1.4: conditional probability and Independence.

→ sample space : $\mathcal{C}$

Def : The conditional prob. of event C_2 given the event C_1 is defined as $p(C_2|C_1) = \frac{p(C_1 \cap C_2)}{p(C_1)}$, provided $p(C_1) > 0$.

---

proposition : Multiplication Rule.

1. $p(C_1 \cap C_2) = p(C_1) \cdot p(C_2|C_1)$, $p(C_1) > 0$.
2. $p(C_1 \cap C_2 \cap C_3) = p(C_1) \cdot p(C_2|C_1) \cdot p(C_3 | C_1 \cap C_2)$, $p(C_1) > 0$ and $p(C_1 \cap C_2) > 0$.

Def : partition :

If $C_1, C_2, \ldots, C_K$ are i. mutually exclusive : $C_i \cap C_j = \phi$, $i \neq j$

ii. exhaustive : $\bigcup_{i=1}^{K} C_i = \mathcal{C}$

such that $p(C_i) > 0$, $\forall i$ so we say $C_1, \ldots, C_K$ Form a partition of $\mathcal{C}$.

---

proposition : The law of total probability.

$p(A) = \sum_{j=1}^{K} p(C_j)\, p(A|C_j)$ for any event $A \subset \mathcal{C}$.

proof : $A \subset \mathcal{C}$

$A \cap \mathcal{C} = A$

بخاصية التوزيع $A \cap (C_1 \cup C_2 \cup \cdots \cup C_K) = A$

$(A \cap C_1) \cup (A \cap C_2) \cup \cdots \cup (A \cap C_K) = A$.

→ $C_1, \ldots, C_K$ partition of $\mathcal{C}$: $C_i \cap C_j = \phi$ $\forall i \neq j$.

→ $(A \cap C_i) \cap (A \cap C_j) = \phi$, $\forall i \neq j$.

→ using (b) of def 7 :

$p(A) = \sum_{j=1}^{K} p(A \cap C_j)$

$\boxed{p(A) = \sum_{j=1}^{K} p(C_j)\, p(A|C_j)}$

Proposition : Bay's Theorem .

let $C_1, \dots, C_K$ partition for $\mathcal{C}$:

$$P(C_j|A) = \frac{P(A \cap C_j)}{P(A)}$$

$$P(C_j|A) = \frac{P(C_j)\,P(A|C_j)}{\sum_{i=1}^{K} P(C_i)\,P(A|C_i)}$$

RMK :

- $P(C_i)$ = Prior probability . قبل ما يصير event
- $P(C_i|A)$ = posterior probability . بعد ما يصير event

Def :

If $P(C_2|C_1) = P(C_2)$ and $P(C_1) > 0$, we say C_1 and C_2 are independent .

Proposition : C_1 and C_2 independent .

1. $P(C_1|C_2) = P(C_1)$, $P(C_2) > 0$
2. $P(C_1 \cap C_2) = P(C_1)\,P(C_2)$, $P(C_1) > 0$, $P(C_2) > 0$

↳ Multiplication Rule for indep. events .

Def : $C_1, C_2, \dots, C_n$ events from $\mathcal{C}$

1. $C_1, C_2, \dots, C_n$ are pairwise independent if, $P(C_i \cap C_j) = P(C_i) \cdot P(C_j)$, $\forall i \neq j$.
2. $C_1, C_2, \dots, C_n$ are mutually independent if, $P(C_{d_1} \cap C_{d_2} \cap \dots \cap C_{d_k}) = P(C_{d_1}) \cdot P(C_{d_2}) \cdot \dots \cdot P(C_{d_k})$.

where $d_1, d_2, d_3, \dots, d_k$ are K distinct integers from $1, \dots, n$ and $2 \leq K \leq n$

where $P(C_i) > 0$ $\forall i$.

example 1 : select at Random without replacement 5 cards From 52 cards, Find the prob. of an all spade hand (C_2) relative to the hypothesis That there are at least 4 spedes in the hand (C_1).

يعني ممكن 4 من spade أو اكثر ← C_1 : at least 4
يعني ممكن احتمال 4 و 5

C_2 : 5 spades. $\quad C_1 \supset C_2 \rightarrow C_1 \cap C_2 = C_2$.

$$\rightarrow P(C_2 \mid C_1) = \frac{P(C_1 \cap C_2)}{P(C_1)} = \frac{P(C_2)}{P(C_1)}.$$

$$\rightarrow P(C_2) = \frac{\binom{13}{5}\binom{39}{0}}{\binom{52}{5}} =$$

$$\rightarrow P(C_1) = \frac{\binom{13}{4}\binom{39}{1} + \binom{13}{5}\binom{39}{0}}{\binom{52}{5}} =$$

$$\rightarrow P(C_2 \mid C_1) = \frac{P(C_2)}{P(C_1)} = \frac{3}{68} = 0.0441.$$

سحبت كرة لونها أحمر فضل عندي 7 كرات بالصندوق عشان هيك $P(C_2) = \frac{5}{7}$

example 2 : A bowl contains eight chips, Three red and Five blue, ..., we want to compute the probability that the first draw results in a red chip (C_1) and that the second draw results in a blue chip (C_2). $P(C_1 \cap C_2)$??

$P(C_1) = \frac{3}{8}$, $P(C_2) = \frac{5}{7}$.

$$\rightarrow P(C_1 \cap C_2) = P(C_1) \cdot P(C_2) = \frac{3}{8} \times \frac{5}{7} = \frac{15}{56}.$$

Example 3 :

Example 4 : select 4 cards Randomly without replacement, From an ordiring deck. Find the prob. of having a spade, a heart, a diamond and a club in that order.

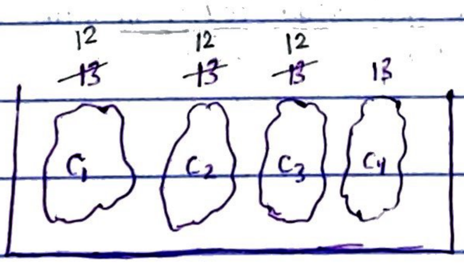
لما اسحب بطلع حسب هذا الترتيب

C_1 : spade
C_2 : heart
C_3 : diamond
C_4 : club.

$$P(C_1 \cap C_2 \cap C_3 \cap C_4) = P(C_1) \cdot P(C_2|C_1) \cdot P(C_3|C_1 \cap C_2) \cdot P(C_4|C_1 \cap C_2 \cap C_3)$$

$$= \frac{13}{52} \cdot \frac{13}{51} \cdot \frac{13}{50} \cdot \frac{13}{49}$$

$$= 0.004$$



52
51
50
49

نقص واحد
لأنه اخترت بطاقة

بدي قانون وفي عندي شرط الترتيب (Multiplication rule)

Example 5 :

C_1	C_2
3 red, 7 blue	8 red, 2 blue
Bowl C_1	Bowl C_2

$1:1:1:1:1:1 \rightarrow$ Fair die

$2:1:1:1:1:1 \rightarrow$ Baised die

اذا واحد بتوقع dice تصرفي انها Fair .

لـ كانت Baised : بحسابلنا انها مرات وبعطي النسبة

R: Red chip

B: Blue chip.

$\rightarrow$ Fair die

$1,2,3,4 \rightarrow C_2 \rightarrow P(C_2) = \frac{4}{6} = \frac{2}{3}$

$5,6 \rightarrow C_1 \rightarrow P(C_1) = \frac{2}{6} = \frac{1}{3}$

$\rightarrow$ Prior prob.

a die is cast, a bowl is selected and from that bowl a chip is selected If the selected chip is Red (C) Find the probability. the bowl selected was C_1.

$$\rightarrow P(C_1 \backslash C) = \frac{P(C_1 \cap C)}{P(C)} = \frac{P(C \backslash C_1) \cdot P(C_1)}{P(C \backslash C_1) \cdot P(C_1) + P(C \backslash C_2) \cdot P(C_2)}$$

$P(C \backslash C_1) = \frac{3}{10}$

عدد الأح من C_1

$P(C \backslash C_2) = \frac{8}{10} = \frac{4}{5}$

$$= \frac{\left(\frac{3}{10}\right) \cdot \left(\frac{2}{6}\right)}{\left(\frac{3}{10}\right) \cdot \left(\frac{2}{6}\right) + \left(\frac{8}{10}\right) \cdot \left(\frac{4}{6}\right)}$$

$$P(C_1 \backslash C) = \frac{3}{19} = 0.1579$$

$$\rightarrow P(C_2 \backslash C) = \frac{P(C_2 \cap C)}{P(C)} = \frac{P(C \backslash C_2) \cdot P(C_2)}{P(C \backslash C_2) \cdot P(C_2) + P(C \backslash C_1) \cdot P(C_1)}$$

$$= \frac{\left(\frac{8}{10}\right) \cdot \left(\frac{4}{6}\right)}{\left(\frac{8}{10}\right)\left(\frac{4}{6}\right) + \left(\frac{3}{10}\right)\left(\frac{2}{6}\right)}$$

$$= \frac{16}{19} = 0.8421$$

Note: $P(C_1) = \frac{2}{6}$, $P(C_2) = \frac{4}{6}$ $\rightarrow$ Prior prob.

$P(C_1 \backslash C) = \frac{3}{19}$, $P(C_2 \backslash C) = \frac{16}{19}$ $\rightarrow$ Posterior prob.

example 6 : The plants C_1, C_2, C_3 : 10% , 50% , 40% of a comany output .

→ C_1 : 1 % , C_2 = 3 % , C_3 = 4 % . All products are sent to a central warehouse .

→ one item selected at Random and observed to be defective say event C .

→ The conditional prob. it comes from plant C_1 is found as follows : $P(C_1|C)$ Find

By hypothesis :

$P(C_1) = 0.1$, $P(C_2) = 0.5$, $P(C_3) = 0.4$ prior prob.

$P(C|C_1) = 0.01$, $P(C|C_2) = 0.03$, $P(C|C_3) = 0.04$ posterior prob.

Bay's Thm

$$\Rightarrow P(C_1|C) = \frac{P(C_1)P(C|C_1)}{P(C_1)P(C|C_1) + P(C_2)P(C|C_2) + P(C_3)P(C|C_3)}$$

$$= \frac{(0.1)(0.01)}{(0.1)(0.01) + (0.5)(0.03) + (0.4)(0.04)}$$

$$= \frac{1}{32}$$

Check 7 on Book .

example 7 : independent events .

C_1 : 4 on the red die .

C_2 : 3 on the white die .

$P(C_1) = \frac{1}{6}$

$P(C_2) = \frac{1}{6}$

1. Find the prob. of order pair (4,3) .

$P([4,3]) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

② we can find the prob. of the sums :

SUM	prob
1	0
2	1\36
3	2\36
4	3\36
5	4\36
6	5\36
7	6\36
12	

2. The prob. that the sum of the up spots of the two dice equal 7 is :

$= P[(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)]$

$= P[(1,6)] + P[(2,5)] + P[(3,4)] + \ldots$

$= \frac{1}{6}\times\frac{1}{6} + \frac{1}{6}\times\frac{1}{6} + \frac{1}{6}\times\frac{1}{6} + \frac{1}{6}\times\frac{1}{6} + \frac{1}{6}\times\frac{1}{6} + \frac{1}{6}\times\frac{1}{6}$

$= \frac{6}{36}$

example 8 : independence

C_i : (H) on the i^{th} toss .

C_i^* : T .

Assume C_i and C_i^* are equally likely $P(C_i) = P(C_i^*) = \frac{1}{2}$

① Find the porb. of an ordered seq. like HHTH :

$P(C_1 \cap C_2 \cap C_3^* \cap C_4) = P(C_1)\,P(C_2)\,P(C_3^*)\,P(C_4)$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$.

أحصل على H في الرمية الثالثة

② the prob. of observing the first head on the third flip ? TTH .

$P(C_1^* \cap C_2^* \cap C_3) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

3. prob. of getting at least one head on four flips is :

$$P(C_1 \cup C_2 \cup C_3 \cup C_4) = 1 - P((C_1 \cup C_2 \cup C_3 \cup C_4)^*)$$

$$= 1 - P(C_1^* \cap C_2^* \cap C_3^* \cap C_4^*)$$

$$= 1 - \left(\frac{1}{2}\right)^4$$

$$= \frac{15}{16} \checkmark$$