae^{j2\pi f_c t} - Ae^{-j2\pi f_c t}\right\} = \frac{1}{2j}\{\delta(f - f_0) - \delta(f + f_0)\}$

$g(t) = \cos(2\pi f_0 t)$



$\mathrm{G}(f)$



Applications of the Delta Function

- **Signum Function**
- $$sgn(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$ $$\Im\{sgn(t)\} = \frac{1}{j\pi f}$$

$$v(t) = \begin{cases} e^{-bt} & t > 0 \\ -e^{bt} & t < 0 \end{cases}$$

$$G(f) = \frac{1}{b+j2\pi f} - \frac{1}{b-j2\pi f} = \frac{-j(2)2\pi f}{b^2+(2\pi f)^2}$$

$$\log_{b\to 0} G(f) = \frac{1}{j\pi f}$$

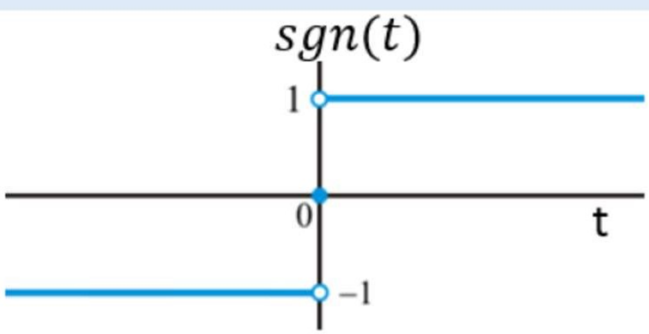
- **Unit Step Function**

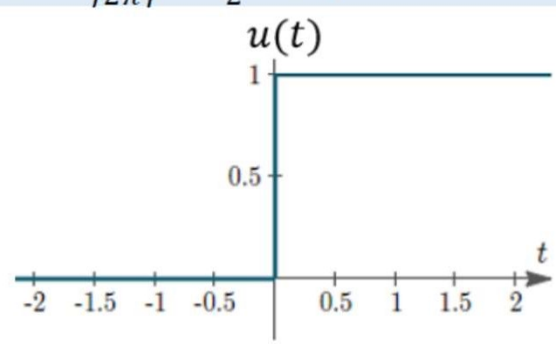
$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$$

$$sgn(t) = 2u(t) - 1$$

$$u(t) = \frac{1}{2}\{sgn(t) + 1\}$$

$$\Im\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\,\delta(f)$$





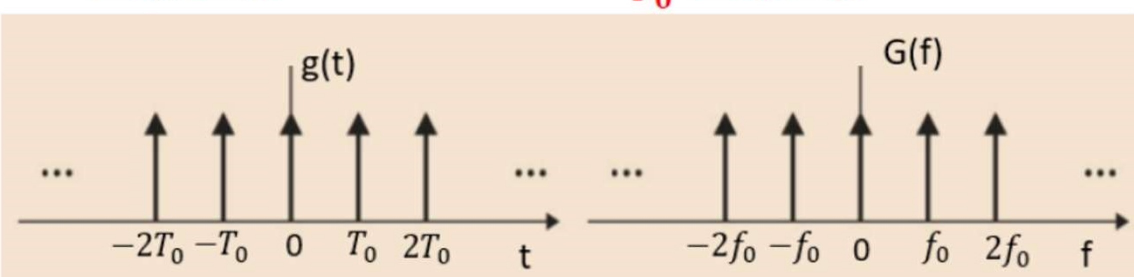
Applications of the Delta Function

- **Periodic Signals**: A periodic signal g(t) is expanded in the complex Fourier Series form as:
- $\boldsymbol{g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \Rightarrow \Im\{g(t) = \sum_{n=-\infty}^{\infty} C_n\, \delta(f - nf_0)}$

Example: Consider the following train of impulses $g(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$

Solution: The Fourier coefficients are obtained by integrating over one period of $g(t)$.

- $C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, e^{-jn\omega_0 t} dt = \frac{1}{T_0} = f_0$; Note that the sifting property has been used.
- Therefore, the complex Fourier series of $g(t)$ is
- $g(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}; \Rightarrow \Im\{g(t)\} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \Im\{e^{jn\omega_0 t}\} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0)$
- $\boldsymbol{\Im \sum_{m=-\infty}^{\infty} \delta(t - mT_0) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0)}$.

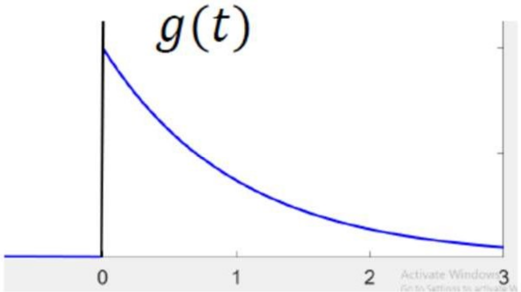


Remark 1: Note that the signal is periodic in the time domain and its Fourier transform is periodic in the frequency domain.

Remark 2: This sequence will be found useful when the sampling theorem is considered later in the course.

Examples

- Let g(t) be given as: $\mathbf{g}(t) = \begin{cases} A\,e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$.



- The Fourier transform of g(t) is: $G(f) = \left(\frac{A}{b+j2\pi f}\right)$

- **Evaluate the following**

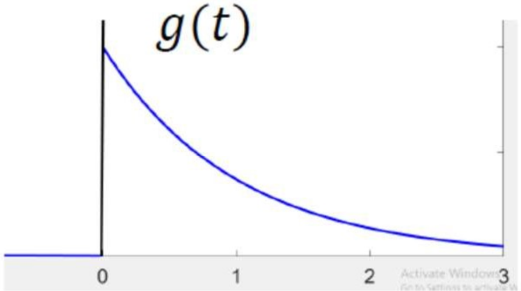
➢ $\mathbf{g}(t)\delta(t-0.5) = \mathbf{g}(t=0.5)\delta(t-0.5) = A\,e^{-0.5b}\delta(t-0.5)$.

➢ $\mathbf{g}(t)\delta(t+1) = \mathbf{g}(t=-1)\delta(t+1) = (0)\delta(t+1) = 0$.

➢ $\mathbf{g}(t) * \delta(t-1) = g(t-1) = \begin{cases} A\,e^{-b(t-1)} & t > 1 \\ 0 & t < 1 \end{cases}$

➢ $\Im\{\mathbf{g}(t) * \mathbf{g}(t)\} = G(f)G(f) = \left(\frac{A}{b+j2\pi f}\right)\left(\frac{A}{b+j2\pi f}\right) = \left(\frac{A}{b+j2\pi f}\right)^2$

Examples



- Let g(t) be given as: $\mathbf{g}(t) = \begin{cases} A\, e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$.
- The Fourier transform of g(t) is: $G(f) = \left(\frac{A}{b+j2\pi f}\right)$
- **Evaluate the following**

➢ $\int_{-\infty}^{\infty} g(t)\delta(t-1)dt = g(t=1) = A\, e^{-b}$; (sifting property)

➢ $\Im\{\mathbf{g}(t) - \mathbf{g}(t-1)\} = \boldsymbol{G}(\boldsymbol{f}) - \boldsymbol{G}(\boldsymbol{f})\boldsymbol{e}^{-j2\pi f} = \frac{A}{b+j2\pi f}(1 - \boldsymbol{e}^{-j2\pi f})$

➢ $\Im\{\mathbf{tg}(t)\} = \begin{cases} At\, e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$

➢ $\Im\{\mathbf{tg}(t)\} = \frac{j}{2\pi}\frac{dG(f)}{df} = \frac{j}{2\pi}\frac{(-)j2\pi}{(b+j2\pi f)^2} = \frac{1}{(b+j2\pi f)^2}$

➢Note: Prove that $\Im\{\mathrm{tg}(t)\} = \left(\frac{j}{2\pi}\right)\frac{dG(f)}{df}$ and $\Im\left\{\frac{dg(t)}{dt}\right\} = (j2\pi f)G(f)$